The Impact of Risk Mismatch on Personal Portfolio Performance

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Abstract

Within the Modern Portfolio Theory framework, personal portfolio choice is driven by the investor's risk aversion. In practice, this criterion is usually replaced by a target volatility level, potentially leading to similar allocation choices. Reconciling these two approaches leads to the design of a performance measure that explicitly allows us to isolate a penalty for the mismatch between the actual and targeted portfolio risks. This penalty is particularly strong for defensive investors and when the market risk premium is high. We also show that the target volatility criterion leads to inadequate portfolio choices when market conditions change or when the investor is confronted with a well-performing active portfolio. We extend this approach with a second dimension that represents the investor's attitude towards asymmetry (skewness) risk. The resulting performance measurement framework involves a penalty for risk unsuitability that can be substantially aggravated, especially for investors who simultaneously exhibit a strong aversion to volatility and skewness risks.

Keywords: personal portfolio management, volatility-managed portfolio, volatility targeting, investor-specific portfolio performance, investor risk appetite.

1. Introduction

Volatility-managed (or "volatility-targeting") portfolios have been extensively analyzed since the publication of the Moreira and Muir (2017) study of their empirical risk-return properties. The many papers investigating this portfolio management approach, which consists of dynamically controlling its volatility around a predefined target level, have largely focused on the "how" (can the objective be achieved) and the "what" (do they bring compared with alternative systems).[1] Taking one step back, we find it useful to address the "why" of this asset management approach, i.e., the economic perspective that justifies the adoption of a sticky volatility budget as a legitimate core mechanism to fulfill the objectives of the investor.

From an institutional investment perspective, a justification is given by the general asset management principle of many lifetime pension funds, as explained by Bégin and Sanders (2024). Considering the structure of their liabilities and the mortality tables of their beneficiaries, a common asset management perspective is to set up a risk appetite framework in which a target risk budget is specified, usually through a maximum volatility, value-at-risk or conditional value-at-risk threshold, depending on the adopted sophistication level. The asset allocation should be piloted such as to approach, but not exceed the risk budget.

For individual investors, the problem is most conveniently tackled within the scope of the Modern Portfolio Theory (MPT) framework. The derivation of the single period Capital Asset Pricing Model (CAPM) hinges on the identification of the market portfolio and its association with the risk-free asset. The asset allocation decision results from expected utility maximization and leads to Tobin's two-fund separation theorem. The investor's characteristics are summarized by a single indicator: the risk aversion coefficient. This is precisely where the notion of "volatility budget" arises. Following the intuition of Fama's (1972) performance decomposition framework, François and Hübner (2024) show the analogy between expected utility maximization and risk-constrained expected return optimization. The volatility budget constraint set by the investor plays the same role as her risk aversion coefficient. It summarizes, through a synthetic value, the investor's risk profile. As suggested by Elkamhi, Lee and Salerno (2023), this approach justifies the relevance of volatility-managed portfolios. Moving away from the volatility target either exposes the investor to an unacceptable level if portfolio risk is too high, or insufficiently consumes her risk appetite if it is too low. Our first intended contribution is to isolate and examine the impact of the difference between the actual and target portfolio risk on its performance.

Under the pragmatic assumption that most individual investors do not have straightforward access to homemade leverage, the performance of an actively managed portfolio must consider their risk appetite as a hard constraint. The Sharpe ratio, which is agnostic to the portfolio volatility level, is thus not adequate in this context. The relevant performance measure associated with a volatility-managed portfolio is the target volatility alpha. It corresponds to the gain in satisfaction perceived by the investor by departing from the optimal passive allocation to adopt the portfolio under review. Out of this performance measure, we derive the "volatility mismatch penalty" that purely measures the cost for the investor, in percentage terms, of missing the risk target. Isolating this penalty from the rest of portfolio performance enables us to study its practical implications. The penalty is linearly related to the market Sharpe ratio, inversely to the target volatility level, and quadratically to the difference between actual and target volatilities. Furthermore, the impact of volatility mismatch appears to be more serious for defensive investors.

Replacing the theoretical risk aversion coefficient with a volatility budget can be viewed as a convenient shortcut. This comes however with a cost, because the connection between both investor-specific parameters is only valid under constant financial conditions, reflected in the Sharpe ratio of the market index. When the *ex post* realized risk premium differs from the expected one, we show that managing a portfolio according to a constant volatility target destroys value for the investor. The allocation becomes too risky when the market risk premium is low, and too defensive when risk is generously rewarded. We also prove that a well-performing active volatility-managed portfolio will always be too defensive according to the true investor's risk appetite derived from her risk profile. A potential solution is to set the target as the *ex post* realized benchmark volatility. This would be particularly detrimental to portfolios with higher risk, whose risk mismatch penalty quickly becomes very negative.

It is tempting to argue that the use of volatility as a risk measure is overly simplistic. In the last part of the paper, we address this potential criticism by introducing the investor's preference for skewness in a parsimonious fashion. The key to the reasoning is (i) to adopt the same isomorphic approach between the investor's utility function and an associated risk budget, while simultaneously (ii) adjusting the portfolio volatility the reflect the impact of its skewness with a synthetic parameter. We show how to implement this approach in practice without undue sophistication. The resulting performance measure has the same interpretation as the original one, but the impact of the risk mismatch can become dramatic for defensive investors with a strong distaste for extreme risks.

2. The penalty for volatility mismatch in the mean-variance framework

In the Modern Portfolio Theory framework, total portfolio risk is represented through the volatility of its returns. When this type of risk is used to appraise portfolio performance, the classical performance measure reported by most portfolio managers is the Sharpe (1966) ratio. Its application hinges on the assumption of a seamless availability of homemade leverage, which might not be the case for most individual (especially retail) investors. This is the reason why performance measures that explicitly take the risk profile of the investor have been developed. Along this line, we develop the analogy between the use of risk aversion, on one side, and the adoption of a volatility target, on the other side, as criteria for portfolio choice. This enables us to isolate the risk mismatch component in performance, and to analyze its implications for the personal portfolio management process.

Theoretical preference-adjusted performance with risk aversion

In the original single period mean-variance framework (from initial investment date $t = 0$ to the final payout date $t = T$), investors are assumed to be risk averse and only consider the standard deviation (aka volatility) of returns as the relevant risk measure for financial assets. As shown by Markowitz (1959), they rationally try to maximize their expected utility of the form:

$$\mathrm{E}\big(U_j(R_P)\big|\Omega_0\big) = \mu_P - \tfrac{1}{2}\gamma_j\sigma_P^2 \tag{1}$$

where $U_j(\cdot)$ is the utility function of investor j, in which $\mathrm{E}(\cdot\,|\Omega_0)$ represents the expectation of the argument conditional on all initially available information about future outcomes (represented by symbol Ω_0). On the right-hand side, γ_j stands for this person's risk-aversion coefficient, whereas $\mu_P \equiv \mathrm{E}(R_P|\Omega_0)$ and $\sigma_P^2 \equiv \mathrm{E}\left(\left(R_P - \mathrm{E}(R_P)\right)^2\middle|\Omega_0\right)$ represent the expectation and variance of the portfolio returns, respectively.

Within the context of the MPT, the investor achieves her objective by choosing the fraction of her wealth to be invested in the market portfolio (considered here as the benchmark B) solving $\max\limits_{w} \mathrm{E}\big(U_j(wR_B + (1-w)R_f)\big|\Omega_0\big) = w\mu_B + (1-w)R_f - \tfrac{1}{2}\gamma_j w^2\sigma_B^2$, where μ_B and σ_B^2 respectively represent the expectation and variance of the market portfolio returns, and R_f is the rate of return of the riskless asset, which is considered constant over the period.

Applying Tobin's two-fund separation theorem, the optimal weight is $w^* = \frac{\sigma_{B^*}}{\sigma_B}$, where B^* is the optimal allocation portfolio, and $\mu_{B^*} = w^*\mu_B + (1 - w^*)R_f$. The straightforward application of the first-order condition yields $w^* = \frac{\mu_B - R_f}{\gamma_j \sigma_B^2} = \frac{1}{\gamma_j \sigma_B}\text{RVR}_B$ where $\text{RVR}_B \equiv \frac{\mu_B - R_f}{\sigma_B}$ represents the market's expected Reward to Variability Ratio. Thus, both the identification of the best possible portfolio B^* and the optimal allocation w^* depend on RVR_B. The investor's expected utility associated with this portfolio is $\text{E}\left(U_j(R_{B^*})\middle|\Omega_0\right) \equiv U_j^* = R_f + \frac{1}{2\gamma_j}\text{RVR}_B^2$ and its associated expected return is simply equal to:

$$\mu_{B^*} = R_f + \frac{1}{\gamma_j}\text{RVR}_B^2 \tag{2}$$

In the world of realized returns (i.e., at the final date $t = T$), we observe the estimates of the moments of the returns distribution of a portfolio through its sample mean m and standard deviation s. The market's RVR is concretized with its Sharpe ratio, denoted $\text{SR}_B = \frac{m_B - R_f}{s_B}$. In this context, appraising the *ex post* performance of a portfolio P over a period of reference involves measuring its utility surplus, denoted $m_P - \frac{1}{2}\gamma_j s_P^2$, over the one of B^*. The **quadratic alpha**, or "Q-alpha" is given by:

$$\text{Q}\alpha_P^j = m_P - R_f - \frac{1}{2}\gamma_j s_P^2 - \frac{1}{2\gamma_j}\text{SR}_B^2 \tag{3}$$

This performance measure features two distinct parts. The first one, represented by $m_P - R_f - \frac{1}{2}\gamma_j s_P^2$, is the certainty equivalent of the portfolio in excess of the risk-free rate. The last negative term $-\frac{1}{2\gamma_j}\text{SR}_B^2$ reflects the additional loss of satisfaction incurred by the investor related to the fact that her optimal portfolio choice is B^* instead of the risk-free asset.

Practical preference-adjusted performance with a volatility budget

The Q-alpha adequately appraises the gain in expected utility obtained by an investor whose risk aversion coefficient γ_j is known to the portfolio manager. However, it is not standard practice to approach the investor's risk profile by estimating her risk aversion coefficient. There are both theoretical and pragmatic reasons for this.

The theoretical underpinnings of the use of a quadratic utility function as implied by equation (1) have been challenged for long. Firstly, as Kahneman and Tversky (1979) experimentally show, most individuals depart from the classical expression of preferences over risk and exhibit distinct attitudes regarding gains or losses, which can be largely explained by behavioral biases in decision making. Second, even if one remains within the scope of the rational expected utility maximization, the departure from normality of financial returns and the investors' preferences for higher moments call for a more sophisticated approach than the use of a penalty for volatility risk. This issue is discussed in detail later in the paper.

Accepting the assumptions of the MPT, and thus the use of γ_j as the risk aversion coefficient, also creates some very practical issues. Since Allais (1953) emphasized, through his famous paradox, how the framing of choices under uncertainty may bias preferences, many other problems have been identified with the estimation of the risk aversion parameter, such as the distortion of objective probabilities also included in Kahneman and Tversky's Prospect Theory (see Machina, 1987, for a review). In short, the way a profiling questionnaire is designed may substantially affect the diagnosis regarding the investor's attitudes toward risk. Furthermore, the notion of "risk aversion coefficient" is not sufficiently intuitive to appeal to most customers and may hinder the effectiveness of the commercial message.

