Hölder Regularity and Fractal Aspects of the Thomae Function

Deuxièmes journées de l'axe AMA

Thomas Lamby

June 24, 2025

"Nowhere Regularity"

- Nowhere Continuous Functions :

"Nowhere Regularity"

- Nowhere Continuous Functions :

$$D(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

"Nowhere Regularity"

- Nowhere Continuous Functions :

$$D(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

- Nowhere Differentiable Functions :

"Nowhere Regularity"

- Nowhere Continuous Functions :

$$D(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

- Nowhere Differentiable Functions :

$$T(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-1} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Thomae's-type functions

Let $\theta > 0$,

$$T_\theta(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

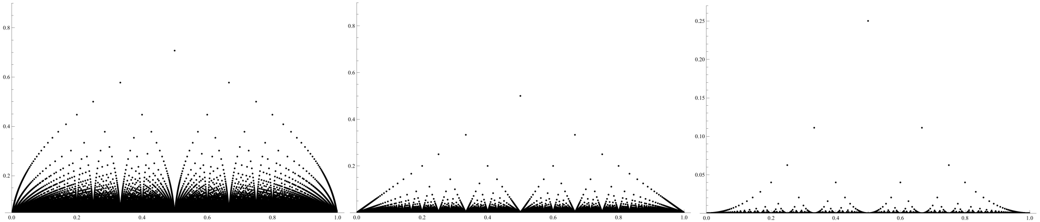

Figure 1: Representation of the function T_θ on $(0,1)$ for $\theta = 1/2$, 1 and 2.

Periodicity

Proposition

The Thomae function is periodic with period 1.

Proof.

- If x is irrational, so is $x + 1$.
- If $x = p/q$, then $x + 1 = \frac{p+q}{q}$. But, p and q coprime implies $(p + q)$ and q coprime, hence the conclusion.

□

Rational-Irrational Dichotomy

Proposition

The function T_θ is discontinuous at rational points and continuous at irrational points.

Proof.

- If x is rational, let s be an irrational number, and define $x_j = x + s/j$ for $j \in \mathbb{N}$. Clearly $x_j \to x$, but since x_j is irrational for all j, $T_\theta(x_j) \not\to T_\theta(x)$.
- If x is irrational, assume $x \in (0,1)$. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ be such that $n^{-\theta} < \varepsilon$. For $j \in \{1, \ldots, n\}$, define $m_j = \sup\{m \in \mathbb{N}_0 : m < jx\}$, and set

$$\delta_j = \inf\{|x - \frac{m_j}{j}|, |x - \frac{m_j + 1}{j}|\}.$$

  Let $\delta = \inf_{1 \leq j \leq n} \delta_j$. If $y = p/q$ is rational and $y \in (x - \delta, x + \delta)$, then $q > n$, so $T_\theta(y) < n^{-\theta} < \varepsilon$. If y is irrational, $T_\theta(y) = 0 < \varepsilon$. Thus, $|x - y| < \delta$ implies $|T_\theta(x) - T_\theta(y)| < \varepsilon$, proving that T_θ is continuous at x.

□

Differentiability

Proposition

Let f be a function on $\mathbb{R}$ that is positive on the rationals and 0 on the irrationals. Then, there is an uncountable dense set of irrationals on which f is not differentiable.

Proof.

Let (r_j) be an enumeration of the rationals and let $x_1 \in \mathbb{Q}$. Find a closed interval I_1 such that for all $x \in I_1$,

$$f(x_1) \geq |x_1 - x|.$$

Having defined I_n and x_n, define x_{n+1} and I_{n+1} such that:

$$I_{n+1} \subset I_n, \quad \mathcal{L}(I_{n+1}) < \frac{1}{n}, \quad r_j \notin I_{n+1} \text{ for } j = 1, \ldots, n, \quad x_{n+1} \in I_{n+1} \cap \mathbb{Q}$$

and for all $x \in I_{n+1}$,

$$f(x_{n+1}) \geq |x_{n+1} - x|.$$

The intervals $(I_n)_{n=1}^{\infty}$ are nested nonempty intervals whose diameters converge to zero. □

Differentiability

Proof (continued).

