Electromagnetic forces and their finite element computation

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Abstract

This paper introduces the concepts of differential geometry that are necessary to establish a systematic definition for electromagnetic forces by means of a natural thermodynamic approach. It is shown that standard electromagnetic force formulae used in finite element computational electromagnetism are particular instances of that general approach. Finally, the paper offers a complete conceptual framework, as well as a mathematical toolbox, to derive electromagnetic force formulae for multi-physics finite element models with complex materials.

1 Introduction

Formulae for the finite element (FE) computation of electromagnetic forces appear to come in the literature as a set of unrelated results, each applicable in specific conditions. One may cite formulae for the force acting on pointwise particles like charges, dipoles and moments [1], formulae for the nodal forces in finite element meshes [2], formulae for the torque acting on the rotor of an electrical machine[3], just to name a few. Besides the extensive and insightful work of Alain Bossavit to whom this article owes a lot [4, 5, 6, 7, 8], very little is however said in the literature about the common theoretical background behind this wealth of seemingly disparate formulae.

The question of electromagnetic forces is situated at the common frontier between three disciplines : electromagnetism, continuum mechanics and thermodynamics. In that context, the definition of electromagnetic force should be something as simple and natural as the partial derivative of the magnetic term in the energy balance with respect to a mechanical state variable, holding the magnetic state variable constant. The question how to realize that partial derivation in practice is however not so trivial as it seems at first sight, and is actually quite far-reaching. One has in particular to make clear and physically sound what it means to hold an electromagnetic field constant whenever it is defined on a deforming domain.

Answering this question requires to invoke concepts from differential geometry that are not part of the mathematcal background of engineers. In consequence, the question has been solved in computational electromagnetism by resorting to various roundabout ways, using techniques more familar to engineers : differentiation of the jacobian of the finite element reference mapping [9], or sensitivity analysis with the adjoint method [10]. These approaches may give useful results in the end but, being not firmly rooted in Physics, they remain conceptually insatisfactory, and their domain of validity is unclear.

We intend in this article to give in a nutshell the concepts of differential geometry that are necessary to theoretically derive a straigthforward definition for electromagnetic forces based on a natural thermodynamic approach. It is then shown that standard electromagnetic force formulae are particular instances of that general approach, obtained by selecting different mechanical state variables (displacement of a rigid body, displacement of a mesh node, etc.). For the sake of clarity and conciseness of the presentation, the focus is placed in this article on magnetic forces in static or quasi-static regime.

Finally, the literature about electromagnetic forces is very abundant and we have not attempted to cover it exhaustively. The bibliography list of this article concentrates essentially on the seminal papers on the topic in the computational electromagnetism community, and in particular on the papers aiming at the finite element computation of magnetic forces.

2 Differential forms

In the early XXth century already, Weyl and Poincaré expressed Maxwell's laws with differential forms. Later, the general relativity text by Misner, Thorne and Wheeler devotes several chapters to electromagnetic theory and differential forms, emphasizing the graphical representation of forms [12]. Systematic treatment of electromagnetism by means of differential geometry concepts is still not standard however, despite patient and didactical presentations [13, 14, 15].

For mathematicians, differential forms are a particular kind of tensor fields. More precisely, a differential form of degree p, or p-form, is a field of p-covectors, the latter being a completely covariant skew-symmetric tensor of order p. Deschamps [14] more simply defines p-forms as expressions that occur in p-fold integrals. Finally, Bossavit sees p-forms as real-valued maps defined on p-dimensional oriented domains [16].

Following Bossavit, Deschamps and many others, and because we are chiefly interested in Physics, we shall adopt the point of view that differential forms are real-valued maps from p-dimensional domains, namely points ($p = 0$), curves ($p = 1$), surfaces ($p = 2$) and volumes ($p = 3$), onto real-valued scalar physical quantities (electric current, magetic flux, electric charge, energy, etc.). As we are limiting ourselves to the 3-dimensional Euclidean space E^3, p-forms exist only up to degree $p = 3$. The adopted definition allows identifying a number of physical fields as differential form, and to attribute them a degree. We limit ourselves essentially to the fields of Electromagnetism that are prevalent for the magnetic force issue.

To start with, 0-forms are simple scalar fields, a well-known concept, so that one can directly move towards differential forms of higher degrees. From the fact that the circulation of the magnetic field strength $\mathbf{h}$ along a curve is equal to an electric current I, it may be inferred that the magnetic field is a 1-form. In a Cartesian coordinate system $\{x, y, z\}$ defined on E^3, a 1-form ($\mathbf{h}$ is taken as an example), expands as

$$\mathbf{h} = h_x \,\mathrm{d}x + h_y \,\mathrm{d}y + h_z \,\mathrm{d}z, \tag{1}$$

in terms of three basis 1-forms $\{\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z\}$ (unit $[\mathrm{m}]$). In the case of the magnetic field, the three scalar components $\{h_x, h_y, h_z\}$ have unit $[\mathrm{A\,m^{-1}}]$, and the magnetic field 1-form itself $\mathbf{h}$ has unit $[\mathrm{A}]$.

The wedge product $\wedge$ is the differential operator that constructs basis p-forms of higher degrees from the basis 1-forms $\{\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z\}$ [17]. The wedge product of two 1-forms is a 2-form, and its main property is antisymmetry, which means that it verifies $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x$, and therefore $\mathrm{d}x \wedge \mathrm{d}x = 0$ (and similarly for the wedge products of other basis 1-forms). In consequence, only three of the nine possible 2-forms $\mathrm{d}x_i \wedge \mathrm{d}x_j, i, j = 1, 2, 3$ are non-zero and linearly independent. One can thus select $\{\,\mathrm{d}y \wedge \mathrm{d}z,\ \mathrm{d}z \wedge \mathrm{d}x,\ \mathrm{d}x \wedge \mathrm{d}y\}$ to form a basis for 2-forms.

The magnetic flux density $\mathbf{b}$, for instance, is associated with the notion of magnetic flux through surface integration, and is therefore a 2-form. As any 2-form, it expands as

$$\mathbf{b} = b_{yz} \,\mathrm{d}y \wedge \mathrm{d}z + b_{zx} \,\mathrm{d}z \wedge \mathrm{d}x + b_{xy} \,\mathrm{d}x \wedge \mathrm{d}y \tag{2}$$

in terms of the three basis 2-forms $\{\,\mathrm{d}y \wedge \mathrm{d}z,\ \mathrm{d}z \wedge \mathrm{d}x,\ \mathrm{d}x \wedge \mathrm{d}y\}$ (unit $[\mathrm{m^2}]$). For the magnetic flux density specifically, the three scalar components $\{b_{yz}, b_{zx}, b_{xy}\}$ have unit $[\mathrm{Wb\,m^{-2}}]$, and the 2-form $\mathbf{b}$ itself has unit $[\mathrm{Wb}]$.

Finally, if Q denotes the electric charge, the 3-form ρ^Q (unit $[\mathrm{Cb}]$) is a volume density of electric charge. As any 3-form, it expands as

$$\rho^Q = \varrho^Q \,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \tag{3}$$

in terms of a unique basis 3-form called volume form (unit $[\mathrm{m}^3]$). Although the volume form, which is the only non-zero wedge product of the three basis 1-forms, is usually noted ω in differential geometry, we nickname it

$$\mathrm{d}\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \tag{4}$$

in this paper so as to obtain for volume integrals the familar notation

$$Q = \int_\Omega \rho^Q = \int_\Omega \varrho^Q \, \mathrm{d}\Omega. \tag{5}$$

In the case of the charge density, the unique scalar component ϱ^Q has unit $[\mathrm{Cb\,m^{-3}}]$.

Notably, one sees that both 1-forms and 2-forms have three components, although they are different kinds of fields, with rather different geometric properties, in particular in deforming domains, as one shall see below. In the Euclidean space E^3, they can be both regarded as vector fields, expanded in a vector basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This means that, for instance, the basis 1-form $\mathrm{d}x$ and the basis 2-form $\mathrm{d}y \wedge \mathrm{d}z$ can both be assimilated with the basis vector $\mathbf{e}_x$, but this assimilation holds only as long as the domain on which the differential forms are defined does not deform, which is no longer true when the question of magnetic forces is addressed. Similarly, 0-forms and 3-forms are both scalar quantities, with also rather different geometric properties. The other physical fields of electromagetism can be classified similarly, see Table 1.

Table 1: Differential forms encountered in electromagnetism with their degrees and units.

degree	p-form name	symbol	unit	integral quantity
1	electric field	$\mathbf{e}$	V	voltage
1	magnetic field strength	$\mathbf{h}$	A	electric current
1	magnetic vector potential	$\mathbf{a}$	Wb	magnetic flux
2	current density	$\mathbf{j}$	A	electric current
2	magnetic flux density	$\mathbf{b}$	Wb	magnetic flux
2	magnetic polarization	$\mathbf{J}$	Wb	
2	electric displacement	$\mathbf{d}$	Cb	electric charge
3	charge density	ρ^Q	Cb	electric charge
3	energy density	ρ^Ψ	J	energy
3	co-energy density	ρ^Φ	J	co-energy

3 The co-moving time derivative

Let Ω be a three-dimensional domain containing only one material (air, steel, etc.), i.e., a domain with no material discontinuity, and let Ψ denote the integral over the domain Ω of the corresponding volume density ρ^Ψ

$$\Psi = \int_\Omega \rho^\Psi . \tag{6}$$

In (6), Ψ is a function of time or of a pseudo-time in case of a virtual displacement, and ρ^Ψ is a 3-form. The domain Ω undergoes a smooth deformation, possibly virtual in the sense of the virtual work principle. One notes $\mathbf{v}$, and calls velocity field, the smooth vector field tangent to the trajectories of the points of Ω.

