Hölder spaces for Lie groups and application to the 2-sphere

S. Nicolay

joint work with A. Molla

Deuxièmes journées de l'axe Analyse Multifractale

Outline

Notations

- We will consider an arbitrary connected Lie group G, equipped with a left-invariant Riemannian metric.

- We will denote the Lie algebra of G by $\mathfrak{g}$ is equipped with $\|X\| = d(\exp X, 1)$ for $X \in \mathfrak{g}$, where $\exp : \mathfrak{g} \to G$ is the exponential map.

- We will mainly address two types of neighborhoods:

$$\mathfrak{B}(r) = \{X \in \mathfrak{g} : \|X\| < r\}$$

and

$$B(r) = \{x \in G : d(x, 1) < r\}.$$

It is clear that $\exp \mathfrak{B}(r) \subset B(r)$ for any $r > 0$ that is sufficiently small.

- For a given $x \in G$, consider the left and right translations as mappings defined on G by $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$, resp.

- We define the right difference operator as $\Delta_h = R_h^* - I$, where I represents the identity operator, and R_h^* signifies the pullback of R_h in the sense that $R_h^* f = f \circ R_h$. Similarly, the left difference operator is defined as ${}_h\Delta = L_h^* - I$.

Hölder spaces for Lie groups

Definition

Given $\sigma > 0$, a function $f : G \to \mathbb{R}$ belongs to the pointwise Hölder space $\Lambda^\sigma(1)$ under the following conditions:

- f belongs to $L^\infty_{\mathrm{loc}}(G)$ with respect to a fixed Haar measure;
- there exists constants $C > 0$, $R > 0$ and an integer $m > \sigma$ st

  $$\sup_{\|X\| \leq r_1} \|\Delta^m_{\exp X} f\|_{B(r_2)} \leq C(r_1 + r_2)^\sigma,$$

  for all $r_1, r_2 \leq R$ (where $\|\cdot\|_{B(r)}$ represents the uniform norm).

The pointwise Hölder space $\Lambda^\sigma(x_0)$ consists of functions $f : G \to \mathbb{R}$ for which the pullback under the left translation $L^*_{x_0} f$, belongs to $\Lambda^\sigma(1)$.

- These spaces remain invariant regardless of the chosen left-invariant Riemannian metric.

- Furthermore, employing ${}_{\exp X}\Delta^m$ instead of $\Delta^m_{\exp X}$ yields the same space.

- The space $\exp^* \Lambda^\sigma(1)$ is connected to the classical Hölder space $\mathcal{C}^\sigma(0)$ on the group $(\mathfrak{g}, +, 0)$: we always have $\exp^* \Lambda^\sigma(1) \subset \mathcal{C}^\sigma(0)$, with equality if G admits a bi-invariant distance.

For $G = \mathbb{R}^n$, we have $\mathfrak{g} = \mathbb{R}^n$ and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\Lambda^\sigma(x_0)$ if there exist $C > 0$, $R > 0$ and $m > \sigma$ st

$$\sup_{|h| \leq r_1} \|\Delta_h^m (f(x_0 + \cdot))\|_{B(r_2)} \leq C(r_1 + r_2)^\sigma,$$

for all $r_1, r_2 < R$. This can be rewritten

$$|\Delta_h^m f(x)| \leq C(|h| + |x - x_0|)^\alpha,$$

for $|h| < r_1$ and $|x - x_0| < r_2$.

Definition

Given $\sigma > 0$, a function $f : G \to \mathbb{R}$ belongs to $\Lambda_r^\sigma(G)$ under the following conditions:

- f belongs to $L^\infty_{\mathrm{loc}}(G)$;
- there exist constants $C > 0$, $R > 0$ and an integer $m > \sigma$ st

$$\sup_{\|X\| \leq r} \|\Delta^m_{\exp X} f\|_G \leq C r^\sigma,$$

for all $r < R$.