Thus,

$$\bigcap_{n=1}^{\infty} I_n = \{a\}$$

where $x_j \to a$ and $a \notin \mathbb{Q}$. If f were differentiable at a, then by irrational approximation of a, the derivative would have to be zero. However, since $a \in I_j$,

$$\frac{|f(x_j) - f(a)|}{|x_j - a|} = \frac{f(x_j)}{|x_j - a|} \geq 1$$

for all $j \in \mathbb{N}$. Thus, f is not differentiable at a. A look at our construction shows that the set A of all points found in this manner is dense.

Differentiability

Proposition

Let $(a_j)_j$ be a sequence of $\mathbb{R} \setminus \mathbb{Q}$. Then there exists a function that is positive on the rationals, zero on the irrationals, and differentiable at each point a_j.

Proof.

For all j, define $g_j(n) = \min\left\{\left|\frac{m}{n} - a_j\right| : m \text{ and } n \text{ are coprime}\right\}$ and $g(n) = \min_{j \leq n} g_j(n)$. The function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ (g(n))^2 & \text{if } x = \frac{m}{n} \text{ with } m \text{ and } n \text{ coprime}, \\ 1 & \text{if } x = 0, \end{cases}$$

is differentiable on $\{a_j : j \in \mathbb{N}\}$. To see this, fix $j \in \mathbb{N}$. For m and n coprime with $n \geq j$,

$$\frac{|f(m/n) - f(a_j)|}{|m/n - a_j|} = \frac{f(m/n)}{|m/n - a_j|} \leq \frac{(g(n))^2}{g_j(n)} \leq \frac{(g_j(n))^2}{g_j(n)} = g_j(n) \xrightarrow[n \to \infty]{} 0.$$

□

Rational Approximations

$\tau(x) = \sup \left\{ u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N} : \left| x - \frac{p}{q} \right| < \frac{1}{q^u} \right\}.$

Dirichlet's Theorem

Let x be a real number and n a positive integer. Then there is a rational number p/q with $0 < q \leq n$, satisfying

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{(n+1)q}.$$

Corollary

Given any real number x, there exists a rational number p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Rational Approximations

Theorem

Let $x \in \mathbb{R}\backslash\mathbb{Q}$, then there are infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Hurwitz's Theorem

(i) Let $x \in \mathbb{R}\backslash\mathbb{Q}$, there are infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}.$$

(ii) If $\sqrt{5}$ is replaced by $C > \sqrt{5}$, then there are irrational numbers x for which statement (i) does not hold.

Rational Approximations

Theorem

Let $\varepsilon > 0$. For almost every $x \in [0, 1]$, there exist only finitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}.$$

Proof.

Set

$$A_q = [0, 1] \cap \bigcup_{p=0}^{q} [\frac{p}{q} - \frac{1}{q^{2+\varepsilon}}, \frac{p}{q} - \frac{1}{q^{2+\varepsilon}}].$$

We want to show that $\mathcal{L}\left(\bigcap_{q\geq 1} \bigcup_{k\geq q} A_k\right) = 0$. Since $\mathcal{L}(A_q) \leq \sum_{p=0}^{q} \frac{2}{q^{2+\varepsilon}} = \frac{2(q+1)}{q^{2+\varepsilon}}$, we get $\sum_q \mathcal{L}(A_q) < \infty$ et we conclude by Borel-Cantelli. $\square$

Differentiability

Proposition

For $\theta \in (0,2]$, T_θ is not differentiable at any point.

Proof.