The co-moving time derivative $\mathcal{L}_\mathbf{v}$ is the derivation operator such that the time derivative of Ψ on the deforming domain Ω writes [17]

$$\dot{\Psi} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho^\Psi = \int_\Omega \mathcal{L}_\mathbf{v}\, \rho^\Psi . \tag{7}$$

If the domain Ω would not move nor deform ($\mathbf{v} = 0$), one could simply derive with respect to time under the integral sign but, because of the deformation of Ω, a more involves differential operator is needed. It is the co-moving time derivative defined by $\mathcal{L}_\mathbf{v} = \partial_t \rho^\Psi + L_\mathbf{v} \rho^\Psi$ in function of the Lie derivative $L_\mathbf{v}$.

The co-moving time derivative $\mathcal{L}_\mathbf{v}$ is a fundamental operator in continuous medium mechanics, where it is usually called material derivative. It is used in fluid dynamics to express the conservation of energy and momentum density in Eulerian coordinates. In particular, it allows expressing mathematically the energy balance in electro-mechanical systems where electromagnetic fields are in interaction with moving and deforming material bodies. The somewhat bulky name 'co-moving time derivative' is preferred to 'material deriavtive' in this context because the operator also applies in empty space, i.e., in regions with electromagnetic fields and no matter.

As shall be discussed in detail in the following of this section, co-moving time derivatives of p-forms of degree $p = 0, 1, 2, 3$, respectively, read as follows

$$\mathcal{L}_\mathbf{v}\, f = \dot{f} \tag{8}$$

$$\mathcal{L}_\mathbf{v}\, \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v})\, \mathbf{h} \tag{9}$$

$$\mathcal{L}_\mathbf{v}\, \mathbf{b} = \dot{\mathbf{b}} - (\nabla \mathbf{v})^T \mathbf{b} + \mathbf{b}\, \mathrm{tr}\, (\nabla \mathbf{v}) \tag{10}$$

$$\mathcal{L}_\mathbf{v}\, \rho = \dot{\rho} + \rho\, \mathrm{tr}\, (\nabla \mathbf{v}) \tag{11}$$

when written with classical notations in a way that underlines their similarities and differences. The trace operator is noted tr, so that $\mathrm{tr}\, (\nabla \mathbf{v}) = \partial_i v_i = \mathrm{div}\, \mathbf{v}$. Einstein implicit summation over repeated indices is assumed and ∂_i denotes the partial derivative ∂_{x_i}. The meaning of the dot operator is $\dot{f} = \partial_t f + (\nabla f)^T \mathbf{v}$ in (8), $\dot{\rho} = \partial_t \varrho + (\nabla \varrho)^T \mathbf{v}$ in (11), and $\dot{\mathbf{h}} = \dot{h}_k \mathbf{e}_k$ and $\dot{\mathbf{b}} = \dot{b}_k \mathbf{e}_k$ in (9) and (10). See Table 2 in the appendix for a summary of the notations used in this paper.

The co-moving time derivative of a 0-form is

$$\mathcal{L}_{\mathbf{v}}\, f = \dot{f} = \partial_t f + (\nabla f)^T \mathbf{v} = \partial_t f + v_k\, \partial_k f. \tag{12}$$

This is known as the material derivative of a scalar field f in fluid dynamics. The co-moving time derivative of 3-form, on the other hand, is

$$\begin{aligned}\mathcal{L}_{\mathbf{v}}\, \rho &= \dot{\rho} + \rho\, \mathrm{tr}\, (\nabla \mathbf{v}) = \{\dot{\varrho} + \mathrm{tr}\, (\nabla \mathbf{v})\}\ \mathrm{d}\Omega \\ &= \left\{\partial_t \varrho + (\nabla \varrho)^T \mathbf{v} + \varrho\, \mathrm{div}\, \mathbf{v}\right\}\ \mathrm{d}\Omega = \{\partial_t \varrho + \mathrm{div}\, (\varrho \mathbf{v})\}\ \mathrm{d}\Omega,\end{aligned}$$

where we have used $\rho = \varrho\ \mathrm{d}\Omega$, and the fact that the basis 3-form $\mathrm{d}\Omega$ is independent of time. This is known as the material derivative of a scalar *density* ϱ in fluid dynamics, times the volume form $\mathrm{d}\Omega$. As the latter is a constant factor common to all terms of conservation equations expressed in Eulerian coordinates, it is in general ignored.

The co-moving time derivatives of 1-forms (9) and of 2-forms (10) are different from each other. The magnetic field strength $\mathbf{h}$ and the magnetic flux density $\mathbf{b}$ can therefore no longer be regarded as being plain vector fields whenever the domain Ω moves or deforms.

After reviewing their differences, one can also highlight some common features in the co-moving time derivative of p-forms, $p = 0, 1, 2, 3$. They all have a dotted term that represents the co-moving time derivative of the scalar coefficients of the p-form. For p-forms of degree $p \geq 1$, there are also additional terms that represents the co-moving time derivative of the basis p-forms. Due to their specific geometric definition, p-forms have their own way to be constant in time. As, for instance, a 2-form $\mathbf{b}$ is a map from the surfaces Σ in Ω to a scalar quantity called the magnetic flux, i.e.,

$$\varphi = \int_\Sigma \mathbf{b},$$

the 2-form is naturally constant in time when

$$\dot{\varphi} = \int_\Sigma \mathcal{L}_{\mathbf{v}}\, \mathbf{b} = 0,$$

which holds when $\mathcal{L}_{\mathbf{v}}\, \mathbf{b} = 0$, and neither when $\dot{\mathbf{b}} = 0$, nor $\partial_t \mathbf{b} = 0$. The same applies for p-forms of other degrees. In consequence, in a deforming domain, the co-moving time derivative of the *coefficients* of a p-form whose co-moving time derivative is zero, does not vanish in general. It is equal to

$$\begin{aligned}\left.\dot{\mathbf{h}}\right|_{\mathcal{L}_{\mathbf{v}}\, \mathbf{h}=0} &= -(\nabla \mathbf{v})\, \mathbf{h} \\ \left.\dot{\mathbf{b}}\right|_{\mathcal{L}_{\mathbf{v}}\, \mathbf{b}=0} &= (\nabla \mathbf{v})^T \mathbf{b} - \mathbf{b}\, \mathrm{tr}\, (\nabla \mathbf{v}) \\ \left.\dot{\rho}\right|_{\mathcal{L}_{\mathbf{v}}\, \rho=0} &= -\rho\, \mathrm{tr}\, (\nabla \mathbf{v})\end{aligned}$$

for 1-forms, 2-forms and 3-forms respectively. Derivation of this result for 1-forms and 2-forms, based on standard vector algebra (vector product and mixed products), can be found in [18] (Eqts. (25) and (32)), [19] (Eqts. (23) and (17)) or in the appendix of [20]. The co-moving time derivative of differential forms has also been introduced and discussed mathematically in details in the context of convection-diffusion problems in [21].

4 Magneto-mechanical power

We have now gathered all elements needed to work on a definition for magnetic forces. Let again Ω be a domain with no material discontinuity, and $\rho^\Psi(\mathbf{b})$ be an energy density functional defined on Ω and depending on the magnetic state variable $\mathbf{b}$. We are interested in this section in the time derivative of the integral of $\rho^\Psi(\mathbf{b})$ over Ω. Starting from (7) and using (11) to make a first subtitution, one has

$$\dot{\Psi} = \int_\Omega \mathcal{L}_\mathbf{v}\, \rho^\Psi = \int_\Omega \left\{ \dot{\rho}^\Psi + \rho^\Psi \,\mathrm{tr}\,(\nabla \mathbf{v}) \right\}. \tag{13}$$

The first term can be rewritten by the chain rule of derivatives

$$\dot{\rho}^\Psi = \dot{\varrho}^\Psi(\mathbf{b})\ \mathrm{d}\Omega = \partial_\mathbf{b} \varrho^\Psi(\mathbf{b}) \cdot \dot{\mathbf{b}}\ \mathrm{d}\Omega = \mathbf{h} \cdot \dot{\mathbf{b}}\ \mathrm{d}\Omega \tag{14}$$

because, as the magnetic field $\mathbf{h}$ is by definition the energy dual of $\mathbf{b}$, the magnetic constitutive relationship in Ω reads $\mathbf{h} = \partial_\mathbf{b} \varrho^\Psi(\mathbf{b})$, with $\partial_\mathbf{b}$ the Fréchet derivative. One now uses (10) to make a second substitution

$$\dot{\mathbf{b}} = \mathcal{L}_\mathbf{v}\, \mathbf{b} + (\nabla \mathbf{v})^T\, \mathbf{b} - \mathbf{b}\,\mathrm{tr}\,(\nabla \mathbf{v}) \tag{15}$$

and obtain

$$\dot{\Psi} = \int_\Omega \left\{ \mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b} + \mathbf{h}^T (\nabla \mathbf{v})^T\, \mathbf{b} - \mathbf{h} \cdot \mathbf{b}\,\mathrm{tr}\,(\nabla \mathbf{v}) + \varrho^\Psi\,\mathrm{tr}\,(\nabla \mathbf{v}) \right\}\ \mathrm{d}\Omega. \tag{16}$$

