Solitons in cyclic and symmetric structures

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Abstract

This research focuses on localised states arising from modulationally unstable plane waves in non-conservative cyclic and symmetric structures. The main application is on vibrations of bladed-disks of aircraft engines experiencing non-linear effects, such as large displacements, friction dissipation, and/or complex fluid-structure interactions. The investigation is based on a minimal model composed of a chain of linearly damped Duffing oscillators under external travelling wave excitation. The computed results are based on two strategies: (1) a Non-Linear Schrödinger Equation (NLSE) approximation; and (2) the periodic and quasi-periodic Harmonic Balance Methods (HBM). In both cases, the results show that unstable plane waves may self-modulate, leading to stable and unstable single and multiple solitons configurations.

Keywords: localised vibrations, solitons, cyclic structures, non-linear Schrödinger equation, harmonic balance methods

1 Introduction

Localisation of vibrations is a very important topic in rotating machines, such as bladed-disks of aircraft engines, due to high cycle fatigue. In the linear regime, localised vibrations arise in ideally periodic structures due to inherent inhomogeneities resulting e.g. from the manufacturing processes or wear. However, in real applications, when structures e.g. experience large deformations induced by strong excitations, their behaviours deviate from the linear regime due to non-linearities. It is well-known that, in the non-linear regime, energy localisation may arise even in perfect period structures due to bifurcations.

This research focuses on the non-linear dynamics of cyclic and symmetric structures excited by travelling waves. This excitation is very common in turbomachinery applications since it can be generated due to unbalances or aerodynamic excitations. The findings are based on two different strategies: (1) a non-conservative NLSE approximation; and (2) a fully numeric periodic and quasi-period HBM approach. The results show that stable solitons may emerge from unstable homogeneous solutions, leading to localised vibrations which move along the structure preserving their shapes.

2 Physical system

The physical system under investigation consists of N_s unitary masses, cyclically connected to each other by linear springs ω_c^2, and attached to the ground by linear springs ω_0^2, viscous dampers γ^2, and cubic springs ξ. The displacement for the nth degree of freedom u_n is written as

$$\ddot{u}_n + \gamma^2 \dot{u}_n + \omega_0^2 u_n - \omega_c^2 (u_{n-1} + u_{n+1} - 2u_n) + \xi u_n^3 = f_n, \tag{1}$$

where f_n represents the external excitation. For the following analysis, it is assumed that f_n has a travelling wave form such as

$$f_n(t) = F_0 \exp\left\{ i\left[k(n-1)a - \omega_f t \right] \right\} + \text{c.c.} \tag{2}$$

where F_0 is the force amplitude, k is the corresponding wave number, $a = 2\pi/N_s$ is the lattice parameter, ω_f is the external frequency, i is the imaginary unit, and c.c. states the complex-conjugate of the first expression. It should be noted that, due to the cyclic symmetry ($f_1 = f_{N_s+1}$), k can only assume integer numbers.

3 Solution approach

In this section, the two proposed solution approaches are presented. Firstly, the requirements to derive a non-conservative NLSE is introduced, and its limitations are discussed. Finally, a periodic and quasi-periodic HBM is proposed to compute homogeneous solutions and localised states emerging from unstable plane waves, respectively.

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3.1 Non-conservative NLSE

In the weakly non-linear regime the displacement u_n is assumed to have the form

$$u_n(t) = \varepsilon \Psi(X,T) \exp\{i[k(n-1)a - \omega_k t]\} + \text{c.c.}, \tag{3}$$

where $\Psi(X,T)$ is an envelope function which modulates the travelling wave response, $\omega_k = \sqrt{\omega_0^2 + 4\omega_c^2 \sin^2\left(\frac{ka}{2}\right)}$ is the linear natural frequency, and ε is a small parameter. In Eq. (3), X and T are continuous variables such as $X = \varepsilon x = \varepsilon(n-1)a$ and $T = \varepsilon t$. After plugging Eq. (3) into Eq. (1), it is possible to obtain

$$i\frac{\partial \Psi}{\partial \tau} + P\frac{\partial^2 \Psi}{\partial \eta^2} + Q|\Psi|^2\Psi = -i\Gamma\Psi - h\exp\{i\delta_\omega \tau\}, \tag{4}$$