Let x be an irrational number. By Hurwitz's Theorem, there exists a sequence $(x_j)_{j\in\mathbb{N}}$ of rational numbers converging to x, such that $x_j = p_j/j$ with p_j and j coprime and $|x - x_j| < \frac{1}{\sqrt{5}j^2}$. Then

$$|\frac{T_\theta(x) - T_\theta(x_j)}{x - x_j}| > \frac{j^{-\theta}}{1/(\sqrt{5}j^2)} = \sqrt{5}\, j^{2-\theta}.$$

This ensures that $DT_\theta(x)$ cannot be equal to zero. However, by irrational approximation, if $DT_\theta(x)$ exists, it must be zero. □

Differentiability at 0

We put $T_\theta(0) = 1$ in order to have the periodicity. Consider

$$\widetilde{T}_\theta(x) = \begin{cases} q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational or } x = 0. \end{cases}$$

As one might expect, $\widetilde{T}_\theta$ becomes continuous at 0 and the dichotomy no longer holds. A more interesting fact is that $\widetilde{T}_\theta$ becomes differentiable at 0 for $\theta > 1$.

Differentiability at 0

We put $T_\theta(0) = 1$ in order to have the periodicity. Consider

$$\widetilde{T}_\theta(x) = \begin{cases} q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational or } x = 0. \end{cases}$$

As one might expect, $\widetilde{T}_\theta$ becomes continuous at 0 and the dichotomy no longer holds. A more interesting fact is that $\widetilde{T}_\theta$ becomes differentiable at 0 for $\theta > 1$. Indeed, if the derivative at 0 exists, it must be equal to 0. Thus, we must show that if $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in (-\delta, \delta) \implies \left| \frac{\widetilde{T}_\theta(x) - \widetilde{T}_\theta(0)}{x - 0} \right| = \left| \frac{\widetilde{T}_\theta(x)}{x} \right| < \varepsilon.$$

If x is irrational, then this difference quotient is equal to $0 < \varepsilon$. Suppose x is a nonzero rational number. There exists a positive integer n such that $\frac{1}{n^{\theta-1}} < \varepsilon$. There exists a $\delta > 0$ such that every nonzero rational number in the interval $(-\delta, \delta)$ has denominator $q > n$. Thus, if $x = \frac{p}{q}$ with $\gcd(p,q) = 1$, then for $|x| < \delta$ we have $q > n$, and hence:

$$\left| \frac{\widetilde{T}_\theta(x)}{x} \right| = \left| \frac{q^{-\theta}}{p/q} \right| = \left| \frac{1}{pq^{\theta-1}} \right| < \varepsilon.$$

Therefore, the difference quotient is less than ε for all $x \in (-\delta, \delta)$, and the derivative of $\widetilde{T}_\theta$ at 0 exists and equals 0.

Regularity of T_θ

- T_θ is discontinuous at rational points and continuous at irrational points.

Regularity of T_θ

- T_θ is discontinuous at rational points and continuous at irrational points.
- For $\theta \in (0, 2]$, T_θ is not differentiable at any point.

Regularity of T_θ

- T_θ is discontinuous at rational points and continuous at irrational points.
- For $\theta \in (0, 2]$, T_θ is not differentiable at any point.
- Exact regularity of T_θ at each of its points ?

Pointwise Regularity

Let $x_0 \in \mathbb{R}^d$, $\alpha > 0$, a function $f \in L^\infty_{\text{loc}}$ is in $\Lambda^\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq C r^\alpha$$

for sufficiently small r.

Pointwise Regularity

Let $x_0 \in \mathbb{R}^d$, $\alpha > 0$, a function $f \in L^\infty_{\text{loc}}$ is in $\Lambda^\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha$$

for sufficiently small r.

Hölder-exponent

$$h_f(x_0) := \sup\{\alpha > 0 : f \in \Lambda^\alpha(x_0)\}.$$

Pointwise Regularity

Let $x_0 \in \mathbb{R}^d$, $\alpha > 0$, a function $f \in L^\infty_{\text{loc}}$ is in $\Lambda^\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha$$

for sufficiently small r.