We use throughout a boldface letters for vectors, $\mathbf{F}$ for a column vector and $\mathbf{F}^T$ for a line vector, so that $\mathbf{F}^T\mathbf{G} = \mathbf{G}^T\mathbf{F} = \mathbf{F} \cdot \mathbf{G}$, $\mathbf{F}^T\mathbf{F} = \mathbf{F} \cdot \mathbf{F} = |\mathbf{F}|^2$, and the dyadic product $\mathbf{F}\mathbf{G}^T$ is a tensor, $(\mathbf{F}\mathbf{G}^T)_{ij} = F_i G_j$. By reorganizing the summations, the tensor $\nabla \mathbf{v}$ can be factorized in the last three terms

$$\dot{\Psi} = \int_\Omega \left\{ \mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b} + \mathbf{b}\mathbf{h}^T\ : (\nabla \mathbf{v}) - \left\{ \mathbf{h} \cdot \mathbf{b} - \varrho^\Psi \right\} \mathbb{I}\ : (\nabla \mathbf{v}) \right\}\ \mathrm{d}\Omega, \tag{17}$$

with $\mathbb{I}$ the identity tensor and $\mathrm{A}\ : \mathrm{B} = \mathrm{A}_{ij}\mathrm{B}_{ij}$ the tensor product. One can thus conclude that the variation of energy due to the dependency in $\mathbf{b}$ of the energy density functional writes

$$\dot{\Psi} = \int_\Omega \{\mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b} + \sigma_{EM} \ : \nabla\mathbf{v}\} \ \mathrm{d}\Omega, \tag{18}$$

with the magnetic stress tensor

$$\sigma_{EM} = \mathbf{b}\mathbf{h}^T - \big\{\mathbf{h} \cdot \mathbf{b} - \varrho^\Psi(\mathbf{b})\big\}\mathbb{I}. \tag{19}$$

So far, it has been only mathematics, but the result can also be analysed from a thermodynamic point of view. The time derivative (18) has been calculated accounting for the fact that ρ^Ψ and $\mathbf{b}$ are differential forms. It has got two terms. The first term involves $\mathcal{L}_\mathbf{v}\, \mathbf{b}$, which represents the time variation of the 2-form $\mathbf{b}$ in the deforming domain. The first term is thus the power exchanged with the magnetic field $\mathbf{b}$, and thus stored as internal energy in the system. The second term in (18), is the result of the action of a stress tensor on the gradient of the velocity field $\mathbf{v}$. It stands for the mechanical power developed by the electromagnetic forces as Ω deforms. The deformation of magnetic flux tubes induced by the deformation of the domain Ω is thus associated with the emergence of a stress tensor σ_{EM} of magnetic origin, not only in material regions but also in empty space or in air.

One may wonder what is the origin of this second term. With only one explicit argument in the functional $\rho^\Psi(\mathbf{b})$, one may indeed have expected to get only one term by application of the chain rule. But, besides its explicit $\mathbf{b}$ argument, the energy density functional $\rho^\Psi(\mathbf{b})$, like any energy density functional, also implicitly always depends on the background geometrical structure of the Euclidean space E^3, namely the metric and the volume form, which yields additional terms in the mathematical expression of the derivative. The good news is that these additional terms can be obtained by means of the double substitution presented in this section, using the formulae (8)-(11), without the need of more involved differential geometry.

The mechanical power developed by the magnetic stress tensor σ_{EM} is so important that it is given an explicit symbol

$$\begin{aligned}\dot{W}_{EM} \equiv -\int_\Omega \sigma_{EM} \ : \nabla\mathbf{v} \ \mathrm{d}\Omega &= -\dot{\Psi} + \int_\Omega \mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b} \ \mathrm{d}\Omega \\ &= \int_\Omega \big\{-\mathcal{L}_\mathbf{v}\, \rho^\Psi(\mathbf{b}) + \mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b}\, \mathrm{d}\Omega\big\}\end{aligned} \tag{20}$$

and a specific name : the magneto-mechanical power. A similar expression is found in [20]. The expression (20) is directly obtained from (18), and the main force formulae used in finite element models are in a way or another derived from it.

A classical definition for magnetic forces is obtained by rewriting (20) as

$$\dot{W}_{EM} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^\Psi(\mathbf{b})\Big|_{\mathcal{L}_\mathbf{v}\,\mathbf{b}=0} = -\,\dot{\Psi}\Big|_{\mathcal{L}_\mathbf{v}\,\mathbf{b}=0}. \tag{21}$$

This says that the magneto-mechanical power $\dot{W}_{EM}$ is equal to minus the variation of the energy Ψ when *holding magnetic fluxes constant*. This is the well-known virtual work principle [9, 4], or virtual power principle [6, 7] with now, thanks to the concepts introduced in sections 2 and 3, an accurate mathematical definition, i.e., $\mathcal{L}_\mathbf{v}\,\mathbf{b} = 0$, of what it means to hold magnetic fluxes constant in a deforming domain.

The virtual work principle also exists in a version based on co-energy [2, 18]. The relationship between the energy Ψ and the coenergy Φ in the domain Ω is given by

$$\Psi + \Phi - \int_\Omega \mathbf{h}\cdot\mathbf{b}\;\mathrm{d}\Omega = \int_\Omega \left\{\rho^\Psi(\mathbf{b}) + \rho^\Phi(\mathbf{h}) - \mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\right\} = 0. \tag{22}$$

To evaluate the time derivative of (22)

$$\dot{\Psi} + \dot{\Phi} - \int_\Omega \mathcal{L}_\mathbf{v}\;\{\mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\} = 0 \tag{23}$$

and establish the virtual work principle in terms of the co-energy, one has first to calculate the co-moving time derivative of the 3-form $\{\mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\}$. Using (11), one has

$$\mathcal{L}_\mathbf{v}\;\{\mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\} = \left\{\dot{\mathbf{h}}^T\mathbf{b} + \mathbf{h}^T\dot{\mathbf{b}} + \mathbf{h}\cdot\mathbf{b}\,\mathrm{tr}\,(\nabla\mathbf{v})\right\}\,\mathrm{d}\Omega, \tag{24}$$

and substituting $\dot{\mathbf{h}}$ and $\dot{\mathbf{b}}$ with their expression in terms of $\mathcal{L}_\mathbf{v}\,\mathbf{h}$, $\mathcal{L}_\mathbf{v}\,\mathbf{b}$ and $\nabla\mathbf{v}$ using (9) and (10), one obtains after a massive term simplification

$$\mathcal{L}_\mathbf{v}\;\{\mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\} = \{\,\mathcal{L}_\mathbf{v}\,\mathbf{h}\cdot\mathbf{b} + \mathbf{h}\cdot\,\mathcal{L}_\mathbf{v}\,\mathbf{b}\}\;\mathrm{d}\Omega. \tag{25}$$

The simplification of all terms depending on $\nabla\mathbf{v}$ does not come as a surprise because differential geometry tells that the product of a 1-form and a 2-form (or vice versa) gives a 3-form independent of the metric and of the volume form. Equation (23) writes now

$$\int_\Omega \mathcal{L}_\mathbf{v}\,\rho^\Psi(\mathbf{b}) + \int_\Omega \mathcal{L}_\mathbf{v}\,\rho^\Phi(\mathbf{h}) - \int_\Omega \{\,\mathcal{L}_\mathbf{v}\,\mathbf{h}\cdot\mathbf{b} + \mathbf{h}\cdot\,\mathcal{L}_\mathbf{v}\,\mathbf{b}\}\;\mathrm{d}\Omega = 0. \tag{26}$$

Hence, using (20), one obtains the identities

$$\dot{W}_{EM} = \int_\Omega -\left\{\,\mathcal{L}_\mathbf{v}\,\rho^\Psi(\mathbf{b}) - \mathbf{h}\cdot\,\mathcal{L}_\mathbf{v}\,\mathbf{b}\,\mathrm{d}\Omega\right\} = \int_\Omega \left\{\,\mathcal{L}_\mathbf{v}\,\rho^\Phi(\mathbf{h}) - \,\mathcal{L}_\mathbf{v}\,\mathbf{h}\cdot\mathbf{b}\,\mathrm{d}\Omega\right\}.$$

(27)

With now $\dot{W}_{EM}$ expressed as a function of the co-energy, one can state

$$\dot{W}_{EM} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \rho^{\Phi}(\mathbf{h}) \Big|_{\mathcal{L}_{\mathbf{v}}\, \mathbf{h}=0} = \dot{\Phi} \Big|_{\mathcal{L}_{\mathbf{v}}\, \mathbf{h}=0}, \tag{28}$$

which says that the magneto-mechanical power $\dot{W}_{EM}$ is equal to the variation of the co-energy Φ when *holding the circulations of the magnetic field strength* $\mathbf{h}$ *along the curves of the domain* Ω *constant*.