To assess an individual's risk tolerance, many financial advisors would favor a more direct and intuitive approach, aiming to capture how much risk the investor is able to bear in specific circumstances. As a simple illustration, we consider a standard method used by many practitioners (re-exploited further in the context of extreme risks). The investor is asked about the worst return that she could afford with a medium frequency. This figure corresponds to the absolute Value-at-Risk (aVaR) limit of the riskiest acceptable portfolio B^j with a probability of x, i.e., $\Pr\left[R_{B^j} \leq \text{aVaR}^{(j)}\right] = x$.[2] To retrieve the volatility budget $\sigma^{(j)}$ corresponding to this portfolio B^j, the advisor posits a Gaussian world in which $\text{aVaR}^{(j)} = \mu_{B^j} + z_x\sigma^{(j)}$, where $z_x < 0$ stands for the x^{th} percentile of the standard Normal distribution. Assuming a set of stylized financial conditions with known values of a risk-free rate R_f and the market Reward-to-Variability Ratio RVR_B, the expected portfolio return stands as $\mu_{B^j} = R_f + \sigma^{(j)}\text{RVR}_B$. Consequently, the volatility budget solves the simple equation $\sigma^{(j)} = \frac{\text{aVaR}^{(j)} - R_f}{\text{RVR}_B + z_x}$. Thus, obtaining information on $\text{aVaR}^{(j)}$ is equivalent to retrieving the volatility budget. For instance, if the investor is asked the maximum loss that she can stand every four out of five years ($x = \frac{1}{5} = 20\%$, thus $z_x = -0.842$), and the answer is -2% ($\text{aVaR}^{(j)}$), assuming a risk-free rate of $R_f = 2\%$ and a market Reward-to-Variability Ratio equal to $\text{RVR}_B = 0.4$ leads to $\sigma^{(j)} = \frac{-2\% - 2\%}{0.4 - 0.842} = 9\%$.

According to this approach, the determination of the optimal portfolio results from a constrained maximization program that replaces the expected utility maximization of equation (1):

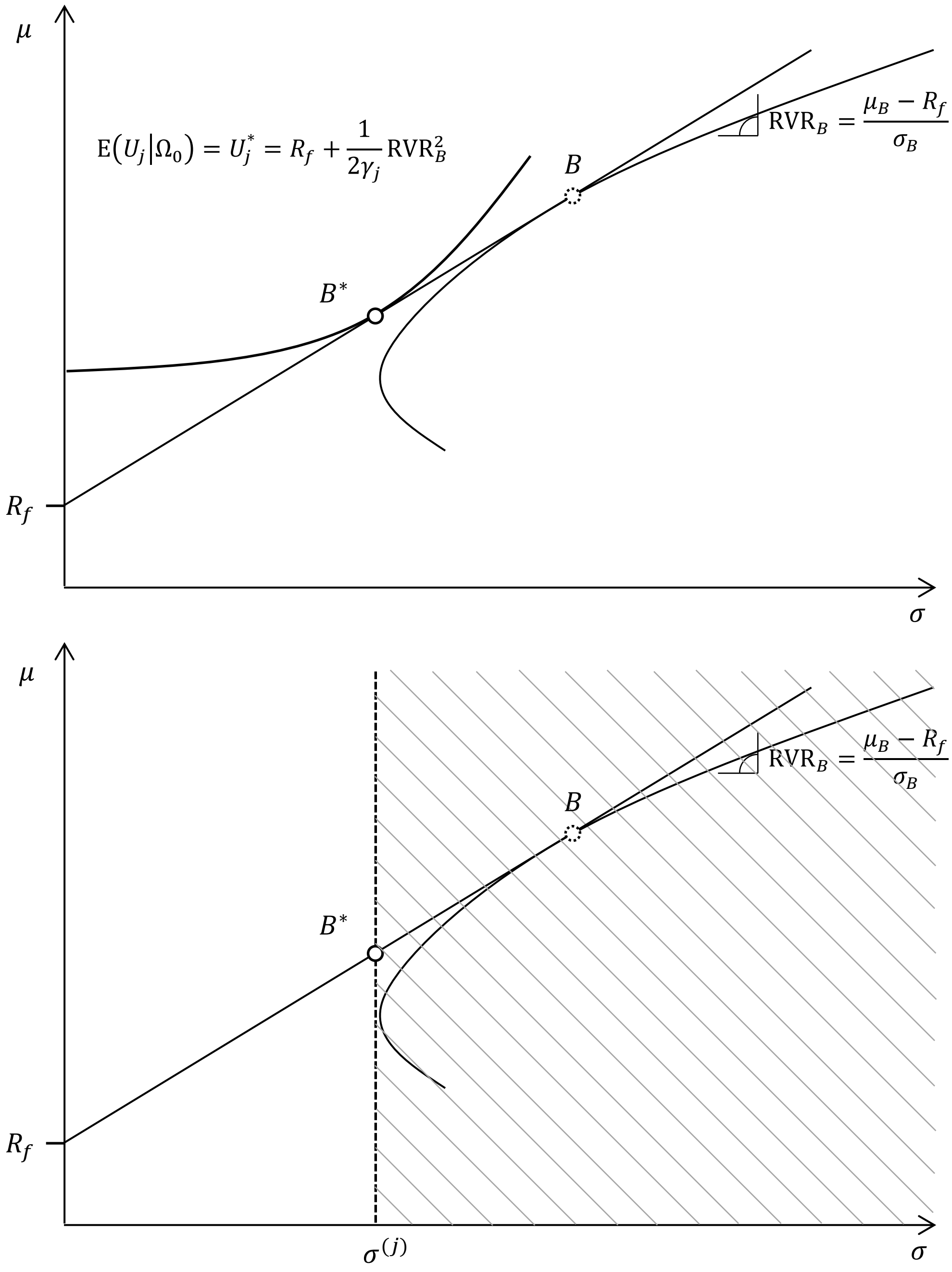
$$\max_{w} w\mu_B + (1 - w)R_f \tag{4a}$$

$$\text{subject to: } w\sigma_B \leq \sigma^{(j)} \tag{4b}$$

The investor-specific information is summarized by the maximum volatility budget $\sigma^{(j)}$, which is supposed to be a substitute to the risk aversion coefficient γ_j.

There are two possibilities regarding the global optimum of this program. It is either an interior solution – in which case the constraint is not binding – or it is a corner solution – in which case the constraint is binding. But since, from an *ex ante* perspective, the expected return of the portfolio is an increasing function of its volatility, the only way to maximize it is to set $\sigma_{B^*} = w^*\sigma_B = \sigma^{(j)}$ and adopt the corner solution. This means that $\sigma^{(j)}$ represents the investor's desired portfolio volatility, playing the exact same role as the risk aversion coefficient in the characterization of her preferences. This analogy is represented in Figure 1.

Figure 1. Graphical equivalence of the expected utility and constrained expected return maximization programs



These twin figures suggest that both routes – expected utility and constrained expected return maximization – can eventually lead to the same optimal portfolio. Setting $\sigma_{B^*} = \sigma^{(j)}$ involves that:

$$\mu_{B^*} = \frac{\sigma^{(j)}}{\sigma_B}\mu_B + \left(1 - \frac{\sigma^{(j)}}{\sigma_B}\right)R_f = R_f + \sigma^{(j)}\mathrm{RVR}_B \quad (5)$$

Matching equations (2) and (5) induces that, for the same optimal portfolio, the link between the investor's risk aversion and volatility budget can be stated as:

$$\sigma^{(j)} = \frac{\mathrm{RVR}_B}{\gamma_j} \tag{6}$$

Under specific market conditions, reflected in the RVR of the market portfolio, the investor's risk aversion and volatility budget contain the same information. This also means that, considering the investor's risk aversion coefficient as a time-invariant intrinsic characteristic, the volatility budget is proportional to the market RVR. If the market risk premium varies over time, so does $\sigma^{(j)}$ too.

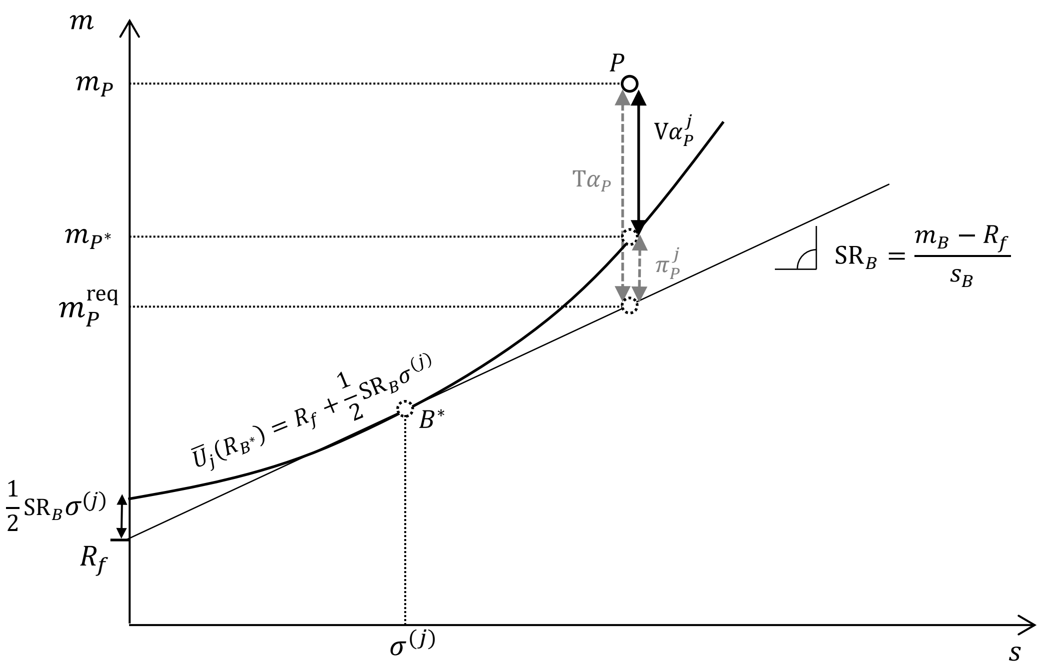
In the *ex post* world, where portfolio P is characterized by the realized risk-return pair (s_P, m_P), François and Hübner (2024) show that realized performance is adequately assessed through the **target volatility alpha** or "V-alpha", denoted $\mathrm{V}\alpha_P^j$. This measure is defined as the difference between the mean portfolio return m_P and the one, denoted m_{P^*}, of another portfolio that lies on the same indifference curve as B^*, but with the same realized volatility as P. To obtain m_{P^*}, note that the average utility obtained by investing in B^* is $\bar{U}_j(R_{B^*}) = R_f + \frac{1}{2\gamma_j}\mathrm{SR}_B^2 = R_f + \frac{1}{2}\mathrm{SR}_B\sigma^{(j)}$, where $\mathrm{SR}_B = \frac{m_B - R_f}{s_B}$ is the Sharpe ratio of the market portfolio. Since, in the mean-variance world, the expected utility is the same as the certainty equivalent, this is also equal to the intercept of the indifference curve. Noting that the generic equation of the investor's indifference curve is $m_x = k + \frac{1}{2}\gamma_j s_x^2$, substituting k with $\bar{U}_j(R_{B^*})$ and s_x with s_P, we get $m_{P^*} = R_f + \frac{1}{2}\mathrm{SR}_B\left(\sigma^{(j)} + \frac{s_P^2}{\sigma^{(j)}}\right)$. The last term represents the investor's required risk premium to accept the same risk as portfolio P.

