Erratum: "Clustering behaviors in networks of integrate-and-fire oscillators" [Chaos 18, 037122 (2008)]

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I. INTRODUCTION

In our manuscript "Clustering behaviors in networks of integrate-and-fire oscillators,"[1] the proof of Theorem 2 is incorrect and, in full generality, the firing map is not a contraction with respect to the quadratic **P**-norm. Adding a mild technical assumption, the present note proves that the firing map is a contraction with respect to an appropriate 1-norm.

II. ERRATUM

In the proof of Theorem 2, the inequality (26)

$$\|\mathbf{DL}(\mathbf{\Phi}-\mathbf{\Psi})\|_{\mathbf{P}} \leq \|\mathbf{D}\|_\infty \|\mathbf{L}(\mathbf{\Phi}-\mathbf{\Psi})\|_{\mathbf{P}}$$

is incorrect. To fix the proof, we consider instead the inequality

$$\|\mathbf{DL}(\mathbf{\Phi}-\mathbf{\Psi})\| \leq \|\mathbf{D}\| \|\mathbf{L}(\mathbf{\Phi}-\mathbf{\Psi})\| \quad (\star)$$

for the vector 1-norm defined as

$$\|\mathbf{x}\| = |x_1| + \sum_{i=1}^{N_g-2} |x_{i+1}-x_i| + |x_{N_g-1}| \quad (\star\star)$$

and the induced matrix norm

$$\|\mathbf{A}\| = \max_{\mathbf{x}} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}.$$

Theorem: Assume that (i) f is concave-up and (ii) $h''(\phi) > 0 \, \forall \, \phi \in [0,1]$. Then the (N_g-1)-dimensional firing map (15) is a contraction for the norm $(\star\star)$.

Proof: The proof follows from $(\star)$ by showing that $\mathbf{L}$ is an isometry and that $\|\mathbf{D}\| = \|\mathbf{D}\|_\infty < 1$.

(a) $\mathbf{L}$ is an isometry: Noting $\mathbf{y}=\mathbf{Lx}$, one verifies

$$\|\mathbf{y}\| = |y_1| + \sum_{i=1}^{N_g-2} |y_{i+1}-y_i| + |y_{N_g-1}|$$
$$= |x_{N_g-1}| + |x_1| + \sum_{i=1}^{N_g-2} |x_{i+1}-x_i| = \|\mathbf{x}\|.$$

(b) $\|\mathbf{D}\| < 1$. The entries of $\mathbf{D}$ are increasing, i.e., $d_i \leq d_{i+1}$. Let $\mathbf{D^{(k)}} = \text{diag}\{d_i^{(k)}\}$, with $d_i^{(k)}=0$ for $i \leq k$ and $d_i^{(k)}=1$ for $i > k$. Any admissible matrix $\mathbf{D}/\|\mathbf{D}\|_\infty$ is a convex combination of the matrices $\mathbf{D^{(k)}}$, which implies

$$\frac{\|\mathbf{D}\|}{\|\mathbf{D}\|_\infty} \leq \max_{k \in \{0,\ldots,N_g-1\}} \|\mathbf{D^{(k)}}\|.$$

When $k=N_g-1$, one has directly $\|\mathbf{D^{(N_g-1)}}\|=0$. For $k<N_g-1$, the norm is obtained by solving

$$\|\mathbf{D^{(k)}}\| = \max_{\mathbf{x}} \frac{|x_{k+1}| + \sum_{i=k+1}^{N_g-2} |x_{i+1}-x_i| + |x_{N_g-1}|}{|x_1| + \sum_{i=1}^{N_g-2} |x_{i+1}-x_i| + |x_{N_g-1}|}.$$

Observing that

$$|x_{k+1}| = \left| x_1 + \sum_{i=1}^{k} (x_{i+1}-x_i) \right| \leq |x_1| + \sum_{i=1}^{k} |x_{i+1}-x_i|,$$

it follows that $\|\mathbf{D^{(k)}}\| \leq 1$ and, consequently, $\|\mathbf{D}\| \leq \|\mathbf{D}\|_\infty$. Choosing a vector $\mathbf{x}$ such that $0 < x_1 \leq \cdots \leq x_n$ yields

$$\frac{\|\mathbf{Dx}\|}{\|\mathbf{x}\|} = \frac{2d_{N_g-1}x_{N_g-1}}{2x_{N_g-1}} = d_{N_g-1} = \|\mathbf{D}\|_\infty,$$

and one obtains $\|\mathbf{D}\| = \|\mathbf{D}\|_\infty < 1$.

III. DISCUSSION

The contraction property of the firing map is thus established using the 1-norm $(\star)$ instead of the quadratic norm of the original proof. Assumption (ii) is a mild additional assumption on h that was not present in the original formulation of Theorem 2. The standard leaky integrate-and-fire (LIF) model with $x_r=0$ and $x_p=1$ satisfies this assumption.

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[1]Mauroy, A. and Sepulchre, R., C*lustering behaviors in networks of integrate-and-fire oscillators*, Chaos **18**, 037122 (2008).