Hölder-exponent

$$h_f(x_0) := \sup\{\alpha > 0 : f \in \Lambda^\alpha(x_0)\}.$$

Hölder-spectrum

$$d_f(h) = \dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : h_f(x) = h\}).$$

Pointwise Regularity of T_θ

Theorem

Let $\theta > 0$; then

$$h_{T_\theta}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ \theta/\tau(x) & \text{otherwise,} \end{cases}$$

where

$$\tau(x) = \sup \left\{ u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N} : \left| x - \frac{p}{q} \right| < \frac{1}{q^u} \right\}.$$

- If $\theta < 2$, T_θ is nowhere differentiable.
- T_2 is nowhere differentiable and $h_{T_2} = 1$ almost everywhere !
- When $\theta > 2$, T_θ is differentiable at x_0 when $\tau(x_0) < \theta$. For example, T_9 is differentiable at algebraic irrationals numbers, $e, \pi, \pi^2, \ln(2)$.

Proof.

We consider $\theta \leq 2$. Since T_θ is not continuous at rational numbers, we can suppose that $x \in (0, 1)$ is irrational.

Proof.

We consider $\theta \leq 2$. Since T_θ is not continuous at rational numbers, we can suppose that $x \in (0,1)$ is irrational. If $y \in (0,1)$ is also irrational, we naturally have $T_\theta(x) - T_\theta(y) = 0$. Let $\tau(x) < \infty$, $\varepsilon > 0$ and $\kappa > 0$ such that $\kappa(\theta/\tau(x) - \varepsilon) < \varepsilon\tau(x)$, if $y_n = \frac{p_n}{q_n} \in (0,1)$ with p_n and q_n coprime, then there exists $N \in \mathbb{N}$ such that for all for all $n \geq N$, one has

$$\frac{|T_\theta(y_n) - T_\theta(x)|}{|y_n - x|^{\theta/\tau(x)-\varepsilon}} = \frac{1/q_n^\theta}{|y_n - x|^{\theta/\tau(x)-\varepsilon}} \leq q_n^{-\theta}\, q_n^{(\tau(x)+\kappa)(\theta/\tau(x)-\varepsilon)},$$

so that $T_\theta \in \Lambda^{\theta/\tau(x)-\varepsilon}(x)$.

Proof.

We consider $\theta \leq 2$. Since T_θ is not continuous at rational numbers, we can suppose that $x \in (0,1)$ is irrational. If $y \in (0,1)$ is also irrational, we naturally have $T_\theta(x) - T_\theta(y) = 0$. Let $\tau(x) < \infty$, $\varepsilon > 0$ and $\kappa > 0$ such that $\kappa(\theta/\tau(x) - \varepsilon) < \varepsilon\tau(x)$, if $y_n = \frac{p_n}{q_n} \in (0,1)$ with p_n and q_n coprime, then there exists $N \in \mathbb{N}$ such that for all for all $n \geq N$, one has

$$\frac{|T_\theta(y_n) - T_\theta(x)|}{|y_n - x|^{\theta/\tau(x)-\varepsilon}} = \frac{1/q_n^\theta}{|y_n - x|^{\theta/\tau(x)-\varepsilon}} \leq q_n^{-\theta}\, q_n^{(\tau(x)+\kappa)(\theta/\tau(x)-\varepsilon)},$$

so that $T_\theta \in \Lambda^{\theta/\tau(x)-\varepsilon}(x)$.If $\theta/\tau(x) = 1$ (which can only occur when $\theta = 2$), it follows that $h_{T_\theta}(x)$ is equal to 1, as T_θ is not differentiable at x. Otherwise, let $\varepsilon > 0$ be such that $\varepsilon + \theta/\tau(x) < 1$ and consider the convergents p_j/q_j of x. For sufficiently large j, we have