To understand the influence of the risk mismatch ($\sigma^{(j)} \neq s_P$) on realized portfolio performance, it is necessary to revert back to Fama's (1972) decomposition framework, and express the mean portfolio return as the sum of three premiums $m_P = \sigma^{(j)}\mathrm{SR}_B + \left(s_P - \sigma^{(j)}\right)\mathrm{SR}_B + (m_P - s_P\mathrm{SR}_B)$. Applying it in the current context, this leads to a simple and interpretable expression for the V-alpha:

$$\mathrm{V}\alpha_P^j = m_P - \left(R_f + s_P\mathrm{SR}_B\right) - \mathrm{SR}_B\frac{\left(s_P - \sigma^{(j)}\right)^2}{2\sigma^{(j)}} = \mathrm{T}\alpha_P - \pi_P^j \tag{7}$$

where $\mathrm{T}\alpha_P = m_P - \left(R_f + s_P\mathrm{SR}_B\right) = m_P - m_P^{\mathrm{req}}$ is the total risk alpha ("T-alpha") in which m_P^{req} is the *ex post* required return on the portfolio, and $\pi_P^j = \mathrm{SR}_B\frac{\left(s_P - \sigma^{(j)}\right)^2}{2\sigma^{(j)}}$ is the **volatility mismatch penalty**. If $s_P = \sigma^{(j)}$, i.e., if the portfolio manager has successfully controlled the risk exposure according to the investor's desired level, then $\pi_P^j = 0$. Otherwise, provided that $\mathrm{SR}_B > 0$, this expression is always positive. This is illustrated in the following figure.

Figure 2. Representation of the target volatility alpha



The components of the V-alpha are represented with grayed arrows on Figure 2. $\mathrm{T}\alpha_P$, which is not investor-specific, represents the performance component due to the abnormal return brought by the manager (on the basis of total risk). It is subtracted with π_P^j, which reflects the sanction for the mismatch between the achieved volatility level s_P and its desired one $\sigma^{(j)}$. This penalty features three components:

(i) The reward for market risk, materialized by its Sharpe ratio SR_B, has a positive and linear influence on the volatility mismatch penalty. Thus, as market conditions become more favorable, the departure from the investor's volatility budget induces a larger penalty, regardless of the direction of this departure. This phenomenon occurs because risk is better remunerated at times of rising markets, which aggravates the consequence of a risk mismatch in the eyes of the investor.

(ii) The realized volatility mismatch, represented by the absolute difference between the realized and budgeted standard deviation of portfolio returns $\left|s_P - \sigma^{(j)}\right|$, has a positive and quadratic influence on the volatility mismatch penalty. The unsatisfaction perceived by the investor increases more than proportionally with the distance from the target. This effect is also symmetric: the penalty does not depend on whether the portfolio has been managed too defensively ($s_P < \sigma^{(j)}$) or aggressively ($s_P > \sigma^{(j)}$).

(iii) The level of the investor's risk budget $\sigma^{(j)}$ has a decreasing and inverse influence on the volatility mismatch penalty. This means that, for the same volatility mismatch, the disutility felt by a defensive investor will be more pronounced than for a more aggressive one. This intuitive phenomenon results from the fact that more risk-averse investors put a greater emphasis on their risk exposures, and thus feel a greater discomfort from the breach of their volatility budget.

We illustrate the influence of these three components on portfolio performance in Figure 3.

Figure 3. Sensitivity of the volatility mismatch penalty to the portfolio volatility

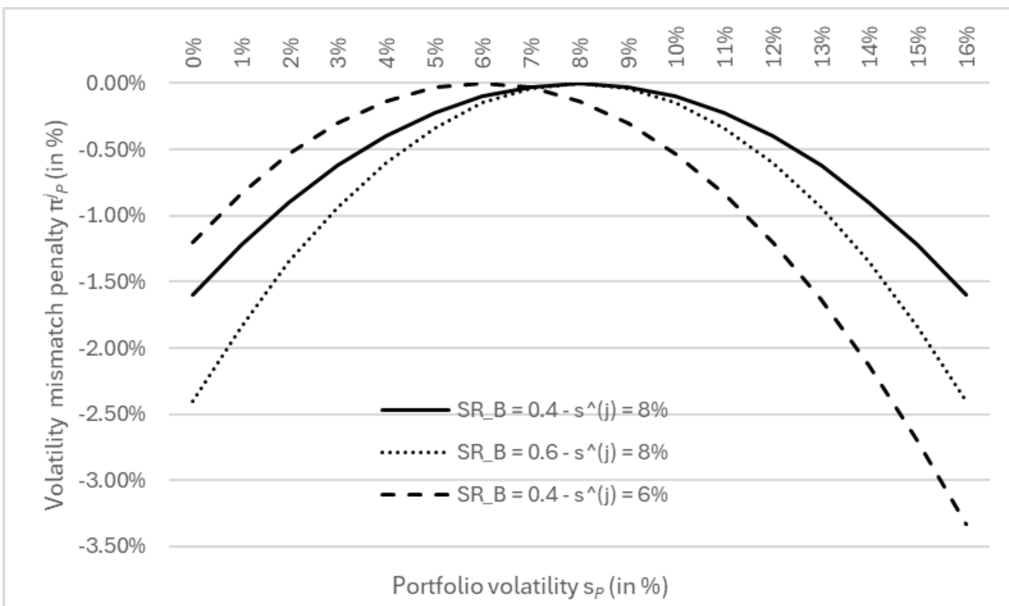


Figure 3 shows the evolution of the volatility mismatch penalty according to the realized volatility of portfolio returns. It has an inverted U-shape as this function is quadratic. Compared to a base case where $\text{SR}_B = 0.4$ and $\sigma^{(j)} = 8\%$ (solid line), the situation of a better market reward with $\text{SR}_B = 0.6$ (thin dashed line) reinforces the curvature of the penalty function, making the risk mismatch comparatively more utility-destroying. On the other hand, considering a more defensive investor with $\sigma^{(j)} = 6\%$ (thick dashed line) shifts the penalty function to the left and increases the curvature, but also introduces an asymmetry between the impact of lower versus higher risk, the latter being much more penalizing for the investor. The loss in utility can become very material: missing the target volatility by 4% (10% instead of 6%) has a "cost" of $\pi_P^j = 0.53\%$, but when the risk increases to 12%, the penalty becomes $\pi_P^j = 1.20\%$, thereby ruining the efforts of an active portfolio manager trying to outperform the market.

For the sake of the illustration, we consider the MSCI ACWI Index as the proxy for the market portfolio, that we denote W. The relevant summary statistics for the monthly net returns over the 10-year window spanning the period July 2014 – June 2024 are, in annualized terms, $m_W = 9.23\%$ and $s_W = 14.83\%$. Considering a holding period of 10 years, the risk-free rate is defined as the 10-year US Treasury bond yield at the starting date (July 1, 2014) and is set at $R_f = 2.54\%$. This induces a realized world index Sharpe ratio equal to $\text{SR}_W = 0.45$, broadly in line with long term averages as documented by Jordà, Knoll, Kuvshinov, Schularick, and Taylor (2019).

We aim at investigating how the distance between the actual portfolio volatility and the one desired by the investor influences performance. To this end, we define three investors' profiles based on their risk aversion coefficients. They are termed "Defensive" (d) with $\gamma_d = 12$, "Balanced" (b) with $\gamma_b = 6$ and "Aggressive" (a) with $\gamma_a = 4$. According to equation (6), these profiles translate into $\sigma^{(d)} = \frac{0.45}{12} = 3.76\%$, $\sigma^{(b)} = 7.51\%$ and $\sigma^{(a)} = 11.72\%$.

For each investor type, the optimal portfolio mixes the index and the risk-free asset so as to exactly reach the desired volatility level. We call these optimized portfolios "Low" (L), "Neutral" (N) and "High" (H). Then, the Volatility mismatch penalty is computed for each pair investor (d, b and a) – portfolio (L, N and H). The results are reported in the following table:

Table 1. Volatility mismatch penalties for three investor types (Defensive, Balanced, Aggressive) and three portfolio risk levels (Low, Neutral, High) with the world market index

				Investor		
				Defensive	***Balanced***	***Aggressive***
			Mean	4.23%	5.93%	7.62%
		Mean	Volatility	3.76%	7.51%	11.27%
Portfolio	***Low***	4.23%	3.76%	**0.00%**	**-0.42%**	**-1.13%**
	Neutral	5.93%	7.51%	**-0.85%**	**0.00%**	**-0.28%**
	High	7.62%	11.27%	**-3.39%**	**-0.42%**	**0.00%**

Notes: The volatility mismatch penalty estimates (last three columns, in bold) are based on 120 monthly returns of the MSCI ACWI Index from 2014/07/01 to 2024/06/30. The risk aversion coefficients for the Defensive, Balanced, and Aggressive investors are set to $\gamma_d = 12$, $\gamma_b = 6$ and $\gamma_a = 4$, respectively.

The magnitude of performance destruction due to volatility misalignment between the target (fourth row) and the actual one (third column) is reported in bold. The main diagonal features zeroes as its cells correspond to a perfect match between the target and observed volatilities. The underperformance remains below 1 percentage point when the mismatch is limited (adjacent profiles), but it can become very material, especially for investors with a defensive profile who are provided with a high risk portfolio. In the Defensive/High case (lower left cell), the magnitude of the loss in utility perceived by the investor (3.39%) almost halves the mean realized portfolio return (7.62%). Thus, achieving a high return is far from being sufficient to satisfy the investor if this performance is realized at the expense of a substantial increase in volatility compared to the customer's appetite for risk.

3. Implications of volatility budgeting for performance maximization

Regarding the investor's suitability assessment method, the key difference between the risk aversion and the volatility budget approaches resides in equation (6). Whereas the risk aversion coefficient γ_j is supposed to be a pure intrinsic trait of the investor and time-invariant, the volatility budget $\sigma^{(j)}$ varies over time according to market conditions, represented by the RVR of the market portfolio. This feature has important implications regarding how the portfolio manager will attempt to fulfill her portfolio management mandate.

To grasp the intuition, consider the *ex ante* determination of the investor's risk appetite through her risk budget $\sigma^{(j)}$, based on expectations of returns. Because it is part of the Investment Policy Statement, this parameter drives the contractual arrangements regarding the portfolio's risk and return objectives. In other words, until the next contract renegotiation, the portfolio manager is tied with this budget constraint, even though the evolution of market conditions (through the Sharpe ratio of the market portfolio) would presumably call for an interim change in $\sigma^{(j)}$. If the contract adaptation does not take place, the *ex post* optimal leverage (i.e., the one that achieves the best possible performance with the realized portfolio returns) will likely differ from the one initially expected.

