Fast-and-flexible decision-making with modulatory interactions

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Abstract— **Multi-agent systems in biology, society, and engineering are capable of making decisions through the dynamic interaction of their elements. Nonlinearity of the interactions is key for the speed, robustness, and flexibility of multi-agent decision-making. In this work we introduce modulatory, that is, multiplicative, in contrast to additive, interactions in a nonlinear opinion dynamics model of fast-and-flexible decision-making. The original model is nonlinear because network interactions, although additive, are saturated. Modulatory interactions introduce an extra source of nonlinearity that greatly enriches the model decision-making behavior in a mathematically tractable way. Modulatory interactions are widespread in both biological and social decision-making networks; our model provides new tools to understand the role of these interactions in networked decision-making and to engineer them in artificial systems.**

I. Introduction

We recently introduced a general, mathematically tractable, nonlinear opinion dynamics (NOD) model of fast-and-flexible multi-agent, multi-option decision-making [1], [2], [3]. The model consists of a network of first-order dynamics with saturated network interactions and exogenous inputs. It is closely related to both bio-inspired [4] and artificial [5] recurrent neural networks models. It is also reminiscent of continuous-time models of gene regulatory networks [6] with linear degradation dynamics and saturated Hill-type [7] molecular interactions. It has been used to model and analyze biological [8] and sociopolitical [9], [3] collective decision-making as well as to engineer a number of decision-making behaviors [10], [11], [12], [13], [14].

In our NOD, a linear term models negative feedback regulation of opinions towards a neutral (unopinionated) state. Saturated network interactions model nonlinear opinion exchanges that can amplify the information brought by exogenous inputs through positive feedback and trigger fast and strong opinion formation. The balance between negative and positive feedback is tuned by an *attention* parameter that models the agents' level of engagement in the decision-making process. When positive and negative feedback are perfectly balanced, the model undergoes a bifurcation at which the neutral state becomes unstable and strong opinions are formed. This opinion-forming bifurcation is the key determinant of decision-making that is *fast* (i.e., indecision is broken as soon as it becomes costly) and *flexible* (i.e., it has tunable sensitivity to inputs) [2].

In the original NOD model [1], [2], [3] network interactions are saturated but otherwise *additive*. We provide ample evidence in Section II that neural, biological, and sociopolitical decision-making networks also use *triadic* [15] interactions to dynamically *modulate* the weights of additive interactions. We introduce such modulatory interactions as *multiplicative* terms between opinions. The NOD model in [1], [2], [3] cannot systematically represent the effect of modulatory interactions. Here, we extend this NOD model to include modulatory network interactions inspired by modulatory interactions observed in nature and society. We analyze how opinion-forming bifurcations are shaped by these interactions, and explore their use to engineer more complex decision-making behavior in autonomous agents in a mathematically tractable way.

The paper contributions are the following. *i)* We define a parametrization of modulatory interactions in the NOD model introduced in [1], [2], [3]. *ii)* We analyze how multiplicative interactions shape the model's opinion-forming bifurcation behavior and we provide a thorough interpretation of the analytical results in terms of fast-and-flexible decision-making. *iii)* We use modulatory interactions to augment a recently introduced NOD-based robotic obstacle avoidance controller [12] and provide it with the capacity to make decisions exclusively under specific conditions, termed here *conditional decision-making*.

The paper is organized as follows. In Section II we review evidence of modulatory interactions in nature and society, and discuss their potential in engineered systems. In Section III we define the general nonlinear opinion dynamics model with modulatory interactions and discuss its interpretation and possible generalizations. In Section IV we illustrate the main ideas and results of the paper on a representative example. Section V presents the main analytical results of the paper and thoroughly discusses their consequence for the mathematical tractability of the model. Section VI applies the main results of the paper to analyze how modulatory interactions shape opinion formation in a multi-agent two-option NOD model of modulatory social influence and in a single-agent multi-option NOD model of robotic navigation.

II. Modulatory interactions in biological, social, and artificial decision-making networks

We describe several examples of the occurrence and relevance of modulatory interactions in biological and social systems, and propose their implementation in artificial

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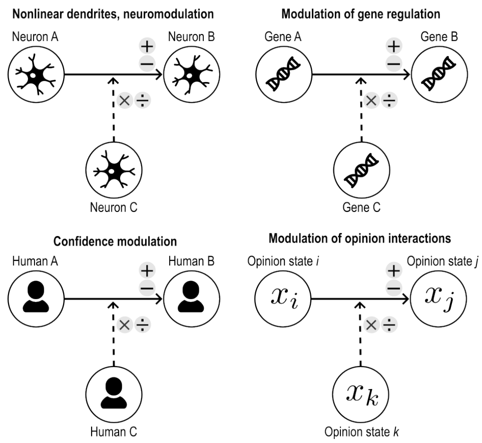

