SUPPLEMENTARY MATERIAL 1

Effective medium theory

Here we demonstrate how to retrieve the expressions for the effective phase velocity and attenuation, respectively denoted as $v_H(\omega)$ and $\alpha_H(\omega)$, for a wave propagating across a 1D-periodic, viscoelastic medium under the long-wavelength, low-frequency approximation (recall the dashed blue curves in Fig. 3 of the manuscript). Let us first introduce the effective stiffness and effective mass density, respectively denoted as $c_H^*(\omega)$ and ρ_H, for a unit cell that consists of two viscoelastic layers in series. Following White and Angona[1], it can be shown that

$$c_H^*(\omega) = \frac{c_1^*(\omega)c_2^*(\omega)}{\phi c_1^*(\omega) + (1-\phi)c_2^*(\omega)} \ , \tag{1a}$$

$$\rho_H = (1-\phi)\rho_1 + \phi\rho_2 \ , \tag{1b}$$

where $c_m^*(\omega)$ and ρ_m, with $m = 1, 2$, are the complex stiffness and mass density of each layer, ϕ being the volume fraction of the layer 2.

The complex stiffness of each layer, $c_m^*(\omega)$, can further be written as

$$c_m^*(\omega) = \rho_m \left(v_m^*(\omega)\right)^2 = \rho_m \left(\frac{\omega}{k_m^*(\omega)}\right)^2 \ , \tag{2}$$

where $v_m^*(\omega)$ and $k_m^*(\omega)$ are the complex phase velocity and complex wave number associated with each layer m. Substituting Eq. (2) into Eq. (1a) yields

$$c_H^*(\omega) = \frac{\rho_1\rho_2\omega^2}{\phi\rho_1(k_2^*(\omega))^2 + (1-\phi)\rho_2(k_1^*(\omega))^2} \ . \tag{3}$$

Likewise, the effective wave number $k_H^*(\omega)$ can also be written as

$$k_H^*(\omega) = \frac{\omega}{v_H^*(\omega)} = \omega\sqrt{\frac{\rho_H}{c_H^*(\omega)}} \ . \tag{4}$$

Therefore, substituting Eqs. (1b) and (3) into Eq. (4) leads to an expression for the effective wave number as a function of the complex wave numbers of each layer m as

$$k_H^*(\omega) = \sqrt{\frac{\left(\phi\rho_1(k_2^*(\omega))^2 + (1-\phi)\rho_2(k_1^*(\omega))^2\right)\left((1-\phi)\rho_1 + \phi\rho_2\right)}{\rho_1\rho_2}} \ . \tag{5}$$

Without loss of generality, the complex wave number associated with each layer m can be written in the form

$$k_m^*(\omega) = \frac{\omega}{v_m(\omega)} - j\alpha_m(\omega) \ , \tag{6}$$

where $v_m(\omega)$ and $\alpha_m(\omega)$ are the phase velocity and attenuation associated with each layer m. Substituting Eq. (6) into Eq. (5) and reorganizing the terms into real and imaginary parts lead to

$$k_H^*(\omega) = \sqrt{I_1(\omega) - 2j\omega I_2(\omega)} \ , \tag{7}$$

where $I_1(\omega)$ and $I_2(\omega)$ are equal to

$$I_1(\omega) = \left(\phi\rho_1\left(\left(\frac{\omega}{v_2(\omega)}\right)^2 - \alpha_2^2(\omega)\right) + (1-\phi)\rho_2\left(\left(\frac{\omega}{v_1(\omega)}\right)^2 - \alpha_1^2(\omega)\right)\right) S \ , \tag{8a}$$

$$I_2(\omega) = \left(\phi\rho_1\left(\frac{\alpha_2(\omega)}{v_2(\omega)}\right) + (1-\phi)\rho_2\left(\frac{\alpha_1(\omega)}{v_1(\omega)}\right)\right) S \ , \tag{8b}$$

with the scalar S being equal to

$$S = \frac{(1-\phi)\rho_1 + \phi\rho_2}{\rho_1\rho_2} \ . \tag{9}$$

Similarly to Eq. (6), the effective wave number can be written as

$$k_H^*(\omega) = \frac{\omega}{v_H(\omega)} - j\alpha_H(\omega) = \sqrt{\left(\left(\frac{\omega}{v_H(\omega)}\right)^2 - \alpha_H^2(\omega)\right) - 2j\omega\left(\frac{\alpha_H(\omega)}{v_H(\omega)}\right)} \ , \tag{10}$$

where $v_H(\omega)$ and $\alpha_H(\omega)$ are the effective phase velocity and attenuation.