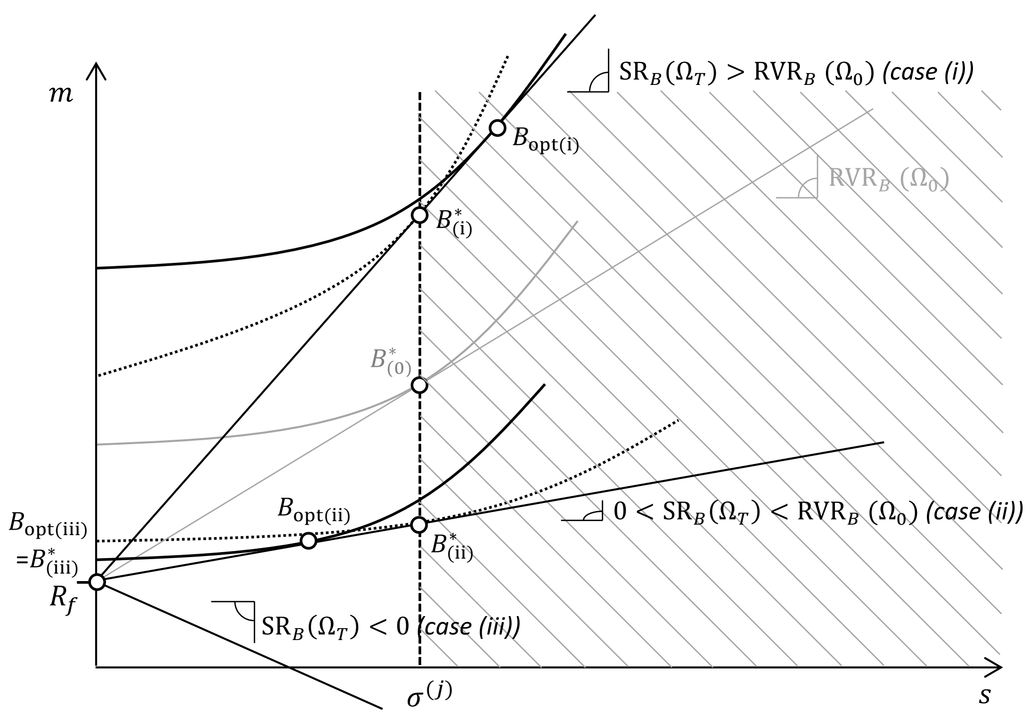
In what follows, we examine the implications of the stickiness of $\sigma^{(j)}$ in three situations: (i) The case of a passive allocation (i.e., only using the market portfolio and the risk-free asset) with constant risk budget; (ii) The case of an active allocation (i.e., selecting a portfolio that achieves a different risk-return tradeoff from the one of the market) with constant risk budget; and (iii) The case of an active allocation with a risk budget equal to the realized benchmark volatility.

Optimal performance for a passive allocation

Consider the situation of a portfolio manager who would have perfect market foresight ability. This (lucky) person would know, from the inception of the portfolio management process, what the market would deliver in terms of risk and return during the whole investment period ending at time T. The information set of this theoretical "crystal ball" manager, that we denote Ω_T, amounts to knowing in advance what the mean and variance of the market portfolio will be, i.e., $m_B \equiv \mathrm{E}(R_B|\Omega_T)$ and $s_B^2 \equiv \mathrm{E}\left(\left(R_B - \mathrm{E}(R_B)\right)^2 \middle| \Omega_T\right)$. Provided that these values differ from the uninformed ones $\mu_B \equiv \mathrm{E}(R_B|\Omega_0)$ and $\sigma_B^2 \equiv \mathrm{E}\left(\left(R_B - \mathrm{E}(R_B)\right)^2 \middle| \Omega_0\right)$, the realized market Sharpe ratio SR_B, known with information set Ω_T, is likely to differ from the anticipated market's expected Reward to Variability Ratio RVR_B, known under Ω_0.

We distinguish three cases that might occur in reality: (i) $\mathrm{SR}_B > \mathrm{RVR}_B$, (ii) $0 < \mathrm{SR}_B < \mathrm{RVR}_B$, and (iii) $\mathrm{SR}_B < 0$. For each one, we identify what would be the optimal attitude of a portfolio manager endowed with information set Ω_T but restricted to apply the constrained optimization program of equations (4a) and (4b) instead of the utility maximization equation (1). The cases are represented in the following figure, that combines information about expected utility and target volatility illustrated in Figure 1.

Figure 4. Optimal passive portfolio allocations when the market Sharpe ratio is (i) better than initially expected, (ii) worse than initially expected, and (iii) negative

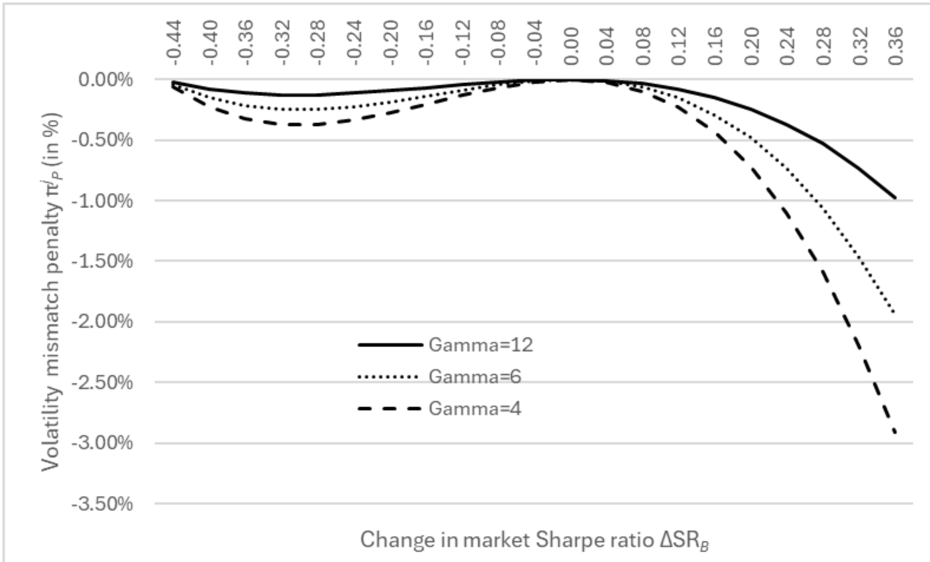


The first two cases, which correspond to a positive market excess return, underline the shortcomings of the volatility budgeting criterion regarding *ex post* portfolio optimization. In the absence of any superior active portfolio, the only hope of the portfolio manager is to deliver a performance in line with the volatility target so as not to disappoint the investor. According to this principle, portfolios $B^*_{(\mathrm{i})}$ (in the case of a high Sharpe ratio) and $B^*_{(\mathrm{ii})}$ (for a low Sharpe ratio) align vertically with the *ex ante* optimal portfolio $B^*_{(0)}$. Their performance, according to the target volatility alpha formula of equation (7), is nil: neither do they deliver any abnormal return, nor are they penalized for volatility mismatch. Nevertheless, keeping the same risk budget *ex post* and *ex ante* induces an implied change in the investor's risk aversion through the relationship $\gamma_j = \frac{\mathrm{SR}_B}{\sigma^{(j)}}$. In case (i), the investor's risk aversion increases (i.e., the investor is supposed to be more risk averse) despite the fact that the market risk premium is more favorable than anticipated. Similarly, in case (ii) the risk aversion coefficient decreases, thus the investor becomes less risk averse, although the market risk premium falls. These counterintuitive outcomes are reflected in the slope of the tangent indifference curves at $B^*_{(\mathrm{i})}$ and $B^*_{(\mathrm{ii})}$, which are respectively steeper and flatter than the initial one at $B^*_{(0)}$. If the portfolio manager had sticked to maximizing the investor's expected utility with the initial risk aversion parameter, she would have reached a more aggressive portfolio in case (i), namely $B_{\mathrm{opt(i)}}$, but it would violate the risk budget constraint and therefore be unsuitable. In case (ii) the optimal portfolio $B_{\mathrm{opt(ii)}}$ would be more defensive and would not bind the budget constraint. Furthermore, both portfolios would deliver a negative performance $\mathrm{V}\alpha^j_P = -\pi^j_P < 0$. Thus, whenever the market Sharpe ratio differs from the expected RVR, a perfectly informed manager has an incentive to propose a sub-optimal asset allocation and destroy value for the investor to maximize her performance.

The third case is more reassuring. When the market Sharpe ratio is negative, which correspond to the decreasing line on the graph, both the expected utility maximization and the constrained optimization with information set Ω_T lead to the same conclusion. The highest indifference curve as well as the highest expected rate of return point to the risk-free rate. The volatility budget constraint is not binding but the investor is satisfied with the avoidance of risk because the associated premium is negative. Thus, the manager's incentive to protect the investor against losses is preserved under the volatility budget approach.

The consequences of the first two cases on the loss in utility for the investor are illustrated in Figure 5.

Figure 5. Sensitivity of the loss in utility to market conditions



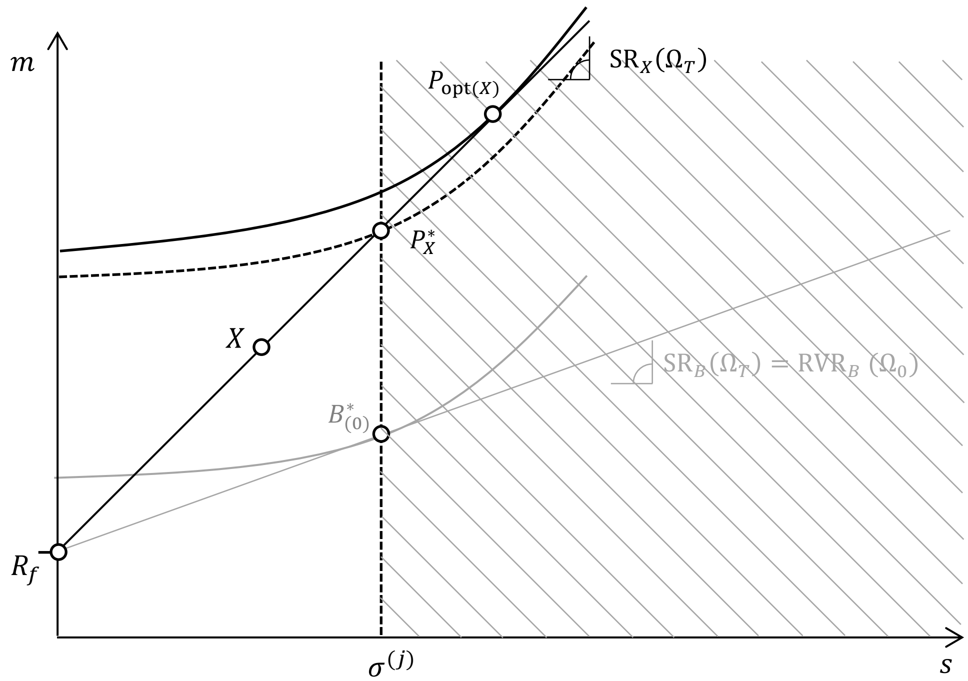
We start with the initial market conditions $R_f = 2.54\%$. and $\mathrm{SR}_B = 0.45$. The function represents the loss of utility experienced by an investor with a constant risk aversion coefficient when the benchmark volatility is kept constant ($\sigma^{(j)} = 3.76\%$ for $\gamma_j = 12$, $\sigma^{(j)} = 7.51\%$ for $\gamma_j = 6$ and $\sigma^{(j)} = 11.27\%$ for $\gamma_j = 12$). When the market Sharpe ratio rises, the penalty inflates in a quadratic fashion and becomes more severe as the investor is increasingly risk averse, as expected. The phenomenon is less pronounced, but more complex as the market Sharpe ratio falls. This is due to the conflicting effects of the pure risk mismatch and the smaller risk-return tradeoff. Eventually, there is no penalty anymore when $\mathrm{SR}_B = 0$ as the optimal portfolio becomes the risk-free asset.