$$\frac{|T_\theta(x) - T_\theta(p_j/q_j)|}{|x - p_j/q_j|^{\frac{\theta}{\tau(x)}+\varepsilon}} = \frac{q_j^{-\theta}}{|x - p_j/q_j|^{\frac{\theta}{\tau(x)}+\varepsilon}} \geq \frac{q_j^{(\tau(x)-\varepsilon)(\frac{\theta}{\tau(x)}+\varepsilon)}}{q_j^\theta} = q_j^{\beta_\varepsilon},$$

with $\beta_\varepsilon > 0$. As $q_j \to \infty$, we get $T_\theta \notin \Lambda^{\frac{\theta}{\tau(x)}+\varepsilon}(x)$. □

Spectrum of T_θ

Let $A \subset \mathbb{R}$, if $\varepsilon > 0$, $\gamma \in \mathbb{R}$ and $\delta \in [0,1]$, we set

$$\mathcal{H}_\varepsilon^{\delta,\gamma} = \inf_R \left(\sum_i \mathrm{diam}(A_i)^\delta \, |\log(\mathrm{diam}(A_i))|^\gamma \right),$$

where the infimum is taken over all coverings R of A by bounded sets $\{A_i\}_{i\in\mathbb{N}}$ whose diameter is less than ε.

For all $\delta \in [0,1]$ and $\gamma \in \mathbb{R}$, we define the (δ,γ)-*Hausdorff outer measure* of A by

$$\mathcal{H}^{\delta,\gamma}(A) = \lim_{\varepsilon \to 0^+} \mathcal{H}_\varepsilon^{\delta,\gamma}.$$

Spectrum of T_θ

Let $A \subset \mathbb{R}$, if $\varepsilon > 0$, $\gamma \in \mathbb{R}$ and $\delta \in [0,1]$, we set

$$\mathcal{H}_\varepsilon^{\delta,\gamma} = \inf_R \left(\sum_i \mathrm{diam}(A_i)^\delta \, |\log(\mathrm{diam}(A_i))|^\gamma \right),$$

where the infimum is taken over all coverings R of A by bounded sets $\{A_i\}_{i\in\mathbb{N}}$ whose diameter is less than ε.

For all $\delta \in [0,1]$ and $\gamma \in \mathbb{R}$, we define the (δ,γ)-*Hausdorff outer measure* of A by

$$\mathcal{H}^{\delta,\gamma}(A) = \lim_{\varepsilon\to 0^+} \mathcal{H}_\varepsilon^{\delta,\gamma}.$$

We set

$$\dim_{\mathcal{H}}(A) = \inf\{\delta \geq 0 : \mathcal{H}^{\delta,0}(A) = 0\} = \sup\{\delta \geq 0 : \mathcal{H}^{\delta,0}(A) = \infty\}.$$

Spectrum of T_θ

Jarnik's Theorem

Let $a, b \in \mathbb{R}$ with $a < b$. For all $\tau \geq 2$, we set

$$E_\tau = \left\{ x \in [a,b] : \left| x - \frac{p}{q} \right| \leq \frac{1}{q^\tau} \text{ for an infinity of coprime pairs } (p,q) \right\}.$$

Then

$$\dim_{\mathcal{H}}(E_\tau) = \frac{2}{\tau} \quad \text{and} \quad \mathcal{H}^{2/\tau,2}(E_\tau) > 0.$$

Spectrum of T_θ

Jarnik's Theorem

Let $a, b \in \mathbb{R}$ with $a < b$. For all $\tau \geq 2$, we set

$$E_\tau = \left\{ x \in [a,b] : \left| x - \frac{p}{q} \right| \leq \frac{1}{q^\tau} \text{ for an infinity of coprime pairs } (p,q) \right\}.$$

Then

$$\dim_{\mathcal{H}}(E_\tau) = \frac{2}{\tau} \quad \text{and} \quad \mathcal{H}^{2/\tau,2}(E_\tau) > 0.$$

Theorem

The Hölder-spectrum is given by

$$d_{T_\theta}(h) = \begin{cases} \frac{2h}{\theta} & \text{if } h \in [0, \theta/2], \\ -\infty & \text{otherwise.} \end{cases}$$

Proof.