Before concluding this section, one notes that the geometrical mechanism at the origin of the magnetic forces is identical for electrostatic forces. The line of reasoning followed above can be replicated to define electrostatic forces (with $\rho^{\Psi}(\mathbf{d})$) or general electromagnetic forces (with $\rho^{\Psi}(\mathbf{b}, \mathbf{d})$). One obtains readily

$$\dot{\Psi} = \int_{\Omega} \{\mathbf{h} \cdot \mathcal{L}_{\mathbf{v}}\, \mathbf{b} + \mathbf{e} \cdot \mathcal{L}_{\mathbf{v}}\, \mathbf{d} + \sigma_{EM} : \nabla \mathbf{v}\}\ \mathrm{d}\Omega, \tag{29}$$

with now the electromagnetic stress tensor

$$\sigma_{EM} = \mathbf{b}\mathbf{h}^T + \mathbf{d}\mathbf{e}^T - \varrho^{\Phi}(\mathbf{h}, \mathbf{e})\, \mathbb{I} \tag{30}$$

in terms of the coenergy density

$$\varrho^{\Phi}(\mathbf{h}, \mathbf{e}) = \mathbf{h} \cdot \mathbf{b} + \mathbf{e} \cdot \mathbf{d} - \varrho^{\Psi}(\mathbf{b}, \mathbf{d}) \tag{31}$$

to generalize (18), (19) and (22).

5 Electromagnetic force density

The magnetic force density is now defined as the divergence of the magnetic stress tensor

$$\varrho^{F}_{EM} = \operatorname{div} \sigma_{EM}. \tag{32}$$

In order to understand the linkage between the magnetic stress tensor σ_{EM} and the magnetic forces density ϱ^{F}_{EM}, it is instructive to apply an integration by part to (20)

$$\begin{aligned} \dot{W}_{EM} &= -\int_{\Omega} \sigma_{EM} : \nabla \mathbf{v}\ \mathrm{d}\Omega = \int_{\Omega} \left\{-\operatorname{div}(\sigma_{EM}\mathbf{v}) + (\operatorname{div} \sigma_{EM})^T \mathbf{v}\right\}\ \mathrm{d}\Omega \\ &= -\int_{\partial\Omega} \mathbf{n}^T \sigma_{EM} \mathbf{v}\ \mathrm{d}\partial\Omega + \int_{\Omega} (\varrho^{F}_{EM})^T \mathbf{v}\ \mathrm{d}\Omega, \end{aligned} \tag{33}$$

where $\partial\Omega$ is the boundary of Ω, and $\mathbf{n}$ the unit exterior normal to the boundary.

As seen above, the magneto-mechanical power $\dot{W}_{EM}$ is expressed with one volume integral (20) involving the magnetic stress tensor σ_{EM} as a coupling quantity. After integration by parts, (33) shows an alternative expression that has got two terms : the mechanical work done by a volume density of magnetic force, $\rho^F_{EM} = \varrho^F_{EM}\, \mathrm{d}\Omega$, acting in the bulk of Ω, and the work done by a magnetic surface force $\mathbf{n}^T\sigma_{EM}$ acting on the boundary $\partial\Omega$. As each material has its own magnetic stress tensor, one has in general a discontinuity of $\mathbf{n}^T\sigma_{EM}$ at interfaces between neighbouring regions occupied by different materials. The jump of $\mathbf{n}^T\sigma_{EM}$ across such material discontinuities

$$\mathbf{n}^T\Delta\sigma_{EM} \equiv \mathbf{n}^T(\sigma^{ext}_{EM} - \sigma_{EM}), \tag{34}$$

where σ_{EM} is the magnetic stress tensor of the material inside the the surface $\partial\Omega$ whose orientation is given by the outer normal vector $\mathbf{n}$, and σ^{ext}_{EM} the magnetic stress tensor of the material in the contiguous exterior domain, is a surface density of magnetic force. This surface force is in general dominant in systems with steel and air regions, and it is called reluctance force.

We now show that the result (30) can also be obtained algebraically from Maxwell's equations

$$\operatorname{div}\mathbf{b} = 0 \tag{35}$$
$$\operatorname{curl}\mathbf{e} + \partial_t\mathbf{b} = 0 \tag{36}$$
$$\operatorname{div}\mathbf{d} - \rho^Q = 0 \tag{37}$$
$$\operatorname{curl}\mathbf{h} - \mathbf{j} - \partial_t\mathbf{d} = 0 \tag{38}$$

without invoking thermodynamic (energy conversion) arguments. The combination of Maxwell's equations

$$0 = (35)\,\mathbf{h} + (36)\times\mathbf{d} + (37)\,\mathbf{e} + (38)\times\mathbf{b} \tag{39}$$

gives

$$\operatorname{curl}\mathbf{h}\times\mathbf{b} + \mathbf{h}\operatorname{div}\mathbf{b} + \operatorname{curl}\mathbf{e}\times\mathbf{d} + \mathbf{e}\operatorname{div}\mathbf{d} = \mathbf{j}\times\mathbf{b} + \rho^Q\mathbf{e} + \partial_t(\mathbf{d}\times\mathbf{b}). \tag{40}$$

In components in a Cartesian setting, it reads

$$\begin{aligned} \operatorname{curl}\mathbf{h}\times\mathbf{b} &= b_j\big(\partial_{x_j}h_i - \partial_{x_i}h_j\big)\,\mathbf{E}_i \\ \mathbf{h}\operatorname{div}\mathbf{b} &= h_i\,\partial_{x_j}b_j\,\mathbf{E}_i, \end{aligned}$$

with $\mathbf{E}_i$ a Cartesian basis vector, so that

$$\begin{aligned} \operatorname{curl}\mathbf{h}\times\mathbf{b} + \mathbf{h}\operatorname{div}\mathbf{b} &= \big(h_i\,\partial_{x_j}b_j + b_j\,\partial_{x_j}h_i - b_j\,\partial_{x_i}h_j\big)\,\mathbf{E}_i \\ &= \big(\partial_{x_j}(b_jh_i) - \delta_{ij}b_k\,\partial_{x_j}h_k\big)\,\mathbf{E}_i. \end{aligned}$$

Similarly,

$$\operatorname{curl} \mathbf{e} \times \mathbf{d} + \mathbf{e} \operatorname{div} \mathbf{d} = \big(\partial_{x_j}(d_j e_i) - \delta_{ij} d_k\, \partial_{x_j} e_k\big)\, \mathbf{E}_i.$$

Summing up and using (40) gives

$$\begin{aligned} &\mathbf{j} \times \mathbf{b} + \rho^Q \mathbf{e} + \partial_t(\mathbf{d} \times \mathbf{b}) \\ &\qquad = \big(\partial_{x_j}\big(b_j h_i + d_j e_i\big) - \delta_{ij}\big(b_k\, \partial_{x_j} h_k + d_k\, \partial_{x_j} e_k\big)\big)\, \mathbf{E}_i. \end{aligned} \tag{41}$$

Considering only the dependency in $\mathbf{h}$ and $\mathbf{e}$ of the coenergy density functional $\rho^\Phi(\mathbf{h}, \mathbf{e})$, one can write

$$b_k\, \partial_{x_j} h_k + d_k \partial_{x_j} e_k = \big(\partial_{h_k} \rho^\Phi\big) \partial_{x_j} h_k + \big(\partial_{e_k} \rho^\Phi\big) \partial_{x_j} e_k = \partial_{x_j} \rho^\Phi(\mathbf{h}, \mathbf{e})$$

so that the right-hand side of (41) can be written as the divergence of a tensor. One has thus

$$\mathbf{j} \times \mathbf{b} + \rho^Q \mathbf{e} + \partial_t(\mathbf{d} \times \mathbf{b}) = \operatorname{div} \sigma_{EM} \tag{42}$$

with

$$(\sigma_{EM})_{ij} = b_j h_i + d_j e_i - \delta_{ij}\, \rho^\Phi(\mathbf{h}, \mathbf{e}), \tag{43}$$

or in tensorial notation

$$\sigma_{EM} = \mathbf{b}\mathbf{h}^T + \mathbf{d}\mathbf{e}^T - \rho^\Phi(\mathbf{h}, \mathbf{e})\, \mathbb{I}. \tag{44}$$

Remarkably, one recovers with (44) the result (30), already obtained by a thermodynamic analysis in section 4. The two results are however not redundant but complementary. The thermodynamic approach in section 4 shows that σ_{EM} is the agent of the electromagneto-mechanical energy conversion. On the other hand, the algebraic derivation made in this section shows that the well-known expressions for Coulomb force, Laplace force, and the time derivative of the electromagnetic momentum $\mathbf{d} \times \mathbf{b}$ are recovered when evaluating the divergence of σ_{EM}. However, instead of postulating the expressions of these forces, as is usually done, the developments presented in this section provide a deductive *definition* for them, and elucidate their thermodynamic origin.

6 Magnetic stress tensor in various materials

The last two sections can be summarized as follows, leaving electrostatic forces aside, and considering only magnetic forces in this section. The energy density functional ρ^Ψ of a system where magnetic forces are acting necessarily exhibits dependencies in some electromagnetic state variables. We have considered $\mathbf{b}$ so far. The variation of energy associated with the $\mathbf{b}$ dependency of ρ^Ψ is made of two parts : a variation of the internal energy associated with the co-moving time derivative $\mathcal{L}_{\mathbf{v}}\,\mathbf{b}$ of the state variable, and a mechanical power $\dot{W}_{EM}$ delivered by the magnetic stress tensor σ_{EM}.

Those results have been presented with a generic energy functional $\varrho^\Psi(\mathbf{b})$ that allows describing the fundamental mechanism of the magnetomechanical energy conversion, but that does not necessarily represent the most general case. In order to be less schematic, real materials should be considered whose energy functionals will present other dependencies than the plain dependency in $\mathbf{b}$ that will also play a part in the expression of the energy balance. The purpose of the next section is to briefly sketch how the general ideas discussed above can be applied to more complex materials commonly encountered in electromagnetic engineering applications. The first step is always to accurately define a realistic expression for the energy density functional ρ^Ψ of the material. The double substitution procedure described in section 4. is applied to the magnetic state variables to derive the magnetic stress tensor σ_{EM}, whereas other dependencies of the energy functional ϱ^Ψ give rise to additional forces (reluctance forces, magnetostriction, etc.).