The takeaway that can be drawn from this graph is that replacing expected utility maximization by target volatility portfolio management is a real issue only when the market is more rewarding than initially expected. This corresponds to case (i) represented in Figure 4.

Optimal performance for an active allocation

The second interesting scenario involves the identification of an outperforming active portfolio X. Again, considering that the portfolio manager has a perfect information set Ω_T, this implies that $\mathrm{SR}_X > \mathrm{SR}_B$. The role of the manager is not only to identify this superior portfolio, but also to allocate it with the risk-free asset so as to maximize the investor's satisfaction through the highest performance level. The situation is relatively similar to case (i) above. For simplicity, we assume that $\mathrm{SR}_B = \mathrm{RVR}_B$, allowing us to focus on information about the active portfolio. The situation is illustrated in Figure 5.

Figure 5. Optimal active portfolio allocation with a superior Sharpe ratio



Because it can be freely leveraged with the riskless asset, the initial volatility level of portfolio X is irrelevant. On the chart, we have chosen $s_X < \sigma^{(j)}$. The manager uses information embedded in the volatility budget $\sigma^{(j)}$ to infer the risk aversion coefficient γ_j and therefore the shape of the indifference curves, which are all parallel. The optimal portfolio leverage should be chosen so as to maximize the value of the target volatility alpha. This is similar to identifying the indifference curve tangent to the capital allocation line drawn from R_f and going through the coordinates of X.

The graph unequivocally shows that the risk of the performance-maximizing portfolio $P_{\text{opt}(X)}$, which also maximizes the investor's expected utility, always exceeds $\sigma^{(j)}$. According to equation (2), its expected return (which is also its average return according to information set Ω_T) is equal to $m_{P_{\text{opt}(X)}} = R_f + \frac{1}{\gamma_j}\text{SR}_X^2 = R_f + \sigma^{(j)}\frac{\text{SR}_X^2}{\text{SR}_B}$, higher than $m_{P_X^*} = R_f + \sigma^{(j)}\text{SR}_X$, and its risk is equal to $s_{P_X^*} = \sigma^{(j)}\frac{\text{SR}_X}{\text{SR}_B} > \sigma^{(j)}$. Thus, provided that the manager finds an active portfolio that outperforms the market, using the volatility budget criterion induces an optimal asset allocation that exceeds the maximum tolerated volatility. The manager is confronted with a cornelian choice: either to stick to the contractual arrangement and propose a sub-optimal allocation, or to adopt the unambiguously best portfolio in the investor's eyes, but at the cost of non-compliance with the risk appetite criterion.

As an illustration, we address the optimal investment in a selected active fund, Fidelity Growth Discovery (FDSVX), whose returns collected between July 2014 and June 2024 provide the following statistics: $m_X = 16.30\%$ and $s_X = 16.55\%$, leading to $\mathrm{SR}_X = 0.83$ using the risk-free rate $R_f = 2.54\%$. The outcome of the analysis is summarized in Table 2.

Table 2. Comparison of risk, return and performance of the optimal allocations to the active fund for three investor types (Defensive, Balanced, Aggressive)

		Portfolio	Weight w_X	Volatility s_{P_X}	Mean m_{P_X}	**T-alpha $\mathrm{T}\alpha_{P_X}$**	**V-alpha $\mathrm{V}\alpha^j_{P_X}$**
Investor	***Defensive***	$P_{\text{opt}(X)}$	41.87%	6.93%	8.30%	**2.64%**	**2.03%**
		P_X^*	22.69%	3.76%	5.66%	**1.43%**	**1.43%**
	Balanced	$P_{\text{opt}(X)}$	83.74%	13.86%	14.06%	**5.28%**	**4.07%**
		P_X^*	45.39%	7.51%	8.79%	**2.86%**	**2.86%**
	Aggressive	$P_{\text{opt}(X)}$	125.62%	20.79%	19.83%	**7.92%**	**6.10%**
		P_X^*	68.08%	11.27%	11.91%	**4.29%**	**4.29%**

Notes: The target volatility alphas are based on the comparison between 120 monthly returns of the MSCI ACWI Index and of the Fidelity Growth Discovery fund (FDSVX) from 2014/07/01 to 2024/06/30. The risk aversion coefficients for the Defensive, Balanced, and Aggressive investors are set to $\gamma_d = 12$, $\gamma_b = 6$ and $\gamma_a = 4$, which correspond to initial target volatilities of $\sigma^{(d)} = 3.76\%$, $\sigma^{(b)} = 7.51\%$ and $\sigma^{(a)} = 11.27\%$, respectively.

To comply with the maximum volatility budget constraint, the allocation to the active fund in P_X^* must almost halve (54%) the one in the utility-maximizing portfolio $P_{\text{opt}(X)}$ for all profiles. Sticking to the original volatility budget induces a sacrifice in performance. When performance is expressed in terms of total risk alpha, the outcome of $P_{\text{opt}(X)}$ appears to be greater than under to target volatility alpha, because of the risk mismatch penalty. Nevertheless, any other allocation that $P_{\text{opt}(X)}$, even those with a higher value of $\mathrm{T}\alpha_{P_X}$, would never exceed a greater $\mathrm{V}\alpha^j_{P_X}$ than the ones reported in the table.

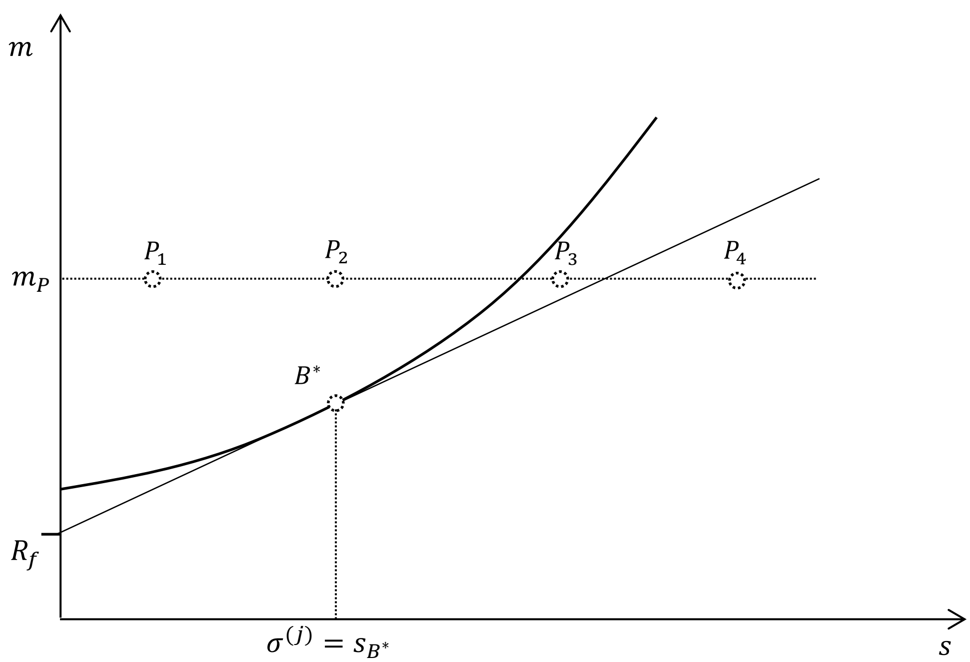
Long horizon allocation with realized volatility budgeting

Reconciling the investor's volatility budget constraint with the risk of an identified reference portfolio is possible if one adopts an *ex post* perspective. Consider that, at time 0, the investor identifies a benchmark asset allocation B^* whose risk is deemed suitable for the duration of her investment horizon. This portfolio is supposed to match the risk budget of the investor if it is held during a sufficiently long period of time, because it is then assumed that $s_{B^*} \cong \sigma_{B^*}$. Doing so, she commits to accepting that the realized benchmark volatility, which is unknown at inception of the portfolio management process, identifies with her risk budget or, in short, $s_{B^*} = \sigma^{(j)}$.

This perspective is consistent with the practical approach of selecting a reference portfolio with constant weights allocated to different asset classes, commonly called the "target allocation". The implied investor risk aversion is reverse-engineered by applying equation (6) with realized returns and is equal to $\gamma_j = \frac{\mathrm{SR}_B}{s_{B^*}} = \frac{m_B - R_f}{w^* \sigma_B^2}$. For instance, using again the MSCI ACWI Index as the proxy for the market portfolio, the implied risk aversion coefficients for investors willing to invest 25%, 50% and 75% in the index would be $\gamma_{w^*=25\%} = \frac{9.23\% - 2.54\%}{0.25 \times 14.83\%^2.} = 12.15$, $\gamma_{w^*=50\%} = 6.08$ and $\gamma_{w^*=75\%} = 4.05$ respectively.

In general, appraising the performance of an active allocation portfolio P consists in measuring its abnormal return (alpha) over its benchmark $\alpha_P = m_P - m_{B^*}$. Taking the investor's risk budget into account changes the perspective, because the portfolio manager has a double objective: (i) beating the average benchmark return and (ii) not exceeding the benchmark volatility. It is tempting to account for this second mission through the total risk alpha $\mathrm{T}\alpha_P = m_P - m_P^{\mathrm{req}}$, but this is not sufficient as the measure ignores the investor's attitude toward risk. Once again, the relevant performance measure is the target volatility alpha. The reasoning is illustrated in the following figure, that builds on Figure 2.