$\Lambda_l(G) = \ldots$ and $\Lambda(G) = \Lambda_r(G) \cap \Lambda_l(G)$.

Proposition

If G is compact, let V be an open neighborhood of 0 in g such that the exponential is a diffeomorphism between V and its image, let $\rho \in D(V)$ be a positive function equal to 1 on $\frac{1}{2}V$ and set $U = \frac{1}{2}V$. Let $x_1, \ldots, x_N \in G$ be such that $G = \cup_{i=1}^N x_i U$ and set

$$F_i(X) = \rho(X) f(x_i \exp X),$$

for all $X \in \mathfrak{g}$.

Then f belongs to $\Lambda^\sigma(G)$ if and only if F_i belongs to $\mathcal{C}^\sigma(\mathfrak{g})$ for all $i \in \{1, \ldots, N\}$.

Theorem

Let $\sigma > 0$; a function f belongs to $\Lambda^\sigma(1)$ if and only if there exist constants $C > 0$, $R > 0$ st

$$\sup_{x \in B(r_1)} \inf_{P \in \mathbb{R}_{[\sigma]}[\mathfrak{g}]} \| f \circ L_x \circ \exp - P \|_{\mathfrak{B}(r_2)} \leq C(r_1 + r_2)^\sigma,$$

for any $r_1, r_2 < R$.

Theorem

Let $\sigma > 0$; a function f belongs to $\Lambda^\sigma(1)$ if and only if there exist constants $C > 0$, $R > 0$ st

$$\sup_{x \in B(r_1)} \inf_{P \in \mathbb{R}_{[\sigma]}[\mathfrak{g}]} \|f \circ L_x \circ \exp - P\|_{\mathfrak{B}(r_2)} \leq C(r_1 + r_2)^\sigma,$$

for any $r_1, r_2 < R$.

Theorem

Suppose that G is endowed with a bi-invariant Riemannian distance and let $\sigma > 0$ such that $\sigma \notin \mathbb{N}$; a function f belongs to $\Lambda^\sigma(G)$ if and only if there exist constants $C > 0$, $R > 0$ and a family of polynomials $(P_x)_{x \in G}$ on $\mathfrak{g}$ of degree at most $[\sigma]$ st

$$\|f \circ L_x \circ \exp - P_x\|_{\mathfrak{B}(r)} \leq C r^\sigma,$$

for all $r < R$ and $x \in G$.

Wavelet transform on compact Lie groups

Definition

Let G be a compact Lie group and U be a neighborhood of the identity such that exp is a diffeomorphism between a neighborhood V of 0 and U. A wavelet on U is a function $\psi \in D(G)$ with support in U. We define the dilated wavelet $\psi_{(j)}$ by

$$\psi_{(j)}(x) = 2^{jn}\psi_{\mathfrak{g}}(2^j \log x)\vartheta(\log x),$$

on U, and 0 everywhere else, for any $j \in \mathbb{N}_0$, where $\psi_{\mathfrak{g}}$ is $\psi \circ \exp$ on V and 0 on $\mathfrak{g} \setminus V$ and ϑ denotes the inverse of the Jacobian of the exponential map at X on an oriented Lie group with a fixed basis of $\mathfrak{g}$.

The dyadic wavelet transform at scale $j \in \mathbb{N}$ is the map

$$W^j_\psi : D'(G) \to C^\infty(G) \quad T \mapsto \psi_{(j)} * T.$$

Definition

A wavelet ψ on U is admissible if it satisfies the following conditions:

- $\int_U \psi(x)\vartheta(\log x)\, dx = 0$,
- the construction formula holds in $D'(G)$ for the weak-$*$ convergence:

$$T = \phi * T + \sum_{j=1}^{\infty} W_\psi^j T, = \sum_{j=0}^{\infty} W_\psi^j T,$$

where $\phi \in D(U)$ is some function satisfying $\psi_{(0)} = \phi_{(1)} - \phi_{(0)}$.

