LIÈGE université

Automatic proofs in combinatorial game theory

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Overview

1. Walnut

2. Wythoff's game

3. A variation of Wythoff's game

Walnut

Walnut

What is `Walnut`?

`Walnut` is a free software system originally created by Hamoon Mousavi.

- $\rightarrow$ extensively used for proving results in combinatorics on words and additive number theory;
- $\rightarrow$ relies on ***Büchi's theorem***: transform first-order logical formulas into finite automata for which decision procedures can be applied.

Hence, if a problem of interest can be expressed in a convenient extension of Presburger arithmetic $\langle \mathbb{N}, + \rangle$, it can then receive an automatic treatment.

Wythoff’s game

Wythoff's game

The rules

Wythoff's game is a 2-player impartial game which is a variation of the game of Nim.

→ 2 piles of tokens,
→ the players play one after another, they may not pass,
→ the first player unable to play loses.

Allowed moves:

→ removing a positive number of tokens from one pile,
→ removing the same number of tokens from both piles.

Wythoff's game

A famous result

Recall that any positive integer can be written as a sum of Fibonacci numbers in a "greedy" way $\rightsquigarrow$ Zeckendorf representation.

Ex: Considering $F_0 = 1, F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, we have

$$11 = 8 + 3 \quad \rightarrow \quad \text{rep}_F(11) = 10100.$$

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Theorem (Fraenkel, 1982)

A pair (a, b) of integers such that $a \leq b$ is a $\mathcal{P}$-position of Wythoff's game if and only if $\text{rep}_F(a)$ ends with an even number of zeroes and $\text{rep}_F(b)$ is a left-shift of $\text{rep}_F(a)$, i.e. $\text{rep}_F(b) = \text{rep}_F(a)0$.

$\rightarrow$ link with numeration systems

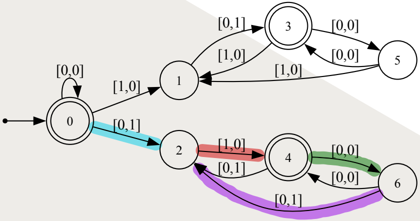
$\rightsquigarrow$ could `Walnut` be used for proving results with combinatorial games?

Wythoff's game

Fraenkel's theorem using `Walnut`

Step 1: Defining the candidate set for $\mathcal{P}$-positions in `Walnut`

```
reg end_even_zeros msd_fib "0*(00|0*1)*":
reg left_shift {0,1} {0,1} "([0,0]|([0,1][1,1]*[1,0]))*":
def ppos_asym "?msd_fib $end_even_zeros(a) & $left_shift(a,b)":
def ppos "?msd_fib $ppos_asym(a,b) | $ppos_asym(b,a)":
```



Ex: we can evaluate `$ppos(6,10)`:

$$\mathrm{rep}_F(6) = 01001$$
$$\mathrm{rep}_F(10) = 10010$$

Wythoff's game

Fraenkel's theorem using `Walnut`

Step 2: Verifying that this set is exactly the set of $\mathcal{P}$-positions

Recall that the sets of $\mathcal{P}$- and $\mathcal{N}$-positions of an impartial acyclic game are uniquely determined by the two following properties:

Stability, i.e. there is no move between two $\mathcal{P}$-positions

```
eval w_stable "?msd_fib Ap,q,r,s
(($ppos(p,q) & $ppos(r,s) & p >= r & q >= s)
=> ((p=r & q=s) | (p>r & q>s & p+s!=q+r)) )":
```

Absorbing, i.e. for each $\mathcal{N}$-position, there exists a move leading to a $\mathcal{P}$-position

```
eval w_absorbing "?msd_fib Ap,q (~$ppos(p,q) => Ex,y
( x<=p & y<=q & $ppos(x,y) & (p+y=q+x | p=x | q=y) )) ":
```

Wythoff's game

Fraenkel's theorem using `Walnut`

Both commands evaluate to `TRUE`, which proves Fraenkel's theorem!

We were able to use `Walnut` because:

- $\rightarrow$ we have a "regular" candidate for the set of $\mathcal{P}$-positions;
- $\rightarrow$ the Fibonacci numeration system is addable.