where $\tau = \varepsilon T$, while $\eta = X - c_g t$ is a frame moving with the group velocity $c_g = \frac{d\omega_k}{dk}$. In Eq (4), the parameter $P = \frac{1}{2}\frac{d^2\omega_k}{dk^2}$ accounts for the dispersion, $Q = -\frac{3\xi}{2\omega_k}$ is the non-linear term, $\Gamma = \frac{\gamma^2}{2}$ models the linear damping effect, $h = \frac{F_0}{2\omega_k}$ is the external force, and $\delta_\omega = \omega_k - \omega_f$ is a detuning parameter. The approach is based on a multiscale approximation and the continuous limit [1, 2, 3]. In practical terms, Eq. (4) is valid when: (1) the solution Ψ varies slowly in space and time compared to a and $2\pi/\omega_f$; (2) the detuning parameter δ_ω is small and the system is close to the resonance; and (3) the system weakly forced and weakly damped. Finally, the carrier wave is assumed to vibrate with ω_f such as $\Psi(\eta, \tau) = \psi(\eta, \tau)\exp\{i\delta_\omega \tau\}$, leading Eq. (4) to an autonomous system.

3.2 Quasi-periodic HBM

In the quasi-periodic HBM approach, the solution u_n is expanded in a truncated Fourier series such as

$$u_n(t) = \sum_{-k_2}^{k_2}\sum_{-k_1}^{k_2} A_{k_2 k_1}\cos(k_2\omega_2 t + k_1\omega_1 t) + B_{k_2 k_1}\sin(k_2\omega_2 t + k_1\omega_1 t), \tag{5}$$

where k_1 and k_2 are the numbers of retained harmonics, $A_{k_2 k_1}$ and $B_{k_2 k_1}$ are the corresponding amplitudes, while ω_1 and ω_2 are the two incommensurable frequencies. It is important to notice that if $k_1=0$ or $k_2=0$ Eq. (5) recovers the standard periodic HBM. The values of $A_{k_2 k_1}$ and $B_{k_2 k_1}$ are obtained by plugging Eq. (5) into Eq. 1, followed by a Galerkin projection, in order to represent the response in Fourier space such as

$$[L]\{Z\} + \{f_{nl}\} - \{g\} = \{0\}. \tag{6}$$

In Eq. (6), $\{Z\}$ is the vector containing the Fourier coefficients $A_{k_1 k_2}$ and $B_{k_1 k_2}$, $[L]$ is a matrix representing the linear dynamics, while the vectors $\{f_{nl}\}$ and $\{g\}$ are the projections of the non-linear and the external forces into the Fourier space, respectively (see [4] for a more detailed discussion). The projection from time domain to Fourier space is calculated using a two-dimensional FFT algorithm, where the two frequencies ω_1 and ω_2 are sampled using a hyper-time approach [4]. Finally, one should note that this paper is interested in quasi-periodic oscillations arising from Neimark-Sacker bifurcations (see Sub. 4). Therefore, the frequency ω_2 is an unknown in Eq. (6), and an additional equation representing a phase condition is added to compute the value of ω_2 (see [5] for a more detailed discussion).

4 Numerical Results

The system described by Eq. (1) is solved assuming $N_s=36$, $\gamma^2=0.01$ kg $\cdot$ s^{-1}, $\omega_0^2=1.0$ kg$\cdot$ s^{-2}, $\omega_c^2=1.0$ kg$\cdot$s^{-2}, $\xi=0.10$ kg$\cdot$m$^{-2}\cdot$s^{-2}. The external force f_n is assumed to vibrate with amplitude $F_0 \sim 0.0225$ N and wave number $k=16$ m^{-1}.

In order to solve the previous example with the NLSE approach, standing solutions $\left(\frac{\partial \psi}{\partial \tau} = 0\right)$ resulting from the autonomous system discussed in Sub. 3.1 is solved with the continuation code AUTO [6]. Figure 1 shows the maximum displacement of u as a function of ω_f. In Panel (a) of Fig. 1 solid lines indicate stable solutions, while dashed lines identify unstable ones (see [7] for the stability analysis of ψ). Panel (b) of Fig. 1 shows the envelope function ψ for four different points identified in Panel (a): P1, P2, P3 show solutions for the branch detaching around $\omega_f \sim 2.2125$ rad/s, corresponding to solitons with one hump; and P4 identifies a solution with two humps corresponding to the branch detaching at $\omega_f \sim 2.2160$ rad/s.