The mathematical trick here consists in performing an identification from the comparison between the terms below the square roots in Eqs. (7) and (10), which leads to a system of two equations with two unknown, *i.e.*, $v_H(\omega)$ and $\alpha_H(\omega)$, as

$$I_1(\omega) = \left(\frac{\omega}{v_H(\omega)}\right)^2 - \alpha_H^2(\omega) \ , \tag{11a}$$

$$I_2(\omega) = \frac{\alpha_H(\omega)}{v_H(\omega)} \ . \tag{11b}$$

Substituting Eq. (11b) into Eq. (11a) and reorganizing the terms yield a fourth-order characteristic polynomial in $v_H(\omega)$ as

$$v_H^4(\omega) + \frac{I_1(\omega)}{I_2^2(\omega)} v_H^2(\omega) - \frac{\omega^2}{I_2^2(\omega)} = 0 \ , \tag{12}$$

which has a biquadratic form and can therefore easily be solved analytically. The solutions for $\alpha_H(\omega)$ can then be straightforwardly derived from Eq. (11b). It should be noted that these solutions could alternatively be numerically obtained from Eq. (7), as $v_H(\omega) = \omega/\Re(k_H^*(\omega))$ and $\alpha_H(\omega) = -\Im(k_H^*(\omega))$.

Numerical example

To serve as an example, the solutions for $v_H(\omega)$ and $\alpha_H(\omega)$ are calculated based on the ultrasound characteristics of the two constituent materials m provided in Fig. 1 of the manuscript, which serve as input data to the effective viscoelastic model. In such a case, $v_H(\omega)$ admits two real (one positive and the other negative) and two imaginary solutions, therefore only the real and positive solution is physically relevant. Likewise, only the corresponding solution $\alpha_H(\omega)$ is retained. The recovered ultrasound characteristics, along with those of the constituent materials, are depicted in Fig. 1.

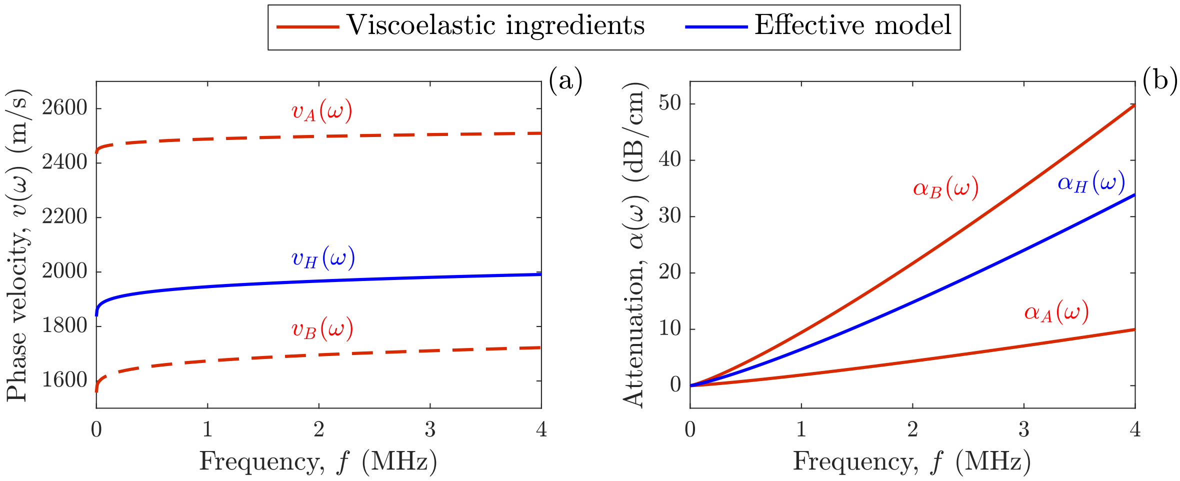

FIG. 1. Effective ultrasound characteristics for the viscoelastic case (continuous blue lines): (a) Phase velocity $v_H(\omega)$ and (b) Attenuation $\alpha_H(\omega)$. The dashed red lines display the viscoelastic ingredients that serve as input data to the effective model.

[1] J. White and F. Angona, J. Acoust. Soc. Am. **27**, 310 (1955).