Fig. 1. **Top left**: Neuron A excites or inhibits Neuron B through additive synaptic inputs. Neuron C modulates the strength of this interaction through multiplicative dendritic and neuromodulatory inputs. **Top right**: Gene A regulates the expression of Gene B through transcription factors. Gene C modulates the strength of this regulation. **Bottom left**: Human A influences the opinion of Human B through verbal information exchange. Human C changes the strength of this influence by modulating the confidence that B has on A. **Bottom right**: In nonlinear opinion dynamics, an opinion state x_i has either a positive or negative effect on another opinion state x_j through saturated additive interactions. Introducing multiplicative interactions, a third opinion state x_k modulates how strong these effects are.

systems by suitably augmenting NOD.

Neural Networks. Recent evidence [16], [17], [18] suggests that multiplicative interactions between neurons (Fig. 1, top left), implemented through nonlinear dendrites and neuromodulation, play a key role in many kinds of neural computation. A fundamental building block of artificial neural networks, the Gated Recurrent Unit (GRU), is also defined by multiplicative interactions [19].

Gene Regulatory Networks. The modulation of the gene transcription machinery, and its effects on gene expression regulation, have been studied both in prokaryotes [20] and eukaryotes [21]. This kind of molecular modulatory interactions can naturally be modeled as multiplicative interactions in gene regulatory network models [6] (Fig. 1, top right).

Social networks. Modulatory interactions between social agents are often introduced into basic social network models like the DeGroot model [22], (Fig. 1, bottom left). Examples include models of asymmetric political polarization in the United States Congress [9], epidemics models with risk aversion [23], and war-peace transitions in neighboring nations [24]. As in the model we introduce in this paper, all the models above use multiplicative terms between opinion states to capture the effects of modulatory interactions.

NOD for autonomous agent control. The nonlinear opinion dynamics in [1], [2], [3] have recently been used for the control of autonomous agents, including self-driving cars [13], robotic obstacle avoidance [12], and unmanned surface vehicles [25]. Inspired by the widespread occurrence of modulatory interactions in biological and social systems, in Section VI-B we show how the NOD with modulatory interactions introduced in the next section (Fig. 1, bottom right) provide the means to enable conditional fast-and-flexible decision-making in the NOD model proposed in [12] for robot navigation.

III. A TRACTABLE NOD MODEL WITH MODULATORY INTERACTIONS

The proposed NOD with modulatory interactions for a group of agents forming opinions about two options is

$$\tau \dot{x}_i = -x_i + b_i + S\left(\sum_{j=1}^{N} a_{ij}\left(u_0 + \sum_{k=1}^{N} m_{ijk} x_k^n\right) x_j\right), \tag{1}$$

$i = 1, \dots, N$, which can also be written in vector form as

$$\tau \dot{\mathbf{x}} = -\mathbf{x} + \mathbf{b} + \mathbf{S}((u_0 \mathbf{1}_N \mathbf{1}_N^T + \tilde{M}(\mathbf{x})) \odot A \cdot \mathbf{x}) \tag{2}$$

where $\mathbf{1}_N = [1, \dots, 1]^T \in \mathbb{R}^N$, $\odot$ represents element-wise multiplication (Hadamard product), τ is the characteristic timescale, $\mathbf{x} = [x_1, \dots, x_N]^T \in \mathbb{R}^N$ represents the agents' opinion states, $x_i > 0$ (resp. $x_i < 0$) means a preference for option 1 (resp. option 2), $\mathbf{b} = [b_1 \ \cdots \ b_N]^T \in \mathbb{R}^N$ are exogenous inputs that can bias the decision-making behavior, and $n \in \mathbb{N}_{>0}$ is the *order* of the modulatory interaction. $A = [a_{ij}]_{i,j=1}^N$ is a matrix of interaction weights that determines the additive effect of the opinion of agent j on the opinion of agent i. The agent's attention $U(\mathbf{x}) = u_0 \mathbf{1}_N \mathbf{1}_N^T + \tilde{M}(\mathbf{x})$ is the sum of the *basal attention* u_0 that an agent is paying to its neighbors' opinions when $\mathbf{x} = \mathbf{0}$ and the modulatory term $\tilde{M}(\mathbf{x}) = [\tilde{m}_{ij}(\mathbf{x})]_{i,j=1}^N$, where $\tilde{m}_{ij}(\mathbf{x}) = \sum_{k=1}^N m_{ijk} x_k^n$, determines the effect of *modulatory interactions* on the attention paid to specific neighbors. In particular, m_{ijk} determines the sign and strength of the modulatory effect that the opinion of agent k has on a_{ij}. That is, the attention paid by agent i to agent j is affected by the state of agent k through the modulatory term $m_{ijk} x_k^n$. Modulatory interactions thus generalize the state-dependent attention mechanism introduced in [1], [26], [11], which is key for speed and flexibility of decision-making. Finally, $\mathbf{S}(\mathbf{x}) = [S(x_1), \cdots S(x_N)]$, with $S : \mathbb{R} \to \mathbb{R}$, is a vector of smooth sigmoidal saturating functions satisfying $S(0) = 0$, $S'(0) = 1$, $S''(0) = 0$. To model symmetry between options in the absence of inputs, it is natural to assume that S is odd-symmetric. Here, we simply assume $S(\cdot) = \tanh(\cdot)$.