The HBM approach is also applied to the same example using an arc-length continuation [4]. Solutions are shown in Panel (a) of Fig. 1 with dotted blue lines. For the plane wave branch Eq. (5) is solved assuming $k_2=0$ and $k_1=3$, while the solitons branches are calculated assuming $k_2=5$ and $k_1=1$. Initial conditions from S1 are used to solve Eq. (1) with a time-marching algorithm. Figure 2 shows the results from the quasi-periodic HBM approach compared to the numerical integration of Eq. (1). The results show great agreement.
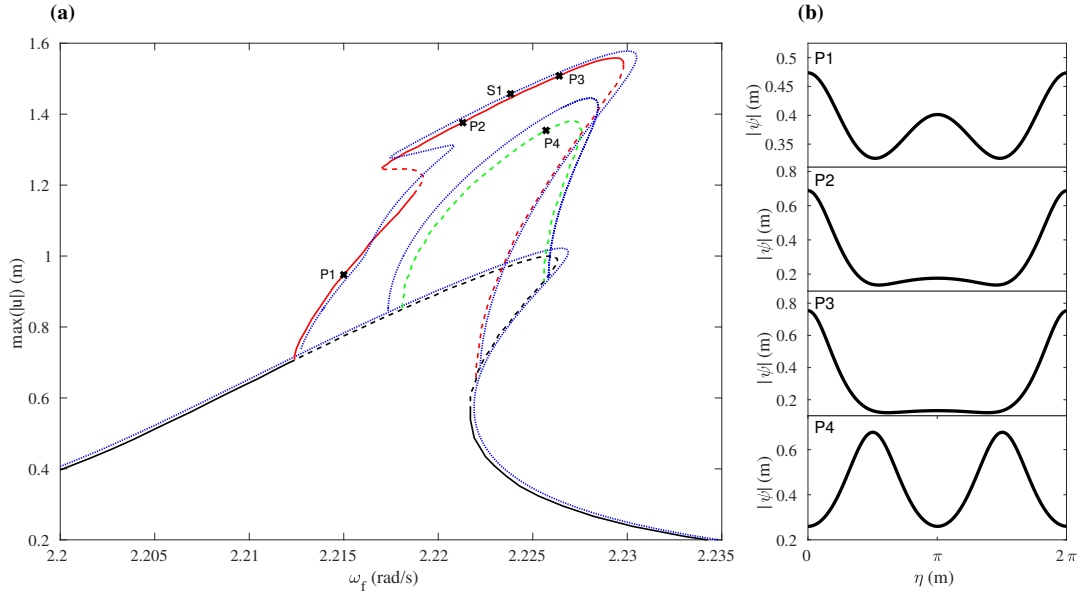
Figure 1: Panel (a) shows the bifurcation diagram computed with the NLSE and the HBM approaches. For the NLSE, solid lines show stable solutions while dashed lines identify unstable ones. Blue dotted lines show results obtained from the HBM approach. Panel (b) shows the envelope functions calculated with the NLSE.
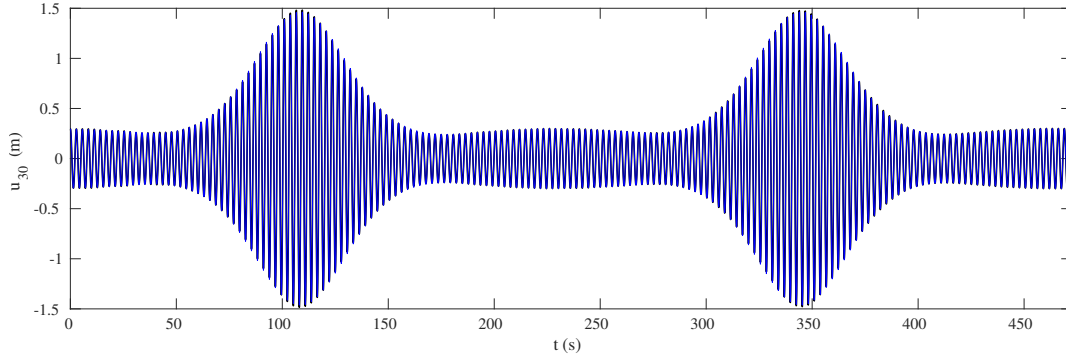

Figure 2: Time evolution of u_{30} calculated with the quasi-periodic HBM (blue) and with a time-marching algorithm (black).

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