Figure 6. Performance of portfolios with equal returns but different risks



The graph reports the coordinates of four portfolios P_1 to P_4 that share the same average return m_P, and thus the same abnormal return α_P, but with different risk levels $s_{P_1} < s_{P_2}(= s_{B^*}) < s_{P_3} < s_{P_4}$. Their visual inspection reveals the impact of the volatility mismatch on performance:

- For P_1 (lower risk than the benchmark), $\mathrm{T}\alpha_{P_1} > \mathrm{V}\alpha^j_{P_1} > \alpha_{P_1} > 0$. The lower risk budget adopted by the portfolio manager enhances the investor's satisfaction, even though this is mitigated by the regret of not having consumed the whole risk budget to get an even better return.
- For P_2 (same risk as the benchmark), $\mathrm{T}\alpha_{P_2} = \mathrm{V}\alpha^j_{P_2} = \alpha_{P_2} > 0$. There is no risk mismatch and thus no associated impact on performance.
- For P_3 (higher risk than the benchmark), $\alpha_{P_3} > \mathrm{T}\alpha_{P_3} > 0 > \mathrm{V}\alpha^j_{P_3}$. Even though the total risk alpha is positive, it is not sufficient to compensate the loss in utility consecutive to the breach of the volatility budget constraint, leading to a negative overall performance.
- For P_4 (much higher risk than the benchmark), $\alpha_{P_4} > 0 > \mathrm{T}\alpha_{P_4} > \mathrm{V}\alpha^j_{P_4}$. The penalty for risk mismatch is very large (proportional to the square of the difference between realized and target volatility). This extreme result (positive abnormal return vs. very negative performance) illustrates how serious the issue of risk mismatch is for a defensive investor.

4. Extending unsuitability to extreme risks

Incorporating asymmetry preferences in preference-adjusted performance

The mean-variance framework does not properly accommodate the investor's attitudes toward extreme risks. It posits either that asset returns follow a Gaussian distribution (or one with similar properties), or that investors only care about portfolio volatility in their assessment of risk. In real life, none of these two assumptions reasonably holds. Investors care much about extreme risks such as market crashes or severe corrections, and portfolio returns materially depart from normality.[3] Their distributions typically display left asymmetry (negative skewness) and fat tails (high kurtosis). In what follows, we will focus on the role of the skewness as this is the second more important source of risk (after volatility) priced on financial markets, as shown a.o. by Harvey and Siddique (2000).[4]

A possible way to deal with this reality from the investor's point of view is to move away from the rationality assumption and posit a behavioral type of utility function. Another approach, which is adopted here, is to incorporate the investor's preferences for skewness through the calibration of the linear-exponential ("linex") utility function, extensively studied by Bell (1988, 1995). Amongst its desirable properties, it displays decreasing risk aversion at all wealth levels and obeys the "one-switch rule" (i.e., there is only one wealth level where the investor switches from less risky to riskier assets).

Considering mild regularity assumptions (see Hlawitschka, 1994), the application of the Taylor series development to this function leads to a formulation for the expected utility of portfolio return under the following form:

$$\mathrm{E}\left(U_j(R_P)\middle|\Omega_0\right) = \mu_P - \frac{1}{2}\gamma_j\sigma_P^2\left(1-\theta_j\nu_P\right) \quad (8)$$

where $\nu_P = \frac{\mathrm{E}[R_P - \mathrm{E}(R_P)|\Omega_0]^3}{\sigma_P^3}$ is the normalized third centered moment of the returns distribution, or "skewness", and $\theta_j \geq 0$ is an investor-specific coefficient reflecting her preference for skewness, that Hübner and Lejeune (2021) associate to the investor's "risk perception" coefficient in the utility function.

Comparing equation (8) with the original expected utility formulation of equation (1) suggests that the factor $\left(1-\theta_j\nu_P\right)$ plays the role of a variance multiplier in the risk penalty term. Because, in general, investors exhibit a positive attitude toward skewness, i.e., $\theta_j > 0$, and most financial return time series display a negative skewness, i.e., $\nu_P < 0$, we can expect this variance multiplier to be greater than 1. To capitalize further with the analogy with equation (1), we define the **perception-adjusted volatility** of portfolio P for investor j as $\tilde{\sigma}_P = \sigma_P\sqrt{1-\theta_j\nu_P}$). Note that, as for the original volatility measure, the perception-adjusted volatility is strictly proportional to leverage, i.e., $\tilde{\sigma}_{\left(wR_P+(1-w)R_f\right)} = w\tilde{\sigma}_P$ because skewness is leverage-invariant.

Following the insight of Kraus and Litzenberger (1976), accounting for the investor's preference would induce three-fund separation for the optimal portfolio choice: the risk-free asset, a mean-variance optimal portfolio V, and a portfolio that optimally trades-off skewness and volatility risks S. From the perspective of a non-professional investor however, the additional complexity compared with the original mean-variance framework is fairly invisible. The underlying reason is that her expected utility maximization would be performed with a four-step sequential procedure:

1. Assess the investor's risk perception coefficient θ_j.

2. Combine portfolios V and S to obtain the optimal Reward to perception-adjusted Volatility Ratio ($\widetilde{\text{RVR}}$) allocation as the solution of $\max_{w} \frac{w\mu_V + (1-w)\mu_S - R_f}{\tilde{\sigma}_{(wR_V + (1-w)R_S)}}$.[5] We can denote this optimal perception-adjusted portfolio B^j and its perception-adjusted volatility $\tilde{\sigma}_{B^j}$.
3. Assess the investor's risk aversion coefficient γ_j'. In that case, for a given risk profile, the estimated coefficient γ_j' in equation (8) will be lower than the original one γ_j in equation (1) by a factor $\gamma_j' = \frac{\gamma_j}{1-\theta_j \nu_{B^j}}$.
4. Finally, combine portfolio B^j with the risk-free asset to maximize the investor's expected utility in the same way as per the previous section, leading to $w^* = \frac{\mu_{B^j} - R_f}{\gamma_j' \tilde{\sigma}_{B^j}^2} = \frac{1}{\gamma_j' \tilde{\sigma}_{B^j}} \widetilde{\text{RVR}}_{B^j}$ and $\mu_{B^*} = R_f + \frac{1}{\gamma_j'} \widetilde{\text{RVR}}_{B^j}^2$.

Consequently, the link between volatility budget, risk perception and risk aversion is given by:

$$\sigma^{(j)} = \frac{\widetilde{\text{RVR}}_{B^j}}{\gamma_j' \sqrt{1-\theta_j \nu_{B^j}}} \tag{9}$$

The second step involves determining, at the portfolio manager's level, a substitute to the mean-variance market portfolio that is suitable regarding the investor's preference for skewness. This step can be performed independently of the investor risk profiling process (steps 1 and 3) and the final allocation proposal (step 4). Depending on the outcome of step 1, the relationship manager can easily select an "off-the-shelf" portfolio allocation, and directly propose to combine it with a fixed income investment according to the investor's appetite for risk.

Once the investor's risk perception and risk aversion coefficients are known, the **perception-adjusted quadratic alpha** simply writes as:

$$\widetilde{\text{Q}\alpha}_P^j = m_P - R_f - \frac{1}{2}\gamma_j' \tilde{\sigma}_P^2 - \frac{1}{2\gamma_j'} \widetilde{\text{SR}}_{B^j}^2 \tag{10}$$

where $\widetilde{\text{SR}}_{B^j} = \frac{m_{B^j} - R_f}{\tilde{\sigma}_{B^j}}$ is the perception-adjusted Sharpe ratio of optimal portfolio B^j for investor j.

Note that, even though the structure of the performance measure is similar to the original one, its implementation differs with the impacts of the third moments of both B^j and P.

Performance with volatility budget and subjective risk assessment

Implementing the expected utility framework with asymmetry preferences in a rigorous fashion is likely to be very arduous because of two obstacles: the estimation of investor-specific parameters and the identification of the optimal perception-adjusted portfolio.

The practical difficulty to directly assess the investor's risk perception coefficient θ_j is presumably much greater than the one for the risk aversion coefficient γ_j estimation. First, risk perception is a second-order dimension of risk in expected utility, and many financial advisors may be tempted to avoid the difficulty of dealing with it for little perceived added value. Second, the concept of asymmetry in returns is much less intuitive than the one of dispersion, and it is not likely that investors with lower financial literacy would provide faithful information from a questionnaire or an interview regarding this notion. Thus, the limitations to expected utility maximization discussed previously would prevail even more acutely here.

Next, adapting the market portfolio to the individual investor's characteristics would involve the identification of two building blocks (portfolios V and S) and their optimal combination according the customer's interests. The market portfolio is not unique anymore, but varies in space (with the diversity of customer profiles) and in time (with the continuous re-estimation of the variances, skeweness, covariances and coskewness). Many financial institutions would not afford, both for operational and commercial reasons, to deal with such a complex task and stick to a single, "one-size-fits-all" risky reference portfolio for all investors.

Hence, how could a wealth manager who wants to reflect risk aversion and risk perception in her investment advice overcome the difficulty in the spirit of the constrained optimization program presented in equation (4)? There are multiple potential answers to this question. We adopt a stance that is in direct continuation of the volatility budget approach. Namely, we posit that each investor expresses her attitudes towards risk according to two dimensions: (i) a maximum volatility budget (irrespective of asymmetry preferences) $\sigma^{(j)}$, and (ii) a subjective perception of extreme risk associated with the optimal portfolio (or any other portfolio) based on its volatility and skewness.

Our objective with the second dimension is to determine the value of the investor-specific risk perception coefficient θ_j in her utility function. We obtain it by replacing the optimal portfolio volatility σ_{B^j} with its perception-adjusted version $\tilde{\sigma}_{B^j} = \sigma_{B^j}\sqrt{1-\theta_j \nu_{B^j}}$ in the investor's assessment of extreme risks. In what follows, we adopt the excess Value-at-Risk (VaR) metric, defined as $\Pr\big[R_P \leq \mu_{B^j} -$

$\text{VaR}^{(j)}\big] = x$ for some confidence level $1 - x$, for this purpose. If the investor strongly believes in the assumption of the Gaussian distribution of returns or if she only cares about volatility as a risk measure, then she would automatically consider that $\text{VaR}^{(j)} = z_{1-x}\sigma^{(j)} = z_{1-x}w\sigma_{B^j}$. Otherwise, the volatility of portfolio B^j being transformed into $\tilde{\sigma}_{B^j}$, then the "perception-adjusted VaR", denoted $\widetilde{\text{VaR}}^{(j)}$, will also be affected in the same manner: $\widetilde{\text{VaR}}^{(j)} = z_{1-x}w\tilde{\sigma}_{B^j}$. This is equivalent to keeping the original portfolio volatility and adapting the volatility multiplier, i.e., $\widetilde{\text{VaR}}^{(j)} = z_{1-x}^{(j)}w\sigma_{B^j}$, which appears to be easier to handle in practice. The constrained optimization program is then slightly adapted with a new condition that corresponds to this second dimension:

$$\max_w \mu_{B^j} + (1-w)R_f \tag{11a}$$

$$\text{subject to: } w\sigma_{B^j} \leq \sigma^{(j)} \tag{11b}$$

$$\text{and: } z_{1-x}^{(j)}w\sigma_{B^j} = \widetilde{\text{VaR}}^{(j)} \tag{11c}$$

To assess the value of $z_{1-x}^{(j)}$, a new dedicated question must be asked to the investor, but this time with a focus on a very low probability event. This second question targets the perceived seriousness of the loss incurred by the investor in a severe crisis and her capacity to absorb it. The form of the question could be "*What is the maximum proportion of wealth $w^{(j)}$ that you could allocate to a portfolio (for instance an all-equity index) with a certain volatility level and with a given loss observed with a very low frequency?*". The difference with the previous situation is that the advisor assumes to know both the benchmark volatility σ_B and its Value-at-Risk VaR_B. The investor's answer determines the VaR of her acceptable portfolio ($\widetilde{\text{VaR}}^{(j)} = w^{(j)}\text{VaR}_B$), which is typically greater than the simple Gaussian volatility multiplication $z_{1-x}\sigma^{(j)}$ because of the investor's perception of risk. This means that the effect of the risk perception on the volatility multiplier is determined by the ratio of these two values, i.e., $z_{1-x}^{(j)} = z_{1-x}\frac{w^{(j)}\text{VaR}_B}{z_{1-x}\sigma^{(j)}} = \frac{\widetilde{\text{VaR}}^{(j)}}{\sigma^{(j)}}$.[6] This is what equation (11c) exactly represents.