Alternatively, model (1) can be interpreted as a modulated NOD for a single agent forming opinions about N options. In this case, x_i is the agent's opinion about option i, where $x_i > 0$ ($x_i < 0$) means that the agent favors (disfavors) option i, a_{ij} is the weight with which the agent's opinion about option i additively affects its opinion on option j, and m_{ijk} is the weight with which the agent's opinion about option k modulates a_{ij}. In the single-agent, multi-option interpretation of (1), S does not need to be odd-symmetric because there is no natural symmetry between favoring or disfavoring an option. Here, we assume S is a shifted $\tanh$ function, i.e., $S(\cdot) = \frac{\tanh(\cdot - s) + \tanh(s)}{1 - \tanh(s)^2}$, $s \in \mathbb{R}$.

Similarly to the original, non-modulated NOD model, the magnitude of the basal attention u_0 tunes the balance between the negative feedback regulation provided by the linear term in (1) and the positive feedback amplification provided by the saturated networked term. For sufficiently large basal attention, positive feedback dominates, which destabilizes the neutral state $\mathbf{x} = \mathbf{0}$ in an opinion-forming bifurcation that organizes the fast-and-flexible decision-making behavior.

As we shall prove in Section V, model (1) is tractable because modulatory interactions do not affect *i)* the location of the bifurcation point in which the neutral state $\mathbf{x} = \mathbf{0}$ loses stability, thoroughly analyzed in [1], [2], [3], and *ii)* the kernel of the Jacobian of model (1) at bifurcation. This determines (to leading order) the opinion pattern observed once indecision is broken and the direction in the input space to which the system is most sensitive close to bifurcation. However, modulatory interactions do affect the shape of the opinion-forming bifurcation branches. This non-local effect is hard to characterize in full generality but it can be analyzed on a case-by-case basis using Lyapunov-Schmidt (LS) reduction [27, Section I.3] together with bifurcation recognition and unfolding theory [27, Chapter 2 and 3].

IV. An Illustrative Example

For two agents (or two options), model (2) reduces to

$$\tau \dot{x}_1 = -x_1 + b_1 + S\Big((u_0 + m_{111}x_1^n + m_{112}x_2^n)a_{11}x_1 + \\ + (u_0 + m_{121}x_1^n + m_{122}x_2^n)a_{12}x_2)\Big) \tag{3a}$$

$$\tau \dot{x}_2 = -x_2 + b_2 + S\Big((u_0 + m_{211}x_1^n + m_{212}x_2^n)a_{21}x_1 + \\ + (u_0 + m_{221}x_1^n + m_{222}x_2^n)a_{22}x_2)\Big). \tag{3b}$$

For the modulated interaction network sketched in Fig. 2A, the additive and modulatory interaction matrices are

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ [M_{ij1}] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ [M_{ij2}] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

$i, j = 1, 2$. Fig. 2B shows the bifurcation behavior of model (3) for $b_1 = b_2 = 0$, without modulatory interactions ($M = 0$) and with modulatory interactions of different orders ($n = 1, 2, 3$), for additive and modulatory interaction matrices defined in (4). In all cases, the *neutral state* $x_1 = x_2 = 0$ loses stability in a bifurcation at $u_0 = 1$.

When $M = 0$, the model exhibits the symmetric *indecision-breaking* or *opinion-forming* supercritical *pitchfork* bifurcation thoroughly studied in [1], [2]. For $b_1 = b_2 = 0$, the existence of this symmetric bifurcation arises from the symmetry of the system with respect to swapping agents and swapping options. Agent symmetry is enforced by a symmetric adjacency matrix. Option symmetry is enforced by the odd-symmetry of the vector field, as ensured by using the odd sigmoidal function $S(\cdot) = \tanh(\cdot)$.