Definition

A function $f \in L^1(G)$ has m vanishing moments if f is supported in the exponential chart U and

$$\int_U f(x)P(\log x)\vartheta(\log x)\, dx = 0,$$

for any polynomial P on $\mathfrak{g}$ with degree at most m.

If f has m vanishing moments on G, then it is supported in the exponential chart, and $\langle f, P\rangle = 0$ for any smooth solution P of the global Fréchet functional equation of order at most m.

Let ρ be a smooth function on $\mathfrak{g}$ compactly supported in a neighborhood $\mathfrak{B}(R/m)$ of 0, where $R > 0$ is such that the exponential is a diffeomorphism between $\mathfrak{B}(R)$ and its image, and set $c = \int_{\mathfrak{g}} \rho(X)\, dX \neq 0$. Define

$$\Phi(X) = \sum_{j=1}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} \frac{1}{j^n} \rho(\frac{X}{j}),$$

so that Φ is supported in $\mathfrak{B}(R)$. The function

$$\varphi(x) = \frac{(-1)^m}{c} \Phi(\log x) \vartheta(\log x)$$

is with support included in $\exp \mathfrak{B}(R)$. The function

$$\psi(x) = (\varphi_{(1)}(x) - \varphi_{(0)}(x)) \mathcal{J}(\log x),$$

where $\mathcal{J}(X)$ is the Jacobian of the exponential map at X, is a dyadic wavelet on G that possesses m vanishing moments

Proposition

Let $\sigma > 0$ and consider an admissible wavelet with $m > \sigma$ vanishing moments. A continuous function f belongs to $\Lambda^\sigma(G)$ iff there exists $C > 0$ st

$$|W_\psi^j f(x)| \leq C 2^{-j\sigma},$$

for any $j \in \mathbb{N}_0$ and any $x \in G$.

Proposition

Let $\sigma > 0$ and consider an admissible wavelet with $m > \sigma$ vanishing moments. If f belongs to $\Lambda^\sigma(1)$, then there exist $C > 0$ and $R > 0$ st

$$|W_\psi^j f(x)| \leq C 2^{-j\sigma}(1 + 2^j r)^\sigma,$$

for $j \in \mathbb{N}_0$, $x \in B(r)$ and $r < R$.

Proposition

Let $\sigma > 0$ and consider an admissible wavelet with $m > \sigma$ vanishing moments. If f belongs to $\Lambda^\sigma(1)$, then there exist $C > 0$ and $R > 0$ st

$$|W_\psi^j f(x)| \leq C 2^{-j\sigma}(1 + 2^j r)^\sigma,$$

for $j \in \mathbb{N}_0$, $x \in B(r)$ and $r < R$.

Proposition

Let $\sigma > 0$ with $\sigma \notin \mathbb{N}$, and consider an admissible wavelet with $m > \sigma$ vanishing moments. If f belongs to L^∞_{loc} and there exists $0 \leq \sigma' < \sigma$, $C > 0$, and $R > 0$ such that

$$|W_\psi^j f(x)| \leq C 2^{-j\sigma}(1 + 2^j r)^{\sigma'},$$

for any $j \in \mathbb{N}_0$, and $x \in B(r)$ for $r < R$, then f belongs to $\Lambda^\sigma(1)$.

Proposition

Let $\sigma > 0$ with $\sigma \notin \mathbb{N}$, and consider an admissible wavelet with $m > \sigma$ vanishing moments. If f belongs to $\Lambda^{\epsilon}_{\text{loc}}(1)$ for some $\epsilon > 0$ and there exist $C > 0$ and $R > 0$ such that

$$|W^j_\psi f(x)| \leq C 2^{-j\sigma}(1 + 2^j r)^\sigma,$$

for any $j \in \mathbb{N}_0$, and $x \in B(r)$ for $r < R$, then there exist $C_0 > 0$ and $R_0 > 0$ st

$$\inf_{P \in \mathbb{R}_{[\sigma]}[\mathfrak{g}]} \|f \circ \exp - P\|_{\mathfrak{B}(r)} \leq C_0 r^\sigma |\log r|,$$

for any $r < R_0$.