Let $a, b \in \mathbb{R}$ with $a < b$, and for all $\tau \geq 2$, set $F_t = \{x \in [a, b] : \tau(x) = t\}$. We want to prove that

$$\dim_{\mathcal{H}}(F_t) = \frac{2}{t} \quad \forall t \in [2, +\infty] .$$

Let $t \in [2, \infty]$, one has

$$F_t = \bigcap_{\tau < t} E_\tau \Big\backslash \bigcup_{\tau > t} E_\tau .$$

Therefore, for all $\tau < t$,

$$\dim_{\mathcal{H}}(F_t) \leq \dim_{\mathcal{H}}(E_\tau) = \frac{2}{\tau}.$$

But, F_t contains the set $G_t = E_t \Big\backslash \bigcup_{\tau > t} E_\tau$. Since the sets E_τ are nested decreasingly, the union $\bigcup_{\tau > t} E_\tau$ can be rewritten as a countable union. By Jarník's theorem, this union consists of sets of zero $\mathcal{H}^{2/t,2}$-measure. Therefore, we have

$$\mathcal{H}^{2/t,2}(F_t) \geq \mathcal{H}^{2/t,2}(G_t) = \mathcal{H}^{2/t,2}(E_t) > 0.$$

By consequence,

$$\dim_{\mathcal{H}}(F_t) = \frac{2}{t}.$$

□

Bigger class of Thomae's type functions

We consider continuous functions $\phi : (0,1) \to (0,\infty)$ such that

$$0 < \underline{\phi}(t) := \inf_{s<1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s<1} \frac{\phi(ts)}{\phi(s)} < \infty,$$

for any $t < 1$. The *lower* and *upper indices* of ϕ are defined by

$$\underline{s}(\phi) = \lim_{t\to 0} \frac{\log \underline{\phi}(t)}{\log t} \quad \text{and} \quad \overline{s}(\phi) = \lim_{t\to 0} \frac{\log \overline{\phi}(t)}{\log t}.$$

Bigger class of Thomae's type functions

We consider continuous functions $\phi : (0,1) \to (0,\infty)$ such that

$$0 < \underline{\phi}(t) := \inf_{s<1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s<1} \frac{\phi(ts)}{\phi(s)} < \infty,$$

for any $t < 1$. The *lower* and *upper indices* of ϕ are defined by

$$\underline{s}(\phi) = \lim_{t\to 0} \frac{\log \underline{\phi}(t)}{\log t} \quad \text{and} \quad \overline{s}(\phi) = \lim_{t\to 0} \frac{\log \overline{\phi}(t)}{\log t}.$$

We define

$$T_\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \phi(1/q) & \text{if } x = p/q, \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

where $\underline{s}(\phi) = \overline{s}(\phi) = \theta \in (0,2]$.

Bigger class of Thomae's type functions

For example, one can consider

$$T_{\log}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{\log(q)}{q} & \text{if } x = p/q, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

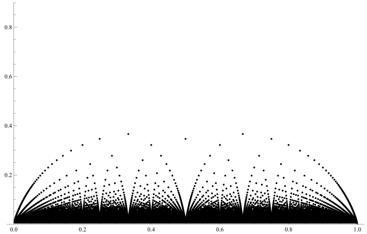

Figure 2: Representation of the function $T_{\log}$ on $(0, 1)$.

Bigger class of Thomae's type functions

- What about $\underline{s}(\phi) < \overline{s}(\phi)$?
- Negatives indices ?
- Interchange the dichotomy ?