6.1 Linear materials

A linear magnetic material is characterized by a uniform magnetic permeability $\mu = \mu_0\mu_r$. The energy density is given by

$$\varrho^\Psi(\mathbf{b}) = \frac{\mathbf{b}\cdot\mathbf{b}}{2\mu} \quad , \quad \mathbf{h} = \partial_{\mathbf{b}}\varrho^\Psi(\mathbf{b}) = \frac{\mathbf{b}}{\mu}, \tag{45}$$

and the derived magnetic stress tensor (19) reads

$$\sigma_{EM} = \frac{1}{\mu}\left\{\mathbf{b}\mathbf{b}^T - \frac{\mathbf{b}\cdot\mathbf{b}}{2}\mathbb{I}\right\} = \mu\left\{\mathbf{h}\mathbf{h}^T - \frac{\mathbf{h}\cdot\mathbf{h}}{2}\mathbb{I}\right\}. \tag{46}$$

In particular, in empty space or in non-magnetic materials ($\mu_r = 1$), one has

$$\sigma_{EM} = \frac{1}{\mu_0}\left\{\mathbf{b}\mathbf{b}^T - \frac{\mathbf{b}\cdot\mathbf{b}}{2}\mathbb{I}\right\} = \mu_0\left\{\mathbf{h}\mathbf{h}^T - \frac{\mathbf{h}\cdot\mathbf{h}}{2}\mathbb{I}\right\}, \tag{47}$$

which is the *Maxwell stress tensor*. We shall reserve this name for the magnetic stress of empty space only.

If now the magnetic permeability $\mu(x)$ is not uniform, the derivative (14) of the energy density

$$\varrho^\Psi(\mathbf{b}) = \frac{\mathbf{b}\cdot\mathbf{b}}{2\mu(x)} \quad , \quad \mathbf{h} = \partial_{\mathbf{b}}\varrho^\Psi(\mathbf{b}) = \frac{\mathbf{b}}{\mu(x)},$$

gives, using (12) with $\partial_t \mu = 0$,

$$\dot{\varrho}^{\Psi}(\mathbf{b}) = \partial_{\mathbf{b}} \varrho^{\Psi}(\mathbf{b}) \cdot \dot{\mathbf{b}} + \nabla \mu^{-1}(x) \cdot \mathbf{v} \, \frac{\mathbf{b}\cdot\mathbf{b}}{2}$$
$$= \mathbf{h}\cdot\dot{\mathbf{b}} - \frac{\nabla\mu(x)\cdot\mathbf{v}}{\mu^2(x)} \, \frac{\mathbf{b}\cdot\mathbf{b}}{2} = \mathbf{h}\cdot\dot{\mathbf{b}} - \frac{\mathbf{h}\cdot\mathbf{h}}{2} \nabla\mu\cdot\mathbf{v}.$$

Compared to (14), the derivative has an additional term that represents the mechanical work developped by the reluctance force

$$\varrho^F_{\text{rel}} = \nabla\mu^{-1} \, \frac{\mathbf{b}\cdot\mathbf{b}}{2} = -\nabla\mu \, \frac{\mathbf{h}\cdot\mathbf{h}}{2}. \tag{48}$$

Besides this additional force, the derivation of σ_{EM} is identical as above and yields (46) with $\mu = \mu(x)$.

Reluctance forces are thus due to the gradient of the magnetic permeability $\mu(x)$. Beware, it is not gradients of magnetic permeability due to saturation, $\mu(\mathbf{b}(x))$, which are dealt with with (54), but a true spatial inhomogenity of the properties of the magnetic material. Such inhomogenities are not common in bulk materials, but they are ubiquitous at material discontinuities where the magnetic permeability $\mu(x)$ presents a jump. Due to this jump, differentiability conditions to directly evaluate the gradient $\nabla\mu$ are not fulfilled and, in order to obtain an applicable formula for reluctance forces at material discontinuities, the divergence theorem must be invoked. Let Ω_a and Ω_b be two regions with magnetic permeability $\mu_a(x)$ and $\mu_b(x)$, respectively. Their common boundary Γ_{ab} represents thus a material discontinuity. Let Ω be a pillbox shaped region with its top face in Ω_a and its bottom face in Ω_b. The pillbox Ω is assumed to be small enough to consider that the fields $\mathbf{b}$ and $\mathbf{h}$, and the magnetic permeabilities μ_a and μ_b are homogeneous in $\Omega_a \cap \Omega$ and in $\Omega_b \cap \Omega$, and that the surface $\Gamma_{ab} \cap \Omega$ is planar. Inside the pillbox, a Cartesian vector basis $\{\mathbf{E}_n, \mathbf{E}_t\}$ can be constructed so that $\mathbf{b} = b_n \mathbf{E}_n + b_t \mathbf{E}_t$ and $\mathbf{h} = h_n \mathbf{E}_n + h_t \mathbf{E}_t$, and that the vectors $\mathbf{E}_n$ match the normal vectors on the top and bottom faces of the pillbox. The vectors $\mathbf{E}_t$ are aligned with the non-normal component of the homogeneous $\mathbf{b}$ field, so that a third basis vector is not needed. In that coordinate system, the magnetic stress of the linear material writes

$$\sigma_{EM} = \begin{pmatrix} b_n h_n & b_n h_t \\ b_t h_n & b_t h_t \end{pmatrix} - \left(\frac{b_n^2}{2\mu} + \frac{\mu h_t^2}{2} \right) \mathbb{I} = \begin{pmatrix} \frac{b_n^2}{2\mu} - \frac{\mu h_t^2}{2} & b_n h_t \\ b_t h_n & \frac{\mu h_t^2}{2} - \frac{b_n^2}{2\mu} \end{pmatrix}, \tag{49}$$

whereas the reluctance force (48) can be rewritten

$$\varrho^F_{\text{rel}} = \nabla\mu^{-1} \frac{|\mathbf{b}|^2}{2} = \nabla\mu^{-1} \frac{b_n^2}{2} - \nabla\mu \, \frac{h_t^2}{2},$$

using $b_t = \mu h_t$. Now, field continuity conditions imply that $b_n^a = b_n^b$ and $h_t^a = h_t^b$. Whereas $|\mathbf{b}|^2 = b_n^2 + b_t^2$ is discontinuous across Γ_{ab}, b_n^2 and h_t^2 are homogeneous over the whole pillbox region Ω, and can be slipped into the gradient to obtain

$$\varrho_{\text{rel}}^F = \nabla \left(\frac{b_n^2}{2\mu} - \frac{\mu h_t^2}{2} \right) = \operatorname{div} \left\{ \left(\frac{b_n^2}{2\mu} - \frac{\mu h_t^2}{2} \right) \mathbb{I} \right\}.$$

The reluctance force at a material discontinuity is so expressed as the divergence of a tensor. By application of the divergence theorem, one has now

$$\int_\Omega \varrho_{\text{rel}}^F \, \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}^T \left(\frac{b_n^2}{2\mu} - \frac{\mu h_t^2}{2} \right) \mathrm{d}\partial\Omega = \int_{\Gamma_{ab}} \mathbf{n}^T (\sigma_{EM}^a - \sigma_{EM}^b) \, \mathrm{d}\Gamma_{ab}. \quad (50)$$

The rightmost identity is obtained by letting the thickness of the pillbox region Ω vanish to zero, and noting that the off-diagonal terms $b_n h_t$ of σ_{EM} (49) are continuous across the material discontinuity. This result is important as it solves the question of evaluating (48) at material discontinuities, and it shows that reluctance forces are actually equal to the difference of the magnetic stress tensors of the materials located on either side of the discontinuity. The interpretation of $\mathbf{n}^T(\sigma_{EM}^a - \sigma_{EM}^b)$ as a surface density of magnetic forces was already mentioned above (34).

6.2 Magnetostrictive materials

Only energy density functionals of the form $\varrho^\Psi(\mathbf{b})$ have been discussed so far. But, strictly speaking, the energy density of a magneto-mechanical system also depends on a mechanical state variable, say a strain tensor ε. In many case however, the structural deformation and the magnetic behaviour are decoupled, which translates into the simplification

$$\varrho^\Psi(\mathbf{b}, \varepsilon) \approx \varrho_1^\Psi(\mathbf{b}) + \varrho_2^\Psi(\varepsilon), \quad (51)$$

and the analysis of magnetic forces can be carried out by considering $\varrho_1^\Psi(\mathbf{b})$ only, as has been done so far. It is then understood that the obtained magnetic stress σ_{EM} and forces ϱ_{EM}^F are to be added with the other stresses and forces acting in the system that are accounted for by the second functional $\varrho_2^\Psi(\varepsilon)$.

The simplification permitted by the decoupling (51) ceases to be correct when dealing with magnetostrictive materials. In that case, one has to work with the complete magneto-mechanical energy density functional of the system, $\varrho^\Psi(\mathbf{b}, \varepsilon)$, including magnetic and structural aspects altogether. This allows representing a magnetic material law

$$\mathbf{h}(\mathbf{b}, \varepsilon) = \partial_{\mathbf{b}} \varrho^\Psi(\mathbf{b}, \varepsilon) \quad (52)$$

that significantly depends on the mechanical strain ε. The stress tensor

$$\sigma(\mathbf{h}, \varepsilon) = \partial_\varepsilon \varrho^\Psi(\mathbf{b}, \varepsilon) \tag{53}$$

on the other hand, is a combination of structural effects and magnetostrictive effects manifested by the dependency in $\mathbf{b}$. Still, the treatment of the dependency in $\mathbf{b}$ of the complete energy density functional $\varrho^\Psi(\mathbf{b}, \varepsilon)$ with the double subtitution presented in section 4 yields a magnetic stress σ_{EM} that is to be added to the mechanical stress (53).