Examples

- The n-dimensional torus $(S^1)^n$ as a subgroup of $\mathbb{C}^n \setminus \{0\}$;
- the orthogonal group $O(n, \mathbb{R})$;
- the special orthogonal group $SO(n, \mathbb{R})$.

Examples

- The n-dimensional torus $(S^1)^n$ as a subgroup of $\mathbb{C}^n \setminus \{0\}$;
- the orthogonal group $O(n, \mathbb{R})$;
- the special orthogonal group $SO(n, \mathbb{R})$.

For the Heisenberg group, $\Lambda^\sigma(x_0)$ does not correspond to the Hölder space defined by S. Seuret and F. Vigneron.

The Weierstraß function

Definition

The Hölder exponent of a locally bounded function $f : G \to \mathbb{R}$ at a point $x \in G$ is defined as

$$h_f(x) = \sup\{\sigma > 0 : f \in \Lambda^\sigma(x)\}.$$

If $h = h_f(x)$, for every $\epsilon > 0$ (st $h - \epsilon > 0$), we have $f \in \Lambda^{h-\epsilon}(x)$ but $f \notin \Lambda^{h+\epsilon}(x)$.

Definition

If the map $x \mapsto h_f(x)$ is constant, then f is said to be unifractal with exponent $h_f(1)$.

Given $\sigma \in (0,1)$, consider the function

$$\mathfrak{W}_\sigma : S^1 \to \mathbb{R} \quad z \mapsto \sum_{j=1}^{\infty} 2^{-\sigma j} \Re z^{2^j}.$$

The exponential chart maps x to e^{ix}, so that

$$\mathfrak{W}_\sigma(z_0 \exp x) = \sum_{j=1}^{\infty} 2^{-\sigma j} \cos(2^j(\theta_0 + x)),$$

with $z_0 = e^{i\theta_0}$, which establishes continuity and nowhere differentiability in a neighborhood of z_0.

It is not difficult to show that $\mathfrak{W}_\sigma \in \Lambda^\sigma(G)$, and $\mathfrak{W}_\sigma \notin \Lambda^{\sigma'}(z_0)$, for any $\sigma' > \sigma$ and $z_0 \in S^1$.

If G is a non-trivial compact Lie group, let $\rho : G \to GL(V)$ be a non-trivial irreducible unitary continuous Lie group representation on a finite-dimensional Hilbert space V.

Definition

The Weierstraß function of parameter $\sigma \in (0, 1)$ is the function

$$\mathfrak{W}_\sigma : G \to \mathbb{R} \quad x \mapsto \sum_{j=1}^{\infty} 2^{-\sigma j} \Re(\mathrm{tr}(\rho(x)^{2^j})).$$

Proposition

$\mathfrak{W}_\sigma$ belongs to $\Lambda^\sigma(G)$.

Proposition

$\mathfrak{W}_\sigma$ belongs to $\Lambda^\sigma(G)$.

Proposition

$\mathfrak{W}_\sigma$ does not belong to $\Lambda^{\sigma'}(x_0)$, for any $\sigma' > \sigma$ and any $x_0 \in G$, .

Proposition

$\mathfrak{W}_\sigma$ belongs to $\Lambda^\sigma(G)$.

Proposition

$\mathfrak{W}_\sigma$ does not belong to $\Lambda^{\sigma'}(x_0)$, for any $\sigma' > \sigma$ and any $x_0 \in G$, .

Corollary

$\mathfrak{W}_\sigma$ is unifractal with Hölder exponent σ.

Hölder-regularity on the sphere S^2

- The Hairy ball theorem implies that S^2 on $\mathbb{R}^3$ does not admit a Lie group structure.