Different indices

For example, define

$$\phi(t) = \begin{cases} t^\alpha & \text{if } t \in (0, s], \\ t^\beta & \text{if } t \in (s, 1). \end{cases}$$

$\rightsquigarrow$ Only few particular points. A more complex example : consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1}, \\ 2^{j_{2n}} 4^{j - j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

The sequence σ oscillates between $(j)_j$ and $(2^j)_j$. By setting

$$\phi(t) = \frac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} \quad \text{if} \quad t \in (2^{-j-1}, 2^{-j}],$$

we have $\underline{s}(\phi) = 0$ and $\overline{s}(\phi) = 1$. $\rightsquigarrow$ Partial Results : $h_{T_\phi}(x) \in [\underline{s}(\phi)/\tau(x), \overline{s}(\phi)/\tau(x)]$ if $x \in \mathbb{R} \backslash \mathbb{Q}$.

$\theta < 0$

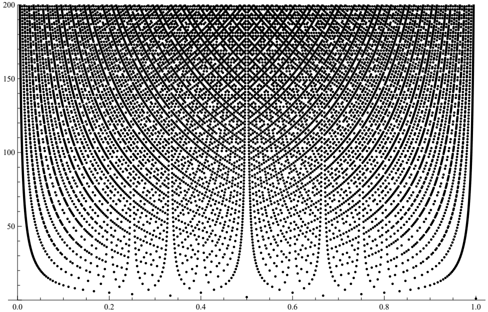

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.

$\theta < 0$

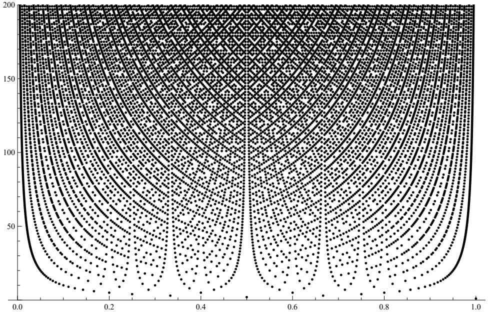

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.
- $\int_{\mathbb{R}} T_\theta(x)dx = \int_{\mathbb{R}\setminus\mathbb{Q}} T_\theta(x)dx = 0.$

$\theta < 0$

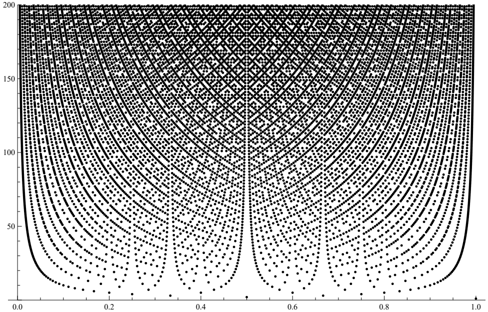

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.
- $\int_{\mathbb{R}} T_\theta(x)dx = \int_{\mathbb{R}\setminus\mathbb{Q}} T_\theta(x)dx = 0$. $\rightsquigarrow$ Notion of p-exponents not adapted.

Interchange the dichotomy?

Is there a function that is continuous on the rational numbers and discontinuous on the irrational numbers ?

Interchange the dichotomy?

Is there a function that is continuous on the rational numbers and discontinuous on the irrational numbers ?
$\rightsquigarrow$ No, since the set of discontinuities of a function $\mathbb{R} \to \mathbb{R}$ is always a F_σ-set.

Interchange the dichotomy?

Is there a function that is continuous on the rational numbers and discontinuous on the irrational numbers ?
$\rightsquigarrow$ No, since the set of discontinuities of a function $\mathbb{R} \to \mathbb{R}$ is always a F_σ-set.
Starting from a F_σ subset of $\mathbb{R}$, $A := \bigcup_n F_n$, we define

$$T_A(x) = \begin{cases} 1/n & \text{if } x \text{ rational and } n \text{ is minimal s.t. } x \in F_n, \\ -1/n & \text{if } x \text{ irrational and } n \text{ is minimal s.t. } x \in F_n, \\ 0 & \text{if } x \notin A. \end{cases}$$

The set of discontinuities of T_A is given by A.

Thank you for your attention !