When $n = 1$, any increase in x_1 makes the modulated inhibitory weight $\tilde{a}_{21} = (u_0 + m_{211}x_1)a_{21}$ more negative. That is, the larger x_1 is the more it inhibits x_2 because $\frac{\partial \tilde{a}_{21}}{\partial x_1} = m_{211}a_{21} < 0$. This modulatory interaction breaks the network symmetry: it favors larger x_1 and smaller x_2 as compared to the non-modulated regime. Formally, such a modulatory interaction breaks both agent and option symmetries by making the modulated adjacency matrix non-symmetric and the modulated vector field non-odd symmetric. This is reflected in the resulting bifurcation diagram: the pitchfork unfolds into a transcritical bifurcation characterized by a region of bistability between the neutral state and an opinionated state $\mathbf{x}^*$ characterized by $x_1^* > 0 > x_2^*$ (the state x_2 is not shown in the bifurcation diagrams).

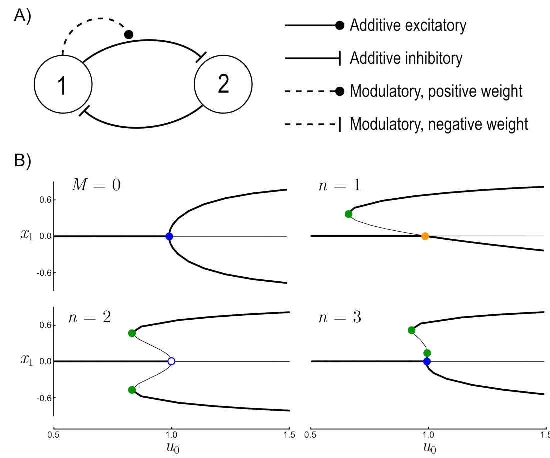

Fig. 2. A) Arrows conventions for additive and modulatory interactions, and an example of a modulated interaction network for a two-agent, two-option decision-making dynamics (3). B) Effect of modulatory interactions of various orders on the bifurcation behavior of (3) with additive and modulatory interactions as in A). Stable (unstable) bifurcation branches are indicated by thick (thin) lines. Supercritical pitchfork bifurcations are indicated by blue dots, subcritical pitchfork bifurcations by blue circles, saddle-node bifurcations by green dots, transcritical bifurcations by orange dots.

When $n = 2$, the modulated inhibitory weight $\tilde{a}_{21} = (u_0 + m_{211}x_1^2)a_{21}$ is non-monotone in x_1. If $x_1 > 0$ ($x_1 < 0$), an increase in x_1 makes $\tilde{a}_{21}$ more (less) negative because $\frac{\partial \tilde{a}_{21}}{\partial x_1} = 2m_{211}a_{21}x_1 < 0$ (> 0). This modulatory interaction amplifies mutual inhibition but does *not* break network symmetry: it favors larger x_1 and smaller x_2 for positive x_1 (and negative x_2) but smaller x_1 and larger x_2 for negative x_1 (and positive x_2). The even order of the modulatory interaction preserves the symmetry of the original system in the network equivariant sense of [28]. This is reflected in the resulting bifurcation diagram: the symmetric pitchfork becomes subcritical and is characterized by a region of bistability between the neutral state and two symmetric opinionated states $\mathbf{x}^*_{up} = -\mathbf{x}^*_{down}$, such that $(\mathbf{x}^*_{up})_1 > 0 > (\mathbf{x}^*_{up})_2$ and $(\mathbf{x}^*_{down})_1 < 0 < (\mathbf{x}^*_{down})_2$.

When $n = 3$, the modulated inhibitory weight $\tilde{a}_{21} = (u_0 + m_{211}x_1^3)a_{21}$ is monotone in x_1, similarly to $n = 1$, because $\frac{\partial \tilde{a}_{21}}{\partial x_1} = 3m_{211}a_{21}x_1^2 \leq 0$. As for $n = 1$, this modulatory interaction breaks both agent and option symmetry and there exists a region of bistability between the neutral state and an opinionated state $\mathbf{x}^*$ characterized by $x_1^* > 0 > x_2^*$. For the same modulatory strength m_{211}, a cubic modulatory interaction locally preserves the supercritical pitchfork of the non-modulated case, while a linear modulatory interaction unfolds it into a transcritical bifurcation.