In the continuity of the example developed in the Gaussian case, consider that the market portfolio has a volatility of $\sigma_B = 20\%$ (and thus $\mu_B = R_f + \sigma_B \text{RVR}_B = 10\%$) and that its VaR at the 95% confidence level is $\text{VaR}_B = 45\%$ (corresponding to an absolute VaR of $10\% - 45\% = -35\%$, a yearly loss only recently exceeded during the 2008 Global Financial Crisis). If returns were Gaussian, this portfolio VaR would be equal to $z_{95\%}\sigma_B = 1.645 \times 20\% = 32.9\%$. The actual VaR is thus greater than this value because of tail risk. The investor' risk budget (in volatility) has already been determined as $\sigma^{(j)} = 9\%$, leading to an associated Gaussian VaR of $z_{95\%}\sigma^{(j)} = 1.645 \times 9\% = 14.8\%$. If she is asked her maximum acceptable allocation in this portfolio and her answer is $w^{(j)} = 35\%$, this induces an acceptable risk of $\text{VaR}^{(j)} = 35\% \times 45\% = 15.75\%$. This corresponds to an investor-specific volatility multiplier of $z_{95\%}^{(j)} = \frac{\text{VaR}^{(j)}}{\sigma^{(j)}} = \frac{15.75\%}{9\%} = 1.75$.

The subjective extreme risk ratio $\kappa^{(j)} \equiv \frac{z_{1-x}^{(j)}}{z_{1-x}}$ fully summarizes the risk-perception dimension of the investor profile. It is directly connected with the risk perception parameter θ_j through a simple identity:

$$\kappa^{(j)} = \sqrt{1 - \theta_j \nu_{Bj}} \text{ and } \theta_j = \frac{1 - \left(\kappa^{(j)}\right)^2}{\nu_{Bj}} \quad (12)$$

Since, in general for the market portfolio, $\nu_B < 0$ and $z_x^{(j)} > z_x$, we expect θ_j to be positive for the average investor.[7] The influence of market conditions on the risk perception coefficient materializes through the skewness ν_B of the market portfolio. The more negative it is, the larger the value of $\kappa^{(j)}$. This is a logical result, as a larger left asymmetry automatically increases the market portfolio VaR, and thus a very skewness-sensitive investor will presumably raise her subjective value $z_{1-x}^{(j)}$ in reaction.

As the global constrained optimization problem is unaffected by condition (11c), the solution is the same as before (equation (5)). This result suggests that, to assess portfolio performance, the original mean-variance analysis should be replaced with a "mean-perception-adjusted variance" framework adapted to the investor. Substituting volatility with the adjusted one on the horizontal axis preserves the original geometric interpretation shown in Figure 2. Because B^* has a volatility of $\sigma^{(j)}$ by definition, its perception-adjusted volatility is equal to $\tilde{\sigma}^{(j)} = \sigma^{(j)}\sqrt{1 - \theta_j \nu_{Bj}} = \sigma^{(j)}\kappa^{(j)}$. Likewise, $\tilde{\sigma}_{Bj} = \sigma_{Bj}\kappa^{(j)}$ and, in general, $\tilde{\sigma}_P^2 = \sigma_P^2 \left(1 + \left(\left(\kappa^{(j)}\right)^2 - 1\right)\frac{\nu_P}{\nu_{Bj}}\right)$.

The final step is to determine the equivalent version of the target volatility alpha in this perception-adjusted framework, using the realized means m, volatilities s and skewnesses n. To avoid any confusion, we directly write the formula using the inputs of the optimization program:

$$\widetilde{\text{V}\alpha}_P^j = m_P - R_f - \frac{1}{2}\text{SR}_{Bj}\left[\sigma^{(j)} + \frac{s_P^2}{\sigma^{(j)}}\left(\frac{1 + \left(\left(\kappa^{(j)}\right)^2 - 1\right)\frac{n_P}{n_{Bj}}}{\left(\kappa^{(j)}\right)^2}\right)\right] \quad (13)$$

Expression (13) that characterizes the adjusted target volatility alpha has the advantage of gathering all risk perception-related elements between the parentheses. Beyond the potential misalignment of volatilities, the second source of unsuitability of portfolio P would be $n_P \neq n_{Bj}$. If the portfolio asymmetry is worse than the optimized one ($n_P < n_{Bj} < 0$), the ratio $\frac{n_P}{n_{Bj}} > 1$ and the associated penalty is reinforced by the investor's sensitivity to extreme risks, represented by factor $\left(\kappa^{(j)}\right)^2 - 1 > 0$.

The counterpart of equation (7) is best expressed directly in terms of adjusted volatilities so as to keep the original interpretation:

$$\widetilde{\mathrm{V}\alpha}_P^j = \widetilde{\mathrm{T}\alpha}_P - \widetilde{\mathrm{SR}}_{Bj} \frac{\left(\tilde{s}_P - \tilde{\sigma}^{(j)}\right)^2}{2\tilde{\sigma}^{(j)}} \tag{14}$$

where, in this expression, the Total risk alpha $\widetilde{\mathrm{T}\alpha}_P = m_P - \widetilde{m}_P^{\mathrm{req}}$ uses the required portfolio return $\widetilde{m}_P^{\mathrm{req}} = R_f + \widetilde{\mathrm{SR}}_{Bj}\tilde{s}_P$ is defined according to the modified risk-return framework.

We revert to the same sample as before to illustrate the consequences of asymmetry mismatch on portfolio performance. At first, for each of the three original investor profiles, we partition them further in three sub-profiles according to their risk perception coefficient. They are termed "Quiet" (q) with $\theta_q = 0$, "Indifferent" (i) with $\theta_i = 0.3$ and "Cautious" (c) with $\theta_c = 0.5$. Observing that the sample skewness of the MSCI ACWI Index equals $n_W = -0.34$, these parameter values translate into perception-adjusted index volatilities $\tilde{s}_W = s_W\sqrt{1 - \theta_j n_W} = 14.83\%, 15.57\%$ and 16.04% for the Quiet, Indifferent and Cautious investors, respectively. Note that, as $\theta_q = 0$ is nothing else than a pure mean-variance investor, who does not care at all about asymmetry risk for any portfolio. Furthermore, under these market conditions, the subjective extreme risk ratios that apply to these investor types are $\kappa^{(q)} = 1$, $\kappa^{(i)} = 1.05$ and $\kappa^{(c)} = 1.08$.

The next step is to determine the *ex post* optimal Reward to perception-adjusted Volatility Ratio that applies to those three investor types. Considering that we use a single index, our proposed solution for this illustration is to create protective put (PP) portfolios by combining the index with at-the-money (ATM) index put options. This overlay creates a positive convexity thanks to the puts' positive gammas, presumably leading to an improvement in global portfolio skewness that suits the investor tastes.

To obtain the option returns, we apply a simple procedure that consists of pricing a one-month ATM put option (with index price normalized to 1) according to the Black-Scholes formula with trailing two-year volatility adjusted for skewness and kurtosis according to the Cornish-Fisher formula.[8] This leads to a price of o_{t-1}, funded with a loan obtained at the one-month rate $R_{f,t}$. Then, one month later, we reprice the option at maturity and repay the loan, leading to a return of the option overlay of $R_{o,t} =$

$\max(0, -R_{W,t}) - o_{t-1}(1 + R_{f,t})$. Finally, the global return of the protective put strategy is given by $R_{W+\lambda o,t} = R_{W,t} + \lambda R_{o,t}$, where $0 < \lambda \leq 1$ is the number of puts purchased to protect the index investment ($\lambda = 100\%$ would mean a full portfolio insurance). The next step is to find suitable combinations of the index and the option overlay for each investor profile. A portfolio offering with a proportion of 0%, 60% and 100% of put options allows a proper segmentation, as shown in Table 3.

Table 3. Perception-adjusted Sharpe ratios for three investor types (Quiet, Indifferent, Cautious) and three portfolio protection levels (0%, 60% and 100% one-month ATM put option overlays) with the world market index

					Investor		
		Mean	Volatility	Skewness	***Quiet***	***Indifferent***	***Cautious***
Portfolio	***Index+0%***	9.23%	14.83%	−0.34	**45.07%**	42.93%	41.66%
	Index+60%	6.78%	10.79%	0.70	39.28%	**44.20%**	48.73%
	Index+100%	5.14%	8.81%	1.41	29.55%	38.87%	**54.27%**

Notes: The perception-adjusted Sharpe ratios estimates (last three columns) are based on 120 monthly returns of the MSCI ACWI Index from 2014/07/01 to 2024/06/30. Put option prices are based on the Black-Scholes formula using a trailing 24-month Cornish-Fisher-adjusted volatility estimate. The risk perception coefficients for the Quiet, Indifferent, and Cautious investors are set to $\theta_q = 0$, $\theta_i = 0.3$ and $\theta_c = 0.5$, respectively.

The last three columns report the perception-adjusted Sharpe ratios of the three protective put portfolios (rows) for the three sub-investor profiles according to their risk perception coefficient (columns). The values reported in bold correspond to the highest values for each investor type. The index with no overlay achieves the best performance for the Quiet investor, hence it is identified as B^q. Likewise, the index with a 60% overlay becomes the optimized portfolio for the Indifferent investor (B^i) and the full protective put with a 100% option overlay is the preferred one for the Cautious investor (B^c).

After this matching exercise between investor types and versions of the index, we have a final set of 9 investor types (3 risk aversion-based × 3 perception-based) and 9 portfolios (3 versions of index with option overlay × 3 levels of volatility). The purpose of the exercise is the same as in the mean-variance framework, namely to exactly determine how the mismatch between the risk of a portfolio and the one underlying the investor profile damages performance. All portfolios are purely passive and based on the same index. Thus, the outcome of the adjusted target volatility alpha measurement exactly indicates the cost of portfolio unsuitability in the eyes of the investor. This is summarized in Table 4.