- Instead of having a group operation, we can consider an action of a group on it and apply directly our results to obtain analogous properties on the sphere. As S^2 is invariant for rotations in $\mathbb{R}^3$, we naturally have an action of the Lie group $SO(3,\mathbb{R})$ on S^2.

- On the Lie algebra $\mathfrak{o}(3,\mathbb{R})$, we equip an $Ad(SO(3,\mathbb{R}))$-invariant inner product given by
$$K : \mathfrak{o}(3,\mathbb{R}) \times \mathfrak{o}(3,\mathbb{R}) \to \mathbb{R} \quad (X,Y) \mapsto -\mathrm{tr}(XY).$$

- The invariant Riemannian metric induced by the Killing form of $\mathfrak{o}(3,\mathbb{R})$ is equal to the Riemannian metric induced by the Euclidean inner product on $\mathbb{R}^3$ up to a positive multiplicative constant. Therefore, the Riemannian metric defined here is essentially the same as the natural one.

Rotations on S^2 play the role that translations do in $\mathbb{R}^n$. The group $SO(3,\mathbb{R})$ acts transitively on S^2, so that S^2 is a homogeneous space, with stabilizer subgroup isomorphic to $SO(2,\mathbb{R})$. This yields the following diffeomorphism of manifolds:

$$S^2 \simeq SO(3,\mathbb{R})/SO(2,\mathbb{R}).$$

The universality property of surjective submersions establishes that the C^p-regularity of a function on S^2 is equivalent to the C^p-regularity of $f \circ T \circ \Pi$ on the group $SO(3,\mathbb{R})$.

Here,

$$T : SO(3,\mathbb{R})/G_{p_0} \to S^2 \quad [A] \mapsto Ap_0,$$

where G_{p_0} is the stabilizer of p_0 and

$$\Pi : SO(3,\mathbb{R}) \to SO(3,\mathbb{R})/G_{p_0} \quad A \mapsto [A],$$

Definition

- The space $\Lambda^\sigma(p)$ consists of locally bounded functions $f : S^2 \to \mathbb{R}$ such that $f \circ T \circ \Pi$ belongs to $\Lambda^\sigma(M)$ in $SO(3, \mathbb{R})$, where M is chosen such that $p = Mp_0$.
- Similarly, $\Lambda^\sigma(S^2)$ is the spaces of locally bounded functions $f : S^2 \to \mathbb{R}$ such that $f \circ T \circ \Pi$ belongs to $\Lambda^\sigma(SO(3, \mathbb{R}))$.

These spaces are well-defined, meaning that their definition is independent of the specific choice of p_0 or M.

Definition

Let $H \in \mathfrak{o}(3,\mathbb{R})$; the spherical finite difference operator of step H is the operator Δ_H defined by

$$\Delta_H f(p) = f(\exp(H)p) - f(p),$$

for $p \in S^2$

For any map $f : S^2 \to \mathbb{R}$, we have

$$\Delta_H^m f(p) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\exp(jH)p).$$

Let $\mathcal{K}$ be the norm on $\mathfrak{o}(3,\mathbb{R})$ induced by K.

Proposition

- A function f belongs to $\Lambda^\sigma(p)$ iff f is locally bounded and there exist $C > 0$, $R > 0$ and an integer $m > \sigma$ such that

$$\sup_{\mathcal{K}(H)<r_1} \|\Delta_H^m f\|_{B(r_2)p} \leq C(r_1 + r_2)^\sigma,$$

  for $r_1, r_2 < R$.

- A function f belongs to $\Lambda^\sigma(S^2)$ iff f is locally bounded and there exist $C > 0$, $R > 0$ and an integer $m > \sigma$ such that

$$\sup_{\mathcal{K}(H)<r} \|\Delta_H^m f\|_{S^2} \leq C r^\sigma,$$

  for $r < R$.

we will make use of the map

$$\exp_p : \mathfrak{o}(3,\mathbb{R}) \to S^2 \quad H \mapsto (\exp H)p,$$

where $\exp : \mathfrak{o}(3,\mathbb{R}) \to SO(3,\mathbb{R})$ is the Riemannian exponential.