V. OPINION-FORMING BIFURCATIONS IN THE PRESENCE OF MODULATORY INTERACTIONS

We start by proving that modulatory interactions do *not* change neither the location of the opinion-forming bifurcation nor the associated critical subspace (the Jacobian kernel at bifurcation) and sensitive subspace (the direction in input space that is amplified nonlinearly along the critical subspace). We then state and interpret our main result that characterizes opinion-forming bifurcation in modulated NOD.

Start by observing that for $\mathbf{b} = \mathbf{0}$ the neutral state $\mathbf{x} = \mathbf{0}$ is an equilibrium of (1) for all $u_0 \in \mathbb{R}$. Let $p_i(\mathbf{x}) = \sum_{j=1}^N a_{ij}\left(u_0 + \sum_{k=1}^N m_{ijk}x_k^n\right)x_j$. Then (1) becomes $\tau\dot{x}_i = -x_i + b_i + S(p_i(\mathbf{x}))$. It follows that the Jacobian $J(\mathbf{x})$ of (1) at $\mathbf{x}$ has components

$$J_{il}(\mathbf{x}) = \begin{cases} S'\big(p_i(\mathbf{x})\big)\partial p_i/\partial x_l(\mathbf{x}) - 1, & \text{if } l = i, \\ S'\big(p_i(\mathbf{x})\big)\partial p_i/\partial x_l(\mathbf{x}), & \text{if } l \neq i, \end{cases} \quad (5)$$

with

$$\frac{\partial p_i}{\partial x_l}(\mathbf{x}) = a_{il}u_0 + \sum_{k=1}^N a_{il}m_{ilk}x_k^n + n\sum_{j=1}^N a_{ij}m_{ijl}x_l^{n-1}x_j\,.$$

We have the following evident but key lemma.

Lemma 5.1: The Jacobian $J(\mathbf{0})$ of (1) at $\mathbf{x} = \mathbf{0}$ is $-I + u_0S'(0)A$. In particular, $J(\mathbf{0})$ does not depend on M.

The following theorem generalizes [3, Theorem 4.2] to model (1),(2). Let $\sigma(A)$ denote the spectrum of A.

Theorem 5.2: Consider model (2). Suppose that A has a strictly leading eigenvalue λ_{max}, i.e., a simple real eigenvalue satisfying $\lambda_{max} > \max_{\lambda_i\in\sigma(A)\setminus\lambda_{max}}\Re(\lambda_i)$. Let $\mathbf{v}_{max}$ and $\mathbf{w}_{max}$ be the right and left eigenvectors associated to λ_{max}, respectively. Let $u_0^* = (S'(0)\lambda_{max})^{-1}$. Then:
1. (Indecision-breaking bifurcation and critical subspace) For $\mathbf{b} = \mathbf{0}$, $\mathbf{x} = \mathbf{0}$ is exponentially stable for $u_0 < u_0^*$, undergoes a bifurcation at $u_0 = u_0^*$, and is unstable for $u_0 > u_0^*$. Furthermore, bifurcation branches emanating from $(\mathbf{0}, u_0^*)$ are tangent to $\mathbf{v}_{max}$.
2. (Input sensitivity subspace) If $\langle\mathbf{b}, \mathbf{w}_{max}\rangle = 0$ and $\|\mathbf{b}\|$ is small enough, there exists a neighborhood $U \ni u_0^*$ such that for all $u_0 \in U$ there exists an equilibrium $\mathbf{x} = \mathbf{x}^*(u_0)$ satisfying $\langle\mathbf{x}^*(u_0), \mathbf{v}_{max}\rangle = 0$ such that $\mathbf{x}^*(u_0)$ is stable (unstable) for $u_0 < u_0^*$ ($u_0 > u_0^*$) and undergoes a bifurcation at $u_0 = u_0^*$. If $\langle\mathbf{b}, \mathbf{w}_{max}\rangle \neq 0$ and $\|\mathbf{b}\|$ is small enough, then the bifurcation unfolds according to its *universal unfolding* (see [27, Chapter 4]).

Proof: Observe that $J(\mathbf{0})$ has a simple leading eigenvalue $-1+u_0S'(0)\lambda_{max}$. Let U be the matrix that puts $J(\mathbf{0})$ in the Jordan form

$$UJ(\mathbf{0})U^{-1} = \begin{pmatrix} -1 + u_0S'(0)\lambda_{max} & \mathbf{0}_{1\times N-1} \\ \mathbf{0}_{N-1} & \tilde{J}(\mathbf{0}) \end{pmatrix}.$$

