p-Regularity of Random Wavelet Series

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Joint work with

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Given a signal, we want to study

- the regularity in each of its points
- the repartition of the different singularities through the Hausdorff dimension of the sets of points sharing a common regularity