Proposition

- A function f belongs to $\Lambda^\sigma(p)$ iff f is locally bounded and there exist a polynomial P of degree at most $[\sigma]$ on $\mathfrak{o}(3,\mathbb{R})$, $C > 0$ and $R > 0$ st

$$\|f \circ \exp_p - P\|_{B_{\mathcal{K}}(r)} \leq C r^\sigma,$$

  for $r < R$.

- A function f belongs to $\Lambda^\sigma(S^2)$ iff f is locally bounded and there exist $C > 0$ and $R > 0$ st

$$\sup_{q \in S^2} \inf_{P \in \mathbb{R}_{[\sigma]}[\mathfrak{o}(3,\mathbb{R})]} \|f \circ \exp_q - P\|_{B_{\mathcal{K}}(r)} \leq C r^\sigma,$$

  for $r < R$.

Wavelets on the sphere S^2

Let us introduce an invariant measure on S^2: if μ is the Haar measure on $SO(3,\mathbb{R})$, we can consider the induced measure ν on S^2 defined as the image measure

$$\nu(A) = \mu(\pi^{-1}A)),$$

for any Borel set A of S^2, where π is the projection map

$$\pi : SO(3,\mathbb{R}) \to S^2 \quad M \mapsto Mp_0,$$

for a fixed point p_0. It is invariant by rotation.

Definition

- Let U be a neighborhood of the identity in $SO(3,\mathbb{R})$ such that the exponential is a diffeomorphism between a neighborhood of 0 in $\mathfrak{o}(3,\mathbb{R})$ and U. A wavelet on S^2 is an admissible wavelet ψ on U.

- The wavelet transform of a locally bounded function $f : S^2 \to \mathbb{R}$ at scale $j \in \mathbb{N}_0$ is the function $W_\psi^j f$ st

$$(W_\psi^j f) \circ T \circ \Pi = W_\psi^j (f \circ T \circ \Pi),$$

  where the operator W_ψ^j in the rhs is the wavelet transform given in the previous definition.

We have the following reconstruction formula in $D'(SO(3,\mathbb{R}))$:

$$f \circ T \circ \Pi = \sum_{j=0}^{\infty} (W_\psi^j f) \circ T \circ \Pi.$$

Proposition

Let $\sigma > 0$ such that $\sigma \notin \mathbb{N}$ and consider a wavelet ψ with $m > \sigma$ vanishing moments. Let $f : S^2 \to \mathbb{R}$ be a locally bounded function.

We have $f \in \Lambda^\sigma(S^2)$ if and only if there exist $C > 0$ and $R > 0$ such that

$$|W_\psi^j f(q)| \leq C 2^{-j\sigma},$$

for $j \in \mathbb{N}_0$ and $q \in S^2$.

Proposition

Let $\sigma > 0$ such that $\sigma \notin \mathbb{N}$ and consider a wavelet ψ with $m > \sigma$ vanishing moments. Let $f : S^2 \to \mathbb{R}$ be a locally bounded function.

- If f belongs to $\Lambda^\sigma(p)$ then there exist $C > 0$ and $R > 0$ such that

$$|W_\psi^j f(q)| \leq C 2^{-j\sigma}(1 + 2^j r)^\sigma, \qquad (*)$$

for $j \in \mathbb{N}_0$ and $q \in B(r)p$, for $r < R$.

- Conversely, assume that there exists $\epsilon > 0$ such that $f \in \Lambda^\epsilon_{\mathrm{loc}}(p)$; if $(*)$ is satisfied, then there exist $C_0 > 0$ and $R_0 > 0$ such that

$$\inf_{P \in \mathbb{R}_{[\sigma]}[\mathfrak{o}(3,\mathbb{R})]} \|f \circ \exp_p - P\|_{B_\mathcal{K}(r)} \leq C_0 r^\sigma |\log r|,$$

for $r < R_0$.