Observe that $[U_{i1}]_{i=1}^N = \mathbf{v}_{max}$ and $[U_{1i}^{-1}]_{i=1}^N = (\mathbf{w}_{max})^\top$. All the $N-1$ eigenvalues of $\tilde{J}(\mathbf{0})$ have negative real part for u_0 sufficiently close to u_0^*. Then the first statement follows from Lyapunov's indirect method [29, Theorem 4.7] and the Center Manifold Theorem [30, Theorem 3.2.1]. The second statement follows by applying the LS reduction [27, Section I.3] at $(\mathbf{x}, u_0) = (\mathbf{0}, u_0^*)$ with respect to right and left singular directions $\mathbf{v}_{max}, \mathbf{w}_{max}$ and noticing that if $\langle\mathbf{b}, \mathbf{w}_{max}\rangle = 0$ the reduced dynamics along $\mathbf{v}_{max}$ do not depend on $\mathbf{b}$, whereas for $\langle\mathbf{b}, \mathbf{w}_{max}\rangle \neq 0$ the branches are predicted by applying unfolding theory [27, Chapter 4] on the resulting scalar equilibrium equation. ■

It follows from Theorem 5.2 that the critical attention value u_0^* is independent of modulatory interactions. It solely depends on the leading eigenstructure of A. Theorem 5.2 also implies that the opinion patterns $\mathbf{x}_{bif}$ along the opinionated bifurcation branches are solely determined by the right leading eigenvector $\mathbf{v}_{max}$, i.e., to leading order, $\mathbf{x}_{bif} = \bar{x}_{bif}\mathbf{v}_{max}$, $\bar{x}_{bif} \in \mathbb{R}$. Note that Theorem 5.2 predicts the opinion-forming bifurcation also in the case $\mathbf{b} \neq \mathbf{0}$ and the bifurcating equilibrium $\mathbf{x}^*$ is not at the origin. More precisely, the right leading eigenvector $\mathbf{v}_{max}$ of the Jacobian $J(\mathbf{0})$ *at the origin* singles out the subspace along which we can perform the LS reduction, and, therefore, predicts the following: *i)* sensitivity to inputs $\mathbf{b} \neq \mathbf{0}$ at bifurcation is independent of modulatory interactions as determined by the left leading eigenvector $\mathbf{w}_{max}$ of $J(\mathbf{0})$; *ii)* the nonlinear effects of inputs and other parameters on the bifurcating branch shape can be predicted by computing higher-order terms of the LS reduction and using bifurcation recognition and unfolding theory [27, Chapters 2 and 3]. See [3, IV.B] for an in-depth discussion on the graph structures for which the assumptions of Theorem 5.2 are guaranteed to hold.

VI. SHAPING GLOBAL INDECISION-BREAKING BIFURCATION THROUGH MODULATORY INTERACTIONS

In the next section we apply Theorem 5.2 and illustrate first in a multi-agent, two-option network and then in a single-agent, multi-option network how to analyze and predict the effect of modulatory interactions on the *shape* of indecision-breaking bifurcation branches along $\mathbf{v}_{max}$.

A. Modulated indecision-breaking in a multi-agent, two-option social influence network

Consider the 5-agent, 2-options, modulated opinion interaction network in Fig. 3A. Let all additive links have unitary weight, all modulatory links have weight $\bar{m} \geq 0$, and the order of the modulatory interactions be $n = 1$. We interpret this network as a social network with first-neighbor additive coupling and the presence of an "influencer" node (node 1) that affects the network discourse by modulating all additive coupling weights as a function of its state.

In model (2) the additive interaction matrix A associated to Fig. 3A is the adjacency matrix of an undirected ring graph, i.e., the circulant matrix generated by the vector $[0, 1, 0, 0, 1]$. Additionally, because agent 1 is the only modulator and it is modulating all interactions, the modulatory interaction matrix M satisfies $M_{\cdot\cdot 1} = \bar{m}A$, $\bar{m} \in \mathbb{R}$, and 0 elsewhere. Invoking Theorem 5.2, there is an opinion-forming bifurcation at $u_0 = u_0^* = (S'(0)\lambda_{max})^{-1}$, where $\lambda_{max} = 2$ is the largest eigenvalue of A. For $S(\cdot) = \tanh(\cdot)$, $u_0^* = \frac{1}{2}$. $\lambda_{max} = 2$ is simple and with right and left