6.3 Ferromagnetic materials

For an isotropic anhysteretic saturable steel, a possible expression for the energy density is

$$\varrho^\Psi(\mathbf{b}) = \int_0^{|\mathbf{b}|} \mu^{-1}(x)\,\mathrm{d}x\ \ \mathrm{d}\Omega \tag{54}$$

and one has

$$\mathbf{h} = \partial_\mathbf{b} \varrho^\Psi(\mathbf{b}) = \partial_{|\mathbf{b}|}\left\{\varrho^\Psi(\mathbf{b})\right\}\ \partial_\mathbf{b}\{\sqrt{\mathbf{b}\cdot\mathbf{b}}\} = \mu^{-1}(|\mathbf{b}|)\,\mathbf{b}. \tag{55}$$

Proceeding with the double substitution, one obtains

$$\sigma_{EM} = \mathbf{b}\mathbf{h}^T - \left(\mathbf{h}\cdot\mathbf{b} - \int_0^{|\mathbf{b}|} \mu^{-1}(x)\,\mathrm{d}x\right)\mathbb{I}, \tag{56}$$

with $\mathbf{h}$ given by (55).

An accurate thermodynamic modelling of ferromagnetic materials (steel and permanent magnets) is however somewhat more involved. It implies decomposing the flux density field $\mathbf{b}$ into an empty space magnetic polarization $\mathbf{J}_0$ and a material magnetic polarization $\mathbf{J}$, with $\mathbf{b} = \mathbf{J}_0 + \mathbf{J}$. The empty space magnetic polarization $\mathbf{J}_0$ always depends linearly and reversibly on the magnetic field strength $\mathbf{h}$, i.e., $\mathbf{J}_0 = \mu_0\mathbf{h}$. The material magnetic polarization $\mathbf{J}$, on the other hand, is due to the orientation of microscopic magnetic moments in the magnetic field $\mathbf{h}$ under the constraints of the microstructure of the ferromagnetic material (grains, Weiss domains, Bloch walls, etc.). It depends on $\mathbf{h}$ in a non-linear (saturation) and irreversible (hysteresis) way. The energy density of such a ferromagnetic material is represented by the functional

$$\varrho^\Psi(\mathbf{J}_0, \mathbf{J}, \varepsilon) = \frac{\mathbf{J}_0\cdot\mathbf{J}_0}{2\mu_0} + \varrho^\Psi_{\text{mat}}(\mathbf{J}, \varepsilon)$$

with an empty space part depending on $\mathbf{J}_0$ and a material part depending on $\mathbf{J}$ and ε. The thermodynamic analysis of the ferromagnetic material is completed by the rate of work functional

$$\dot{W} = \int_\Omega \{\mathbf{h} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{b} + \sigma \,:\, \nabla\mathbf{v}\} \ \mathrm{d}\Omega$$

and a dissipation functional

$$\dot{Q} = -\int_\Omega \mathbf{h}_i \cdot \mathcal{L}_\mathbf{v}\, \mathbf{J} \ \mathrm{d}\Omega$$

where the irreversible magnetic field $\mathbf{h}_i$ is given, for instance, by the energy-based hysteresis model described in [22] and the references therein. Application of the double substitution to the magnetic state variables $\mathbf{J}_0$ and $\mathbf{J}$ yields

$$\dot{\Psi} = \int_\Omega \left\{ \frac{\mathbf{J}_0}{\mu_0} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{J}_0 + \partial_\mathbf{J} \varrho^\Psi_{\text{mat}} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{J} + \sigma_{EM} \,:\, \nabla\mathbf{v} + \partial_\varepsilon \varrho^\Psi_{\text{mat}} \,:\, \nabla\mathbf{v} \right\} \ \mathrm{d}\Omega \quad (57)$$

with

$$\sigma_{EM} = \mathbf{J}_0 \frac{\mathbf{J}_0^T}{\mu_0} + \mathbf{J}\mathbf{h}_r^T - \left\{ \frac{\mathbf{J}_0 \cdot \mathbf{J}_0}{\mu_0} + \mathbf{h}_r \cdot \mathbf{J} - \varrho^\Psi \right\} \mathbb{I} \quad (58)$$

and

$$\mathbf{h}_r = \partial_\mathbf{J} \varrho^\Psi_{\text{mat}}. \quad (59)$$

The factor $\dot{\varepsilon}$ could be replaced by $\nabla\mathbf{v}$ in (57), because the mechanical stress tensor $\partial_\varepsilon \varrho_{\text{mat}}$ is symmetrical.

Stating the thermodynamic equilibrium of this material

$$\dot{\Psi} = \dot{W} + \dot{Q}$$

yields

$$0 = \int_\Omega \Big\{ (\mathbf{h} - \frac{\mathbf{J}_0}{\mu_0}) \cdot \mathcal{L}_\mathbf{v}\, \mathbf{J}_0 + (\mathbf{h} - \mathbf{h}_r - \mathbf{h}_i) \cdot \mathcal{L}_\mathbf{v}\, \mathbf{J} \\ + (\sigma - \sigma_{EM} - \partial_\varepsilon \varrho^\Psi_{\text{mat}}) \,:\, \nabla\mathbf{v} \Big\} \ \mathrm{d}\Omega$$

and hence the constitutive relationships of the ferromagnetic material since, as the variations of the state variables are arbitrary, all quantities between parenthesis must vanish. One sees in particular that the magnetic stress tensor σ_{EM} adds up naturally with the mechanical stress $\partial_\varepsilon \varrho^\Psi_{\text{mat}}$ to form the total stress tensor σ.

Permanent magnets are a particular case of ferromagnetic materials. The anhysteretic magnetisation curve is used in its linear part when the permanent magnet is magnetised, so that one has $\mathbf{J} = \mu\mathbf{h}_r$. Once magnetised, the material magnetic polarization $\mathbf{J}$ remains fixed, i.e., $\mathcal{L}_\mathbf{v}\,\mathbf{J} = 0$. It follows that

$$\sigma_{EM} = \mathbf{J}_0\frac{\mathbf{J}_0^T}{\mu_0} + \mathbf{J}\frac{\mathbf{J}^T}{\mu} - \left\{\frac{\mathbf{J}_0 \cdot \mathbf{J}_0}{\mu_0} + \frac{\mathbf{J} \cdot \mathbf{J}}{\mu} - \varrho^\Psi\right\}\mathbb{I} \tag{60}$$

and, since $\mu >> \mu_0$ and $\mathbf{J}_0 = \mu_0\mathbf{h}$, the magnetic stress tensor

$$\sigma_{EM} = \mu_0\mathbf{h}\mathbf{h}^T - \mu_0\frac{\mathbf{h}\cdot\mathbf{h}}{2}\mathbb{I} \tag{61}$$

can be used in good approximation for rigid permanent magnets.

6.4 Discussion

Other materials could be considered following a similar line of reasoning. The bottom line of the previous sections is that the whole question of the definition of electromagnetic forces is reduced to that of the accurate identification of the energy functional of the materials, as a function of an appropriate choice of multiphysics state variables. Then, the application of conservation principles delivers the constitutive relationships and expressions for all forces acting in the system.

The selection of an appropriate coordinate systems to describe a magnetomechanical problem (Eulerian or Lagrangian approaches, small or large deformations, ...), and making correct differentiations in that framework may rise some mathematical questions that are not addressed in this paper. Such unresolved technical issues are however not harmful in practice for two reasons. Firstly, in all models where material pieces can be considered as rigid bodies (motors, actuators, ...), the magnetic stress tensors of magnetic materials are muted in the energy balance because they are multiplied by a zero $\nabla\mathbf{v}$ factor, due to the rigid-body motion they undergo. Secondly, in applications where the deformation of magnetic materials is studied (e.g., noise and vibration analyses) the magnetic stress tensor of the magnetic materials is, as a rule, very weak compared to the Maxwell stress tensor (which is the magnetic stress tensor of empty space). For these reasons, they can nearly always be neglected, and the magnetic forces and stresses acting in the system can be computed in good approximation by considering only the Maxwell stress tensor and the eggshell method, as explained in the next section.

Still, a sufficient knowledge of the magnetic stress tensor of magnetic materials is useful in principle to validate the assumption that they are indeed negligible. This might not be the case, for instance, in the heavily saturated stator teeth of an electrical motor. That is the reason why this section has been devoted to sketching the main lines of how they can be determined in electrical engineering materials.

The definition of the magnetic force density ρ^F_{EM} has also been deferred until quite late in the paper because the differentiability condition to directly evaluate the divergence (32) are very often not fulfilled. Magnetic force density is thus not the most useful concept in practice. Force formulae directly based on $\dot{W}_{EM}$ and σ_{EM} (20) are to be preferred as shown in section 6.1. They involve the divergence in the weak variational sense.