- Similar constructions can be carried out on other homogeneous spaces, such as hyperbolic space with the action of the Lorentz group, or any other homogeneous space where computations on the associated Lie groups are practically feasible.

- We can develop a generalized "microlocal" theory (for unimodular groups G) based on an appropriate "Littlewood–Paley" decomposition.

- I did not discuss the local Hölder spaces, but they are essential and can be defined similarly to the approach of S. Seuret and J. Lévy Véhel.

- There exists a discrete wavelet transform suitable for the characterization of $\Lambda^\sigma(G)$ and $\Lambda^\sigma(x_0)$ (conventional discrete wavelets applied at the level of the Lie algebra).

- These wavelets on S^2 are markedly different from those proposed by J.P. Antoine and P. Vandergheynst or S. Dahke and P. Maass for example. These are not well suited for analyzing Hölder regularity.

Discrete wavelet transform on compact Lie groups

Definition

Let G be a Lie group and V be the biggest open neighborhood of 0 in $\mathfrak{g}$ such that $\exp : V \to G$ is a diffeomorphism to its image denoted U.
Define a function $\rho \in D(V)$ such that it is positive on its support and equal to 1 in a neighborhood of 0. The ρ-tangent lift of $f : G \to \mathbb{R}$ is defined as the map

$$f_\rho : G \times \mathfrak{g} \to \mathbb{R} \quad (x, X) \mapsto \rho(X) f(x \exp X).$$

The tangent lift of a function is defined on a vector bundle, which has a notion of dilation defined on each of its fiber. It is clear that the restriction of f_ρ to the zero section gives us back f, so there is no loss of information. The idea now is to define the wavelet transform separately on each fiber $\{x\} \times \mathfrak{g}$.

Definition

Assume that G is a compact Lie group and $V \subset \mathfrak{g}$. A finite set $\{x_1, \ldots, x_N\}$ is called a good sampling set of G with respect to V if V is an open neighborhood of 0 in $\mathfrak{g}$ such that the exponential is a diffeomorphism between V and its image and such that $G = \cup_{i=1}^{N} x_i U$, where $U = \exp V$.

Definition

Assume that the Lie group G is compact with a bi-invariant distance and $\mathfrak{g}$ is given with an orthonormal basis $E_1, \ldots, E_n$ and the associated Lebesgue measure. With respect to this basis, we have the grid $\mathbb{Z}^n$ that is well-defined on $\mathfrak{g} \simeq \mathbb{R}^n$.

A finite family $\{\psi^{(\epsilon)} : \epsilon \in E\}$ (E finite) of functions in $L^2(\mathfrak{g})$ are called wavelets with scaling function $\varphi \in L^2(\mathfrak{g})$ if their translated-dilated functions

$$\{\psi_{m,j}^{(\epsilon)} = 2^{-nj/2}\psi^{(\epsilon)}(2^j \cdot -m) : m \in \mathbb{Z}^n, j \in \mathbb{N}\}$$

along with

$$\{\varphi_m = \varphi(\cdot - m) : m \in \mathbb{Z}^n\}$$

form an orthonormal basis of $L^2(\mathfrak{g})$.

Definition

The discrete wavelet transform of $f \in L^2(G)$ is defined as follows: Choose a tangent lift of f given by $\rho \in D(V)$, where V is a neighborhood of 0 in $\mathfrak{g}$ such that the exponential map is a diffeomorphism from V to its image. Assume that ρ is equal to 1 on $\frac{1}{2}V$. Let $\{x_1, \ldots, x_N\}$ be a good sampling set of G with respect to $\frac{1}{2}V$. Then set

$$c_{i,m,j}^{(\epsilon)}(f) = \langle f_\rho(x_i, \cdot), \psi_{m,j}^{(\epsilon)} \rangle \quad \text{and} \quad C_{i,m}(f) = \langle f_\rho(x_i, \cdot), \varphi_m \rangle,$$

for $i \in \{1, \ldots, N\}$, $\epsilon \in E$, $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$.