Table 4. Perception-adjusted target volatility alphas for nine investor types (Defensive, Balanced, Aggressive combined with Quiet, Indifferent, Cautious) and nine portfolio risk and protection levels (Low, Neutral, High combined with 0%, 60% and 100% one-month ATM put option overlays) with the world market index

						Investor								
						Defensive			***Balanced***			***Aggressive***		
						Quiet	***Indiff.***	***Cautious***	***Quiet***	***Indiff.***	***Cautious***	***Quiet***	***Indiff.***	***Cautious***
					Mean	4.23%	4.02%	3.65%	5.93%	5.49%	4.76%	7.62%	6.97%	5.87%
					Volatility	3.76%	3.76%	3.76%	7.51%	7.51%	7.51%	11.27%	11.27%	11.27%
		Put Overlay	Mean	Volatility	Skewness	−0.34	0.70	1.41	−0.34	0.70	1.41	−0.34	0.70	1.41
Portfolio	***Low***	***0%***	4.23%	3.76%	−0.34	**0.00%**	−0.07%	−1.05%	**−0.42%**	−0.30%	−0.51%	**−1.13%**	−0.86%	−0.70%
		60%	4.02%	3.76%	0.70	−0.22%	0.00%	−0.30%	−0.64%	−0.37%	−0.24%	−1.35%	−0.98%	−0.59%
		100%	3.65%	3.76%	1.41	−0.58%	−0.17%	0.00%	−1.01%	−0.64%	−0.28%	−1.71%	−1.28%	−0.74%
	Neutral	***0%***	5.93%	7.51%	−0.34	**−0.85%**	−1.47%	−5.93%	**0.00%**	−0.15%	−2.10%	**−0.28%**	−0.20%	−1.20%
		60%	5.49%	7.51%	0.70	−1.28%	−0.74%	−2.47%	−0.44%	0.00%	−0.59%	−0.72%	−0.25%	−0.34%
		100%	4.76%	7.51%	1.41	−2.01%	−0.68%	−0.55%	−1.17%	−0.34%	0.00%	−1.45%	−0.71%	−0.18%
	High	***0%***	7.62%	11.27%	−0.34	**−3.39%**	−4.92%	−15.18%	**−0.42%**	−1.03%	−5.89%	**0.00%**	−0.22%	−3.16%
		60%	6.97%	11.27%	0.70	−4.04%	−2.95%	−7.07%	−1.08%	−0.37%	−2.15%	−0.65%	0.00%	−0.89%
		100%	5.87%	11.27%	1.41	−5.13%	−2.27%	−2.22%	−2.17%	−0.57%	−0.28%	−1.75%	−0.50%	0.00%

Notes: The perception-adjusted target volatility alpha estimates (last nine columns) are based on 120 monthly returns of the MSCI ACWI Index from 2014/07/01 to 2024/06/30. Put option prices are based on the Black-Scholes formula using a trailing 24-month Cornish-Fisher-adjusted volatility estimate. The risk perception coefficients for the Quiet, Indifferent, and Cautious investors are set to $\theta_q = 0$, $\theta_i = 0.3$ and $\theta_c = 0.5$, respectively.The risk aversion coefficients for the Defensive, Balanced, and Aggressive investors are set so as to match a target portfolio volatility of $\sigma^{(d)} = 3.76\%$, $\sigma^{(b)} = 7.51\%$ and $\sigma^{(a)} = 11.27\%$, respectively.

The link between this table and Table 1 is represented by the nine values in bold, each one corresponding to the "Quiet" investor, that replicate the target volatility alpha estimates of the original mean-variance analysis. As in the 3×3 version, all main diagonal elements are set to 0 as they correspond to a perfect investor-portfolio match. The cells are grayed with an intensity that gets darker as the performance becomes more negative.

There are two ways to read the table: the "within" and the "between" dimensions. The impact of skewness preference is best assessed by looking at the nine 3×3 grids corresponding to situations where the portfolio volatility matches the investor needs (e.g., Low/Defensive etc.). At that "within" level, the starting point is the upper-left cell, in bold. In general, the values on the main diagonal, which correspond to a similar portfolio profile regarding target skewness (e.g., 60% overlay/Indifferent etc.) lead to a performance improvement. This means that the damage created by a volatility mismatch can be to some extent alleviated for skewness-matching portfolios with more favorable asymmetry. Regarding the off-diagonal cells, the results can be very different from one situation to another. The most interesting cases are the ones for which the volatility profile is adequate (upper left, middle and lower right zones of the table). The largest damage to performance is observed when the investor has a strong interest for skewness ("Cautious") but when the portfolio displays unfavorable skewness (0% overlay). This situation is the most likely to arise in reality: many investors have a desire for protection, at the expense of the portfolio's RVR (lower Sharpe ratio), but the manager neglects to buy protection to match the customer's requirements. For aggressive portfolios, the loss in investor's utility amounts to -3.16% in the illustration, i.e., more than one third of the portfolio realized return (7.62%). In general, the destruction of performance due to skewness mismatch is more pronounced for aggressive investors. This is understandable as this investor profile puts most emphasis on return, and the sacrifice felt through the cost of the put protection is all the more sensitive in this case.

For the "between" case, we find it useful to focus on the visual inspection of the grayed intensity of the cells. Ne note an asymmetry in the penalty felt by the investor whose target volatility and skewness are simultaneously unmet, depending on whether her profile is defensive or aggressive. The former customer, who wants a low volatility portfolio, feels already strongly penalized if her portfolio becomes more aggressive (from −0% to −0.85%, then to −3.39% for the mean-variance investor). Her discomfort becomes even more severe when she has a strong perception for skewness risk and the portfolio manager ignores it. Whereas the defensive/cautious investor loses 1.05% in performance if the portfolio has the right volatility but the wrong skewness (0% overlay), the penalty becomes −15.18% if the portfolio has the highest volatility and the worst skewness. The most negative values, which can be found in the lower left triangle of the table, generally mean that the combination of a too aggressive and negatively skewed portfolio could lead to a perceived performance that becomes more negative than the mean portfolio return. Thus, the accumulation of an unsuitable level of risk (too high volatility) and inadequate level of protection (too low skewness) is shown to generate a dissatisfaction level that is so high that the perceived portfolio return becomes negative.

5. Conclusions and implications

Since setting a risk budget is equivalent to optimizing expected utility, imposing a volatility level to be targeted in the portfolio management process is a legitimate request for individual investors, at least in the mean-variance framework. This is also a standard investor profiling approach in the wealth management industry. Starting from this perspective, we have shown that, instead of the Sharpe ratio, the relevant performance measure in the personal portfolio management process is the target volatility alpha, with a well-defined expression for the volatility mismatch penalty.

For managers of active allocation funds or discretionary portfolios, the implications of changing the perspective on performance go further than simply adding a line “target volatility alpha” on a factsheet. This calls for fundamentally shifting from a product-driven to a customer-driven point of view on performance, by explicitly accounting for the client’s desire to not only “beat the benchmark”, but also “ fulfilling the risk appetite”. The task is made more complex, because it involves being simultaneously judged on realizing an objective (maximizing return) and respecting a constraint (consuming the risk appetite, no more, no less).

We stress that, from a performance perspective, replacing risk aversion with a risk budget is not a straightforward swap of criteria. For a constant risk aversion level, the risk budget should be positively related with the market risk premium, and thus it is the responsibility of the relationship manager to regularly reassess the risk appetite of the customer. Failing to do so might result in managing a portfolio in good faith, but destroying value for the investor. In our analysis, we have proposed a solution for long horizons, namely replacing an *ex ante* risk target by an *ex post* realized volatility criterion. This makes the job of the manager even more challenging, because it is not sufficient to control the level of active risk on an interim basis (through an Information ratio criterion, for instance) to obtain a satisfactory long term performance. Thus, we acknowledge this limitation: if an investor requests a portfolio to be managed according to a specific risk budget, which is something that can perfectly happen (and the customer is king or queen), we provide in this paper a tool to measure the efficiency of the process, but not the recipe to get the best out of it.

The extension of the framework to the investor's preferences for volatility and asymmetry risks that we propose opens up the way to a potential redesign of the investor profiling approach and the associated portfolio management process. Investors do care about tail risk, and it must be possible to provide them with a proper method to translate their concerns into a "risk budgeting" approach. In the last section, we have proposed a consistent, end-to-end framework in which an intuitive way to question a customer regarding her attitudes toward "normal" as well as "extreme" risks can be projected into a definition of a risk budget, a portfolio management approach, and a performance measurement system. It is a strong belief of the author that adopting this kind of point of view would contribute to a future-proof strategy of managing a customer's assets, without the need to promise mountains and wonders regarding the levels of abnormal returns.

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[1] The paper by Moreira and Muir (2017) has opened a very lively debate about the way volatility targeting should be implemented (the "how") so as to generate positive performance (the "what") while mitigating its drawbacks (high turnover, excessive leverage, large drawdowns, tail risks …). Recent examples include a.o. Harvey, Hoyle, Korgaonkar, Rattray, Sargaison, Van Hemert (2018), Bongaerts, Kang and van Dijk (2020), Cederburg, O'Doherty, Wang and Yan (2020), and Barroso and Detzel (2021).

[2] The use of VaR corresponds to profiling questions such as "*What is the maximum loss that you would be ready to bear during a given year, provided that it is not catastrophic?*". There are other ways to assess the investor's attitudes towards risk, like "*What is the maximum loss that you would be ready to bear during a catastrophic year?*" (conditional VaR) or "*Once very how many years are you ready to lose more than x% of your wealth?*" (probability of extreme losses). These alternative framings do not change the qualitative results of the process.

[3] See Sharpe (2007) for a discussion of the shortcomings of mean-variance expected utility optimization.

[4] Our discussion can be extended with the inclusion of kurtosis without technical hindrance but it would not change the approach while complexifying the computations and introducing a third dimension of investor risk attitudes which is seldom encountered in practice.

[5] Formally, the perception-adjusted variance of this portfolio writes:

$$\tilde{\sigma}^2_{(wR_V+(1-w)R_S)} = \left(w^2\sigma_V^2 + (1-w)^2\sigma_S^2 + 2w(1-w)\mathrm{Cov}(R_V R_S)\right) \times \left(1 - \psi_j\left(\frac{w^3 n_V \sigma_V^3 + 3w^2(1-w)\mathrm{Cov}\left(R_V^2 R_S\right) + 3w(1-w)^2\mathrm{Cov}\left(R_V R_S^2\right) + (1-w)^3 n_S \sigma_S^3}{\left(w^2\sigma_V^2 + (1-w)^2\sigma_S^2 + 2w(1-w)\mathrm{Cov}(R_V R_S)\right)^{1.5}}\right)\right).$$

[6] This perception-adjusted volatility multiplier is a subjective notion that should not be confused, for instance, with the Cornish-Fisher VaR that provides an approximation of the true VaR based on the sample higher moments of the returns distribution by replacing $z_{1-x}\sigma_{B^j}$ with $z_{1-x}^{\text{C-F}}\sigma_{B^j}$ regardless of the investor risk profile.

[7] If $\nu_B = 0$, the distribution is symmetrical and it is not possible to assess the preference for skewness, making the whole exercise irrelevant.

[8] More specifically, for each observation date t, we apply the Cornish-Fisher (C-F) approximation with a confidence level $1 - x = 97.5\%$ using the index sample skewness and kurtosis over a two-year period (24 lagged observations), and then define the C-F adjusted index volatility as $\sigma_{W,t}^{\text{C-F}} = \frac{z_\alpha^{\text{C-F}}}{z_\alpha}\sigma_{W,t}$. Detailed calculations are available upon request.