7 Computation of resultant forces and torques

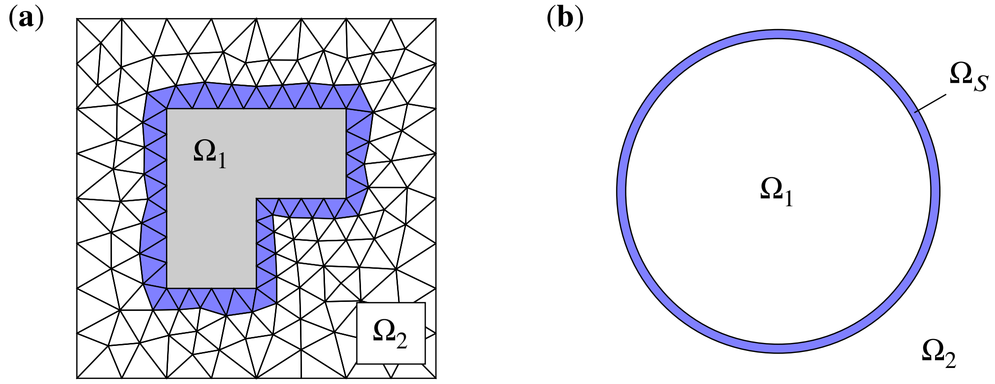

Figure 1: **(a)** Rigid body Ω_1 in a air domain Ω_2. The eggshell Ω_S is the highlighted layer of finite elements of Ω_2 upon which one evaluates the integral (63) to determine the resultant magnetic force acting on Ω_1. **(b)** Application of the eggshell method to evaluate the resultant torque acting on the rotor Ω_1 with an eggshell domain Ω_S contained in the air gap of the machine.

The evaluation of *resultant* magnetic forces and torques acting on rigid bodies is discussed in this section. The issue of contact forces being not addressed in this paper, the rigid body can be assumed to be completely surrounded by air, be it an extremely thin airgap at some part of its periphery.

Practical formulae for the calculation of magnetic resultant forces acting on rigid bodies have been known for long, but due to the lack of a straigthforward differential geometry background (the concepts of Lie derivative and co-moving time derivative), they were established decades ago by following indirect approaches : differentiation with respect to node displacement of the Jacobian matrix of the discrete finite element mapping [9], or continuum sensitivity analysis [10]. In this paper, the same result is obtained more directly by particularizing the expression of the magneto-mechanical power (20) with a crafted auxiliary velocity field

$$\mathbf{v} = \gamma\,\dot{\mathbf{X}} \quad , \quad \nabla\mathbf{v} = (\nabla\gamma)\dot{\mathbf{X}}^T \tag{62}$$

where the vector $\dot{\mathbf{X}}$ represents the velocity of the virtual translation of the rigid body Ω_1 upon which one wants to evaluate the exerted magnetic force, and γ is a continuous scalar field whose value is $\gamma = 1$ on the moving body Ω_1, and that arbitrarily smoothly decays to zero in the air domain Ω_2 surrounding the it, see Fig. 1(**a**). The support of the gradient $\nabla\mathbf{v}$, which we call *eggshell* [23], is thus a region contiguous to the body Ω_1, and that is completely included in the air domain Ω_2.

In a FE implementation, γ is represented by a scalar field defined over $\Omega_1 \cup \Omega_2$. It is discretized with Lagrange finite elements whose nodal degrees of freedom are set to one at all nodes of the object Ω_1, and set to zero at all other nodes. The eggshell is thus in the discrete case the layer of finite elements in the surrounding air domain Ω_2 that have at least one node on the surface of the object Ω_1. It is denoted by Ω_S, $\Omega_S \subset \Omega_2$, and highlighted in blue in Fig. 1(**a**).

By identification of the magneto-mechanical power (20) with the power $\mathbf{F} \cdot \dot{\mathbf{X}}$ delivered to the body Ω_1 by the magnetic force $\mathbf{F}$, one gets

$$\begin{aligned}\dot{W}_{EM} &= -\int_{\Omega} \sigma_{EM} : (\nabla\mathbf{v})\ \mathrm{d}\Omega = -\int_{\Omega} \sigma_{EM} : (\nabla\gamma\,\dot{\mathbf{X}}^T)\ \mathrm{d}\Omega \\ &= -\int_{\Omega_S} (\nabla\gamma)^T \sigma_{EM} \dot{\mathbf{X}}\ \mathrm{d}\Omega = \mathbf{F} \cdot \dot{\mathbf{X}}.\end{aligned}$$

Elimination of the virtual displacement factor $\dot{\mathbf{X}}$ gives a definition for the resultant magnetic force associated with the rigid-body movement represented by the velocity field (62)

$$\mathbf{F}^T = \int_{\Omega_S} -(\nabla\gamma)^T \sigma_{EM}\ \mathrm{d}\Omega. \tag{63}$$

Note that, as $\nabla\mathbf{v}$ is identically zero outside the eggshell Ω_S, the integration can be limited to Ω_S, which is a part of the air domain Ω_2. This implies that only the magnetic stress tensor of empty space, i.e., the Maxwell stress tensor (47), plays a role in the calculation the resultant force acting on a rigid body.

One might be puzzled by the fact that the magnetic force (63) is the effect of a stress tensor acting on the deformation of a region Ω_S devoid of any matter. But the velocity field $\mathbf{v}$ inside Ω_S is actually only an auxiliary quantity. This is shown applying an integration by parts to (63). It comes that

$$\mathbf{F}^T = \int_{\Omega_S} (\varrho^F_{EM})^T \gamma\ \mathrm{d}\Omega - \int_{\partial\Omega_S} \mathbf{n}_S^T \sigma_{EM}\,\gamma\ \mathrm{d}\partial\Omega$$

with $\mathbf{n}_S$ the exterior normal vector to the eggshell region Ω_S. The volume integral vanishes because there are no currents in the eggshell Ω_S, and therefore no volume magnetic force. The integral over the outer part of $\partial\Omega_S$ also vanishes because $\gamma = 0$ on that surface. Only the integral over the inner part of $\partial\Omega_S$ thus remains, which is the boundary of the rigid-body Ω_1 where one has set $\gamma = 1$ and with an opposite exterior normal vector $\mathbf{n} = -\mathbf{n}_S$. One has thus the interesting identity

$$\mathbf{F}^T = \int_{\Omega_S} -(\nabla\gamma)^T \sigma_{EM} \ \mathrm{d}\Omega. = \int_{\partial\Omega_1} \mathbf{n}^T \sigma_{EM} \ \mathrm{d}\partial\Omega \tag{64}$$

that shows that the force (63) computed by integration over the eggshell Ω_S and the resultant of the action of the Maxwell stress tensor on the material surface $\partial\Omega_1$ of the rigid body are always identical. Even if the distribution of the velocity field (62) in the eggshell Ω_S seems to play a role in the calculated force because of the explicit presence of the function γ in (63), the mathematical identity (64) proves that the way γ decreases from one to zero across the eggshell, provided it does it smoothly, has no impact on the calculated force. The virtual velocity field (62), which fringes outside the rigid body, can thus be regarded as an auxiliary quantity to evaluate the flux of the Maxwell stress tensor through the material surface of the rigid body.

Moreover, (64) is again a classical magnetic force formula [24], but under the form of a surface integral this time. The eggshell approach (63) is however easier to implement in a FE code because the magnetic fields $\mathbf{b}$ and $\mathbf{h}$ are readily available in the volume elements of Ω_S whereas, to evaluate the surface integral (64), one has to evaluate the fields in the volume elements contiguous to the surface elements of $\partial\Omega_1$. It is also more accurate [25].

The torque in electrical rotating machines can also be obtained by the eggshell approach. Let Ω_1 be a circular domain of radius R_i containing the rotor, and Ω_2 an annular domain of inner radius $R_o > R_i$ containing the stator, see Fig. 1(**b**). They are both centered at the origin. The thin annular domain Ω_S in between Ω_1 and Ω_2 is entirely contained in the air gap of the machine and can thus serve as an eggshell region for the evaluation of the resultant torque acting in the rotor. Consider a smooth velocity field $\mathbf{v}$ representing a rigid rotation of Ω_1 of angular velocity $\omega\mathbf{e}_z$, and leaving Ω_2 motionless. In the eggshell Ω_S, this velocity field and its gradient are given by

$$\mathbf{v} = \omega \ \mathbf{e}_z \times r \frac{R_o - r}{R_o - R_i} \mathbf{e}_r \quad , \quad \nabla\mathbf{v} = \omega \frac{-r}{R_o - R_i} \ \mathbf{e}_r \mathbf{e}_\theta, \tag{65}$$

in cylindrical coordinates. Substitution of $\nabla\mathbf{v}$ into (20), identification of $\dot{W}_{EM}$ with the power $-T\omega$ delivered by the magnetic torque, and elimination of the virtual factor ω gives Arkkio's formula [3] for the torque

$$T = \int_{\Omega_S} (\sigma_{EM})_{r\theta} \frac{r}{R_o - R_i} \, \mathrm{d}\Omega = \int_{\Omega_S} \frac{b_r b_\theta}{\mu_0} \frac{r}{R_o - R_i} \, \mathrm{d}\Omega. \tag{66}$$

8 Computation of local forces and deformations

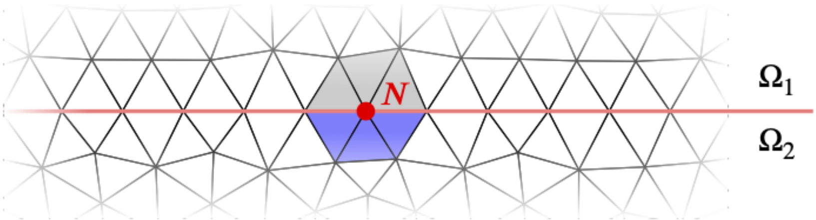

Figure 2: The eggshell to compute the nodal magetic force at an interface node.