The discrete wavelet transform of f is given by the coefficients $c_{i,m,j}^{(\epsilon)}(f)$ and $C_{i,m}(f)$.

We have a local reconstruction formula

$$\rho(X)f(x_i \exp X) = \sum_{m\in\mathbb{Z}^n} C_{i,m}(f)\varphi_m(X) + \sum_{\epsilon\in E}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} c^{(\epsilon)}_{i,m,j}(f)\psi^{(\epsilon)}_{m,j}(X)$$

in $L^2(\mathfrak{g})$.

If f is also continuous, we have

$$f(x_i \exp X) = \sum_{m\in\mathbb{Z}^n} C_{i,m}(f)\varphi_m(X) + \sum_{\epsilon\in E}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} c^{(\epsilon)}_{i,m,j}(f)\psi^{(\epsilon)}_{m,j}(X),$$

for all $i \in \{1, \ldots, N\}$ and all $X \in \frac{1}{2}V$.

Proposition

Let $\sigma > 0$ st $\sigma \notin \mathbb{N}$. If G is a compact Lie group, $\{\psi^{(\epsilon)} : \epsilon \in E\}$ is a family of wavelets in $L^2(\mathfrak{g}) \cap S(\mathfrak{g})$ having at least $[\sigma]$ vanishing moments, $\varphi \in S(\mathfrak{g})$ is its scaling function and $\{x_1, \ldots, x_N\}$ is a good sampling set of G with respect to a sufficiently small neighborhood of 0 in $\mathfrak{g}$.

Then, a continuous function f belongs to $\Lambda^\sigma(G)$ iff there exists $C > 0$ st

$$|C_{i,m}(f)| \leq C \quad \text{and} \quad |c^{(\epsilon)}_{i,m,j}(f)| \leq C 2^{-\sigma j},$$

for all $i \in \{1, \ldots, N\}$, $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$.

Proposition

Let $\sigma > 0$ st $\sigma \notin \mathbb{N}$; $f \in L^\infty_{\mathrm{loc}}(G)$, p, $N > \sigma$ and $\{\psi^{(\epsilon)} : \epsilon \in E\}$ be compactly supported4 wavelets of class C^p with N vanishing moments and scaling function φ. Let V be a neighborhood of 0 in $\mathfrak{g}$ such that the exponential is a diffeomorphism from V to its image. Let $\rho \in D(V)$ be such that it is equal to 1 in $\frac{1}{2}V$ and $0 \leq \rho \leq 1$.
A function f belongs to $\Lambda^\sigma(1)$, then there exist $R > 0$, $C > 0$ and $J \in \mathbb{N}$ st

$$|c^{(\epsilon)}_{m,j}(f_\rho(1,\cdot))| \leq C(2^{-j} + |2^{-j}m|)^\sigma,$$

if $j \geq J$ and $|2^{-j}m| < R$.

Proposition

For the converse result, assume that G has a bi-invariant Riemannian metric and that $f \in \Lambda^\eta(1)$ for $\eta > 0$ and suppose that there exist $C > 0$, $R > 0$ and $J \in \mathbb{N}$ st

$$|c_{m,j}^{(\epsilon)}(f_\rho(1,\cdot))| \leq C(2^{-j} + |2^{-j}m|)^\sigma,$$

for $j \geq J$ and $|2^{-j}m| < R$.
Then, we can find constants $C_0 > 0$, $R_0 > 0$ and a polynomial P of degree at most $[\alpha]$ on $\mathfrak{g}$ st

$$|f(\exp X) - P(X)| \leq C_0 r^\sigma |\log r|,$$

for all $r < R_0$, $\|X\| < r$.