Wythoff's game

Other results about Wythoff's game

Thanks to `Walnut`, we aslo managed to:

- → prove a 15-year-old conjecture describing the set of ***forbidden moves*** in Wythoff's game (i.e., the set of moves which would allow to play between two $\mathcal{P}$-positions);
- → show that all allowed moves in Wythoff's game are ***non-redundant*** (a move is said to be ***redundant*** if removing it from the rule-set does not affect the set of $\mathcal{P}$-positions).

Wythoff’s game

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What more can we do with this software?

A variation of Wythoff’s game

A variation of Wythoff's game

A new game

Same rules as Wythoff's game, but a player may now remove $k > 0$ tokens from one heap and $\ell > 0$ from the other, provided that $|k - \ell| < m$ for a fixed integer $m \geq 1$.

Note that $m = 1 \rightsquigarrow$ Wythoff's game.

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Note that $m = 1 \rightsquigarrow$ Wythoff's game.

Here, we make use of *Ostrowski numeration systems*: consider the quadratic irrational

$$[1, \overline{m}] = \frac{2 - m + \sqrt{m^2 + 4}}{2}$$

$\rightarrow$ the convergents give two numeration systems: the p-system (using numerators) and the q-system (using denominators).

Rem.: for $m = 1$, $[1, \overline{m}] = \varphi$ the golden ratio $\rightsquigarrow$ Fibonacci numeration system.

A variation of Wythoff's game

A new game

We get the same result as the one for Wythoff's game, but with the associated Ostrowski numeration system:

Theorem (Fraenkel, 1982)

A pair (a, b) of integers such that $a \leq b$ is a $\mathcal{P}$-position of m-Wythoff's game if and only if $\mathrm{rep}_p(a)$ ends with an even number of zeroes and $\mathrm{rep}_p(b)$ is a left-shift of $\mathrm{rep}_p(a)$, i.e. $\mathrm{rep}_p(b) = \mathrm{rep}_p(a)0$, where $\mathrm{rep}_p(x)$ is the representation of x in the p-system associated to $[1, \overline{m}]$.

$\rightarrow$ we can handle such systems with `Walnut`, and thus prove results automatically!

A variation of Wythoff's game

Using `Walnut` to find a new conjecture

Again here, we can ask what are the redundant moves of m-Wythoff's game. Using `Walnut`, we get the following conjecture:

Conjecture (Mignoty, R., Rigo, Whiteland, 2025+)

Let $m \geq 2$. The set of redundant moves of the variation of Wythoff's game where one is allowed to remove $k > 0$ and $\ell > 0$ provided that $|k - \ell| < m$ is

$$\bigcup_{1 \leq i < m} \{(n, n+i), (n+i, n) \mid n \geq m - i + 2\}\,.$$

$\rightarrow$ proved for $m = 2, 3, 4, 5$;

$\rightarrow$ for $m = 5$, we quickly run out of memory $\rightsquigarrow$ use `Walnut` differently.

Thank you for your attention!

Another variation of Wythoff's game

Same rules as Wythoff's game, but a player may now remove $k > 0$ tokens from one heap and $\ell > 0$ from the other, provided that $0 < k \leq \ell < sk + m$ for two positive integers m, s.

For $s = 1 \rightsquigarrow m$-Wythoff's game.

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We do get the same result:

Theorem (Fraenkel, 1998)

A pair (a, b) of integers such that $a \leq b$ is a $\mathcal{P}$-position of the generalized Wythoff's game with parameters (s, m) if and only if $\mathrm{rep}_U(a)$ ends with an even number of zeroes and $\mathrm{rep}_U(b)$ is a left-shift of $\mathrm{rep}_U(a)$, where U is the numeration system defined hereinafter.

Another variation of Wythoff's game

Here, we use the following numeration system: we define the linear recurrence sequence $(U_i)_{i\geq 0}$ by

$$U_0 = 1, \quad U_1 = s + m \quad \text{and} \quad U_i = (s + m - 1)U_{i-1} + sU_{i-2} \; \forall i \geq 2.$$

We then have that each integer $n > 0$ has a unique representation $d_\ell \cdots d_0$ such that

$$n = \sum_{i=0}^{\ell} d_i U_i, \quad \text{with } d_\ell \neq 0.$$

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$$n = \sum_{i=0}^{\ell} d_i U_i, \quad \text{with } d_\ell \neq 0.$$

We are in the ***Pisot*** case, which means

→ we have a regular candidate,

→ we have an addable system.

So, in principle, we can use `Walnut`!