One has seen in the previous section that the notion of resultant magnetic force or torque is intimately related with the assumption of rigid body, and that, in that case, only the Maxwell stress tensor (i.e., the magnetic stress tensor of empty space) is playing a role. In contrast, the magnetic stress tensor of other materials (steel, permanent magnets, etc...) plays a role when it comes to the computation of local magnetic forces.

The standard formulae to compute local magnetic forces in computational electromagnetism are obtained by application of the virtual work principle (21) or (28) according to whether one works with a magnetic vector potential $\mathbf{a}$ formulation or a magnetic fied $\mathbf{h}$ formulation. Bossavit and Kameari [5, 20] advocated that the magnetic vector potential $\mathbf{a}$ and the magnetic field $\mathbf{h}$ had to be discretized with Whitney edge elements to properly reflect their nature of differential 1-forms. Then, on that basis, the line of thought of Coulomb [9], based on the differentiation with respect to node displacement of the jacobian matrix of the discrete finite element mapping, was taken over by Ren and Razek [2] to propose a method to compute local nodal magnetic forces in 3D.

Here again, an equivalent but more straightforward formulation is obtained, based on (20) and the eggshell idea, by taking this time the nodal shape functions γ_N of the mesh, in 2D or in 3D, as γ function in (63). One has immediately

$$\mathbf{F}_N^T = \int_{\Omega_S(N)} -(\nabla \gamma_N)^T \sigma_{EM} \, \mathrm{d}\Omega, \tag{67}$$

where the eggshell $\Omega_S(N)$ of node N is the union of the finite elements contiguous to that node. For internal nodes in a domain occupied by a given material, the nodal force (67) only depends on the magnetic stress tensor of that material. For nodes on material discontinuities, on the other hand, the egsshell expands on both sides of the interface and is thus composed of two parts. For instance, if Ω_1 is occupied by saturable steel and Ω_2 by air, see Fig. 2, the egsshell is composed of a part in steel Ω_S^1 highlighted in grey, and a part in air Ω_S^2 highlighted in blue. Due to the large value of μ_0^{-1}, the contribution of the integral on Ω_S^2 largely dominates the contribution of the integral on Ω_S^1, which can therefore in general be neglected. Here again, and integration by parts shows that the integral over Ω_S^2 is equal to the contribution to node N of the Maxwell stress tensor applied at the material interface.

A naive implementation of (67) would be to proceed by a double loop on the nodes in the domain on which one wants to compute the node forces, and then on the list of finite element adjacent to each node. This is however rather inefficient.

One of the main messages of this paper is that the magnetic stress tensor σ_{EM} is a true stress tensor. It can therefore be used as a source term in the structural equations of a coupled magneto-mechanical model

$$\mathrm{div}\,(\sigma(\varepsilon) + \sigma_{EM}) + \varrho^F = 0 \tag{68}$$

where σ is the structural stress tensor, $\varepsilon = (\nabla \mathbf{v} + \nabla \mathbf{v}^T)/2$ the strain tensor, and ϱ^F is the applied force density, excluding magnetic forces. A more efficient implementation then consists in having the standard assembly algorithm of the structural finite element code assemble the term (67) element by element. In GetDP [26], one proceeds as follows. The unknown displacement field $\mathbf{v}$ is defined not only in the material domain Ω_1, but also in the complementary air domain Ω_2, and the nodal degrees of freedom of this field are set to zero at all internal nodes of Ω_2, which are all nodes only surrounded by air elements. The following weak form is assembled

$$\int_{\Omega_1} \sigma^T(\varepsilon)\, \nabla\gamma_N \,\mathrm{d}\Omega + \int_{\Omega_1 \cup \Omega_S} \sigma^T_{EM}(\mathbf{b})\, \nabla\gamma_N \,\mathrm{d}\Omega - \int_{\Omega_1} \varrho^F \gamma_N \,\mathrm{d}\Omega = 0 \tag{69}$$

for all nodes N of the material domain Ω_1. This is the usual variational weak form of (68), except that the domain of integration of the magnetic term is extended to also include the eggshells $\Omega_S \subset \Omega_2$ of the material regions. If now ϱ^F is set to zero, the right-hand side of the assembled linear system (69) is the vector of nodal magnetic forces. Moreover, by solving the linear system, one also obtains the deformations induced by the magnetic forces, which are a much more intuitive way to visualize the effect of magnetic forces than a direct representation of the node forces $\mathbf{F}_N$ or of the magnetic stress tensor σ_{EM}.

In [6], Bossavit claims that the Maxwell stress tensor is logically weaker than the Virtual Power Principle, and that it fails when magnetostricive effects exist.

His assertion is however based on a definition of the Maxwell stress tensor narrower than ours, as an expression in terms of the fields $\mathbf{b}$ and $\mathbf{h}$ only. In our approach, the magnetic stress tensor is derived in principle from the complete multi-physics energy density of the materials involved. In case of a magnetostrictive material, it would therefore depend not only on the field $\mathbf{b}$ (and $\mathbf{h}$) but also on the strain ε [27].

9 Conclusion

The message conveyed by this paper holds in the following points. Electromagnetic fields are differential forms, and their time variation in a deforming domain is given by a specific operator called the co-moving time derivative of differential forms, for which well-established formulae exist. With this, it is shown that the variation of the energy in a thermodynamic system depending on electromagnetic state variables not only contains a variation of the internal energy, i.e., a term of the form $\mathbf{h} \cdot \mathcal{L}_{\mathbf{v}}\,\mathbf{b}$, but also a magneto-mechanical conversion $\dot{W}_{EM}$ of the form $\sigma_{EM} : \nabla\mathbf{v}$, which is the rate of mechanical work developed by the magnetic forces.

Each material has its own magnetic stress tensor σ_{EM} whose expression can be derived from the expression of energy balance of the material. The double substitution procedure explained in the paper is used for the state variables that are differential forms. The issue of the definition of magnetic forces is thus shifted to the more fundamental question of identifying a realistic expression for the energy density ρ^{Ψ} of the material.

The magnetic force density ρ^F_{EM} is the divergence of the magnetic stress tensor σ_{EM}. It has not only a bulk component in magnetic regions, but also surface contributions at material discontinuities that are in general dominant in magnetic systems with steel pieces. At material discontinuities, the magnetic stress is discontinuous, so that the differentiability conditions to explicitly evaluate the divergence are not fulfilled. Force formulae directly based on $\dot{W}_{EM}$ and σ_{EM} are to be used in that case as they contain the divergence in the weak variational sense.

Classic magnetic force formulae used in finite element models have been shown to be different instances of the evaluation of $\dot{W}_{EM}$ with different kinds of virtual velocity fields. As this auxiliary virtual velocity field $\mathbf{v}$ must be continuous, it fringes outside the domain on which the force is computed, and the integration of $\sigma_{EM} : \nabla\mathbf{v}$ must be extended over a thin egsshell region around it.

A Notations

Table 2: Notations used in the paper for the co-moving time derivatives of scalar fields, vector fields, and differential forms in Cartesian coordinates. Implicit summation on repeated indices is assumed, $\mathbf{E}_i$ is a Cartesian basis vector.

	$(\nabla\mathbf{G})\,\mathbf{F}$	$=$	$(\partial_{x_i}G_j)F_j\;\mathbf{E}_i$
	$(\nabla\mathbf{G})^T\mathbf{F}$	$=$	$(\partial_{x_j}G_i)F_j\;\mathbf{E}_i$
	$\mathbf{F}^T\mathrm{A}\mathbf{G}$	$=$	$\mathbf{F}\mathbf{G}^T\;:\mathrm{A}$
	$\mathbf{F}^T\mathrm{A}^T\mathbf{G}$	$=$	$\mathbf{G}\mathbf{F}^T\;:\mathrm{A}$
	$(\nabla\mathbf{F}^T-\nabla\mathbf{F})\mathbf{G}$	$=$	$\operatorname{curl}\mathbf{F}\times\mathbf{G}=G_j(\partial_{x_j}F_i-\partial_{x_i}F_j)\;\mathbf{E}_i$
scalar field	$\dot{f}$	$=$	$\partial_t f+\mathbf{v}^T\nabla f=\partial_t f+v_k\,\partial_{x_k}f$
vector field	$\dot{\mathbf{F}}$	$=$	$\partial_t\mathbf{F}+\mathbf{v}^T\nabla\mathbf{F}=(\partial_t F_i+v_k\,\partial_{x_k}F_i)\;\mathbf{E}_i$
0-form	$\mathcal{L}_{\mathbf{v}}\,f$	$=$	$\dot{f}=\partial_t f+\mathbf{v}^T\nabla f$
1-form	$\mathcal{L}_{\mathbf{v}}\,\mathbf{h}$	$=$	$\dot{\mathbf{h}}+(\nabla\mathbf{v})\mathbf{h}$
2-form	$\mathcal{L}_{\mathbf{v}}\,\mathbf{b}$	$=$	$\dot{\mathbf{b}}-(\nabla\mathbf{v})^T\mathbf{b}+\mathbf{b}\,\operatorname{tr}\nabla\mathbf{v}$
3-form	$\mathcal{L}_{\mathbf{v}}\,\rho$	$=$	$\dot{\rho}+\rho\,\operatorname{tr}\nabla\mathbf{v}$

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