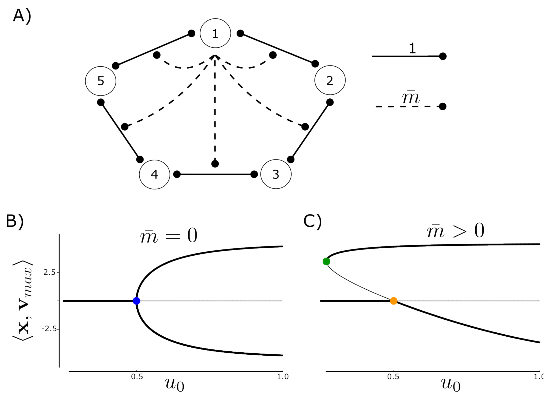

Fig. 3. A) The modulated opinion interaction network under study. B) Bifurcation diagram in the absence of inputs and $\bar{m} = 0$. The model undergoes a supercritical pitchfork at $u_0 = \frac{1}{2}$ along the consensus subspace generated by $\mathbf{v}_{max}$. C) Bifurcation diagram projection along $\mathbf{v}_{max}$ for $\bar{m} = 0.5$. The pitchfork unfolds into a transcritical bifurcation (yellow dot). The associated bistability region reflects the effects of the influencer node in shaping the network discourse toward option 1.

eigenvectors proportional to $\mathbf{v}_{max} = \mathbf{w}_{max} = [1, 1, 1, 1, 1]^T$. The opinionated branches emerging at the opinion-forming bifurcation are tangent to $\mathbf{v}_{max}$, that is, to leading order, they correspond to consensus opinion formation. Similarly, only inputs such that $\langle \mathbf{b}, \mathbf{w}_{max} \rangle = \sum_{i=1}^{5} b_i \neq 0$ affect the opinion forming behavior.

To characterize how the influencer modulation shapes opinionated branches we use the LS reduction with right and left singular eigenvectors $\mathbf{v}_{max}, \mathbf{w}_{max}$. Let v be the reduced variable along $\mathbf{v}_{max}$ and $g(v, u_0)$ be the LS reduced equation. Then at $(v, u_0) = (0, \frac{1}{2})$:

$$g = \frac{\partial g}{\partial v} = \frac{\partial g}{\partial u_0} = \frac{\partial^2 g}{\partial u_0^2} = 0,$$
$$\frac{\partial}{\partial v}\frac{\partial g}{\partial u_0} = \frac{\partial}{\partial u_0}\frac{\partial g}{\partial v} = 3, \quad \frac{\partial^2 g}{\partial v^2} = 4\bar{m}, \quad \frac{\partial^3 g}{\partial v^3} = -2.$$

The recognition problem for the pitchfork [27, Prop. 9.2] then implies that for $\bar{m} = 0$ the opinion forming bifurcation is a supercritical pitchfork (Fig. 3B). When $\bar{m} > 0$, the influencer acts in favor of option 1 and even in the absence of inputs the pitchfork unfolds into a transcritical bifurcation (Fig. 3C; [27, Section III.7]). Similarly to Fig. 2B ($n = 1$), the influencer node modulation of the network discourse creates a pre-bifurcation bistable region in which the group can switch from neutral to option 1 even in the absence of inputs. For $\mathbf{b} \neq \mathbf{0}$, the bifurcation unfolds in favor of the option indicated by the distributed input $\langle \mathbf{b}, \mathbf{w}_{max} \rangle$.

B. Conditional decision-making through modulation in a single-agent multi-option network for robot navigation

The single-agent, multi-option, modulated opinion interaction network in Fig. 4A provides an extension of the NOD-based robot navigation controller introduced in [12]. It consists of two mutual inhibitory NOD subnetworks made by nodes $\{1, 2\}$, 'drive or stay' , and $\{3, 4\}$, 'steer left or steer right', respectively. The two subnetworks are disconnected at the additive interaction level but node 1 of subnetwork $\{1, 2\}$ modulates the mutual inhibition strength of subnetwork $\{3, 4\}$ with modulatory weight $\bar{m}$. Although preferences for the two mutually-exclusive options in each subnetwork could be modeled with a single (reduced) opinion state, the use of one state per option permits to explicitly define the modulatory role of each node and makes the model interpretation more transparent without hampering its mathematical tractability. The analysis for this example is summarized by Fig. 4B.

The mutual inhibition strength of subnetwork $\{1, 2\}$ is α. Invoking Theorem 5.2, for $u_0 < \alpha^{-1}$ the neutral state $x_1 = x_2 = 0$ (idle state) is stable. For $u_0 > \alpha^{-1}$ the neutral state becomes unstable in a pitchfork bifurcation giving rise to two opinionated stable equilibria characterized by $x_1 > 0 > x_2$ ('drive') and $x_1 < 0 < x_2$ ('stay'), respectively.

