Nonequivalence between absolute separability and positive partial transposition in the symmetric subspace

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Abstract. The equivalence between absolutely separable states and absolutely positive partial transposed (PPT) states in general remains an open problem in quantum entanglement theory. In this work, we study an analogous question for symmetric multiqubit states. We show that symmetric absolutely PPT (SAPPT) states (symmetric states that remain PPT after any symmetry-preserving unitary evolution) are not always symmetric absolutely separable by providing explicit counterexamples. More precisely, we construct a family of entangled five-qubit SAPPT states. Similar counterexamples for larger odd numbers of qubits are identified.

Keywords: Quantum entanglement, Absolute separability, Entanglement detection

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1 Introduction

In quantum information, absolutely separable (AS) states are those that remain separable under any global unitary transformation. Their full characterization remains open and known as the *separability from spectrum* problem. On the other hand, absolutely PPT (APPT) states, defined analogously via the positive partial transpose (PPT) criterion [1], are fully characterized. Thus, because of the close connection between separable states and PPT states, a promising way to solve the *separability from spectrum* open problem would be to show that the APPT and AS sets are equivalent, as was undertaken in Ref. [2]. Although AS implies APPT according to the PPT criterion, the converse remains an open question.

In this work, we investigate this equivalence for symmetric states, where the symmetry constraint introduces a new structure. Here, the analogous relevant notions become Symmetric Absolutely Separable (SAS), and Symmetric Absolutely PPT (SAPPT) states. A state is SAS if it remains separable after the action of any symmetry-preserving unitary. Similarly, a symmetric state ρ is said to be SAPPT if it remains PPT under all unitary transformations that preserve symmetry. Although SAS and SAPPT are equivalent in the case of two and three qubit systems, where the PPT criterion is both necessary and sufficient [1], the general case remains unresolved. To disprove the general equivalence, it suffices to provide a counterexample, that is, an entangled SAPPT state. This question represents much more than a simplified version of the nonsymmetric case; it is of significant independent interest because of the many physical quantum systems constrained by permutation symmetry such as Bose-Einstein condensates [3, 4, 5] and multiphoton systems [6]. More generally, entanglement in bosonic systems plays an important role as a resource in quantum metrology and quantum information [7]. In this work, we show that SAPPT and SAS states are not equivalent by giving an explicit family of entangled SAPPT states [8].

2 Overview of main results

We consider the convex mixture

$$\rho(p) = p\,\rho_0 + (1-p)\,|\psi_0\rangle\langle\psi_0| \tag{1}$$

of the maximally mixed state in the symmetric sector, ρ_0, with probability p, and a pure symmetric state $|\psi_0\rangle\langle\psi_0|$ with probability $1-p$, where $p \in [0,1]$. We begin by providing for the first time the eigendecomposition of $\rho_0^{T_A}$ for any bipartition $k|N-k$, see Ref. [8]. In particular, we prove that its minimum eigenvalue is given by

$$\lambda_{\min}(\rho_0^{T_A}) = \left[(N+1)\binom{N}{k}\right]^{-1}. \tag{2}$$

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After deriving the minimal eigenvalue of $(|\psi_0\rangle\langle\psi_0|)^{T_A}$, we derive a necessary and sufficient condition on the value of the critical value $p_{\min}$ beyond which the state (1) is SAPPT.

Theorem 1 *Any symmetric N-qubit state ρ with a spectrum composed of $N+1$ non-zero eigenvalues of the form $\left(1-\frac{Np}{N+1}, \frac{p}{N+1}, \ldots, \frac{p}{N+1}\right)$ is SAPPT if and only if $p \in [p_{\min}, 1]$ with*

$$p_{\min} = \frac{1}{1+2\left[(N+1)\binom{N}{\lfloor N/2 \rfloor}\right]^{-1}}.$$

Theorem 1 is an improvement over a SAPPT criterion derived in Ref. [9] using invertible linear maps of operators.

For $N=5$, we find that the SAPPT state $\rho(p_{\min})$ with $|\psi_0\rangle$ the Greenberger-Horne-Zeilinger (GHZ) state, $|\mathrm{GHZ}_5\rangle$, is detected to be entangled by not having a 2-copy PPT symmetric extension of the second party for the bipartition $1|4$ (see Ref. [10] for more details). This method, whenever the state is entangled, allows one to obtain additionally an entanglement witness W_5 [10]. This operator W_5 also detects entanglement of $\rho(p)$ for values of p other than $p_{\min}$ (see Figure 1, bottom panel), thus providing a uniparametric family of SAPPT bound entangled states $\rho(p)$ for $p \in [p_{\min}, p_{\mathrm{ent}}^{W_5}]$. A larger family can be obtained using the reformulation of the separability problem as a truncated moment problem (see Ref. [11] for more details), which can be implemented as a semidefinite optimization.

In a similar way, we searched for entangled SAPPT states for a number of qubits up to $N=10$. For an odd number of qubits, the state (1) with $|\psi_0\rangle = |\mathrm{GHZ}_N\rangle$, is SAPPT and detected as entangled for values of p in the range $[p_{\min}, p_{\mathrm{ent}}]$ (see Table 1). On the other hand, for even N, we find that the state $\rho(p)$ is always separable for any $p \in [p_{\min}, 1]$. So we could not find an example of an entangled SAPPT state for an even number of qubits.

3 Conclusion

In this work, we established a sufficient condition for certain symmetric states of N-qudit systems to be SAPPT (symmetric absolutely PPT). For qubits, we showed that this condition is also necessary. In the course of this proof, we analytically determined the spectrum of $\rho_0^{T_A}$ where ρ_0 is the maximally mixed state in the symmetric subspace. Based on this, we proved the existence of entangled SAPPT states for

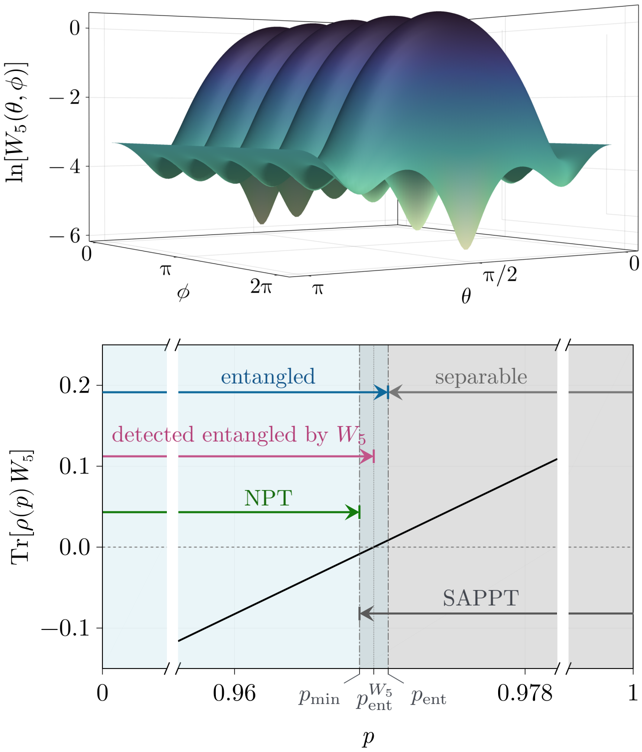

Figure 1: Top panel: Logarithm of the expectation value of the entanglement witness W_5 over the pure symmetric product states, showing its positivity over separable states. Bottom panel: Expectation value of the entanglement witness W_5 in $\rho(p)$ as a function of p (black oblique straight line). Entangled SAPPT states lie within the overlap between the blue (entangled states) and gray (SAPPT states) areas, ranging from $p = p_{\min} = 30/31 \approx 0.96774$ to $p_{\mathrm{ent}} = 0.96953$. Those detected by the witness W_5 lie between $p_{\min}$ and $p_{\mathrm{ent}}^{W_5} \approx 0.96862 < p_{\mathrm{ent}}$.

an odd number of qubits from $N=5$ by constructing explicit entanglement witnesses. These results resolve an open question concerning the equivalence between SAPPT and SAS states, by showing that this equivalence does not hold in general, although it does apply to 2-qubit and 3-qubit systems.

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N	$p_{\min}$	$p_{\text{ent}}^{W_N}$	p_{ent}
4	$\frac{15}{16}$	/	$\frac{15}{16}$
5	$\frac{30}{31} \approx 0.96774$	0.96862	0.96953
6	$\frac{70}{71}$	/	$\frac{70}{71}$
7	$\frac{140}{141} \approx 0.99291$	0.99302	0.99329
8	$\frac{315}{316}$	/	$\frac{315}{316}$
9	$\frac{630}{631} \approx 0.99842$	0.99845	0.99849
10	$\frac{1386}{1387}$	/	$\frac{1386}{1387}$

Table 1: Particular values of the probability p defining the state $\rho(p)$ given by Eq. (1) with $|\psi_0\rangle = |\text{GHZ}_N\rangle$, which has an eigenspectrum $\left(1 - \frac{Np}{N+1}, \frac{p}{N+1}, \ldots, \frac{p}{N+1}\right)$. First column: number of qubits. Second column: minimum value of p, $p_{\min}$ as it appears in Theorem 1, for $\rho(p)$ to be SAPPT. Third column: maximum value $p_{\text{ent}}^{W_N}$ such that the witnesses W_N given in [8] for $N = 5, 7$ and 9, respectively, detect that $\rho(p)$ is entangled. Fourth column: value p_{ent} below which the state $\rho(p)$ is found to be entangled using the method described in Ref. [11].

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