The mutual inhibition strength of subnetwork $\{3, 4\}$ is $\beta + \bar{m}x_1$, so in this subnetwork the opinion-forming pitchfork bifurcation happens for $u_0 = (\beta + \bar{m}x_1)^{-1}$. Hence, if $\bar{m} > 0$, the more positive (negative) is x_1, the smaller (larger) the basal attention u_0 is needed for the opinion-forming pitchfork of subnetwork $\{3, 4\}$ to happen. For $u_0 < (\beta + \bar{m}x_1)^{-1}$ the neutral state $x_3 = x_4 = 0$ ('no steering') is stable. For $u_0 > (\beta + \bar{m}x_1)^{-1}$ the neutral state is unstable and two stable opinionated equilibria appear, characterized by $x_3 > 0 > x_4$ ('steer left') and $x_3 < 0 < x_4$ ('steer right'), respectively.

In the network model of Fig. 4A it is natural to assume $\alpha > \beta$ in such a way that, as attention increases, the decision to drive (or stay) happens before the decision to steer. If $\bar{m} = 0$, the steering pitchfork of subnetwork $\{3, 4\}$ happens for the same basal attention value $u_0 = \beta^{-1}$, independently of the decision state of the drive-or-stay subnetwork $\{1, 2\}$. Conversely, for $\bar{m} > 0$ and recalling that $x_1 > 0$ ($x_1 < 0$) along the driving (staying) branch, the critical attention value to trigger a steering decision is lower (higher) when subnetwork $\{1, 2\}$ is in the drive (stay) state. This ensures faster and more sensitive steering decisions when driving as opposed to staying. It also hampers the occurrence of unrealistic or unwanted opinion states such as 'stay and turn'. Both are desirable properties for efficient navigation.

The modulation of the basal attention needed to trigger a decision as a function of another decision outcome can be understood as a form of soft or flexible conditional decision-making: the sensitivity of a subordinate decision (e.g. steering) is conditioned on the outcome of a primary decision (e.g. drive-or-stay). The continuous state and parameter nature of modulated NOD, and the organizing role of its bifurcations, makes this kind of soft conditional decision-making tunable and adaptable according to the principles of fast-and-flexible decision-making [2]. As in the previous example, the presence of inputs (e.g., $b_{\mathrm{drive}} > 0$ representing a green light) would unfold the bifurcations in favor of the suggested action.

VII. Discussion

Inspired by examples in neuroscience, molecular biology, and social sciences, we proposed a new NOD with

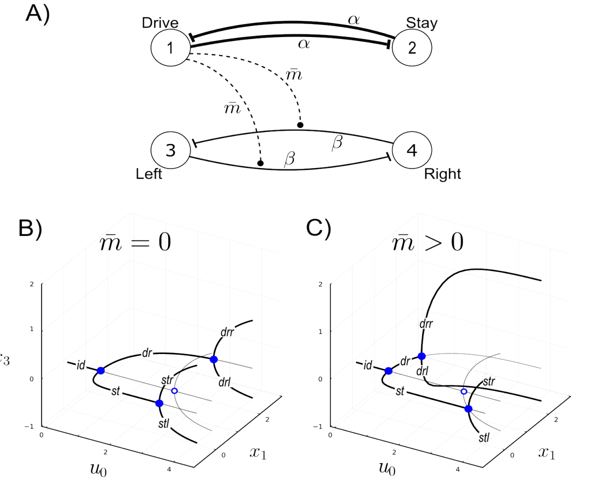

Fig. 4. A) The modulated opinion interaction network under study. B) Bifurcation diagram for $\alpha = 1, \beta = 0.3, \bar{m} = 0$. Branches are labelled with the associated robot navigation commands (id: idle; dr: drive; st: stay; drr: drive+steer right; drl: drive+steer left; str: stay+steer right; stl: stay+steer left) C) Same as B but for $\bar{m} = 2$. In the presence of modulatory interactions, the drive+steer bifurcation (drr,drl branches) happens for lower u_0 than the stay+steer bifurcation (str,stl branches).

modulatory interactions, rigorously characterized its opinion-forming bifurcation behavior, and illustrated possible applications through social network and robotic navigation examples. As in the original NOD model, the modulated model can be extended to opinion dynamics with N_a agents and N_o options. Additionally, replacing the modulatory term $m_{ijk}x_k^n$ by a general polynomial in x_k would lead to a variant of (1) reminiscent of polynomial network dynamics on hypergraphs [31]. Such a polynomial version of the model could also be used to identify the order of modulatory interactions in cases in which it is not known *a priori*, e.g., by fitting the model to experimental data. A weakness of the proposed approach, inherent to the used bifurcation-theoretical methods, is that it could be difficult to characterize in its full generality the effect of modulatory interactions for arbitrary adjacency and modulatory matrices. This can however always be done on a case-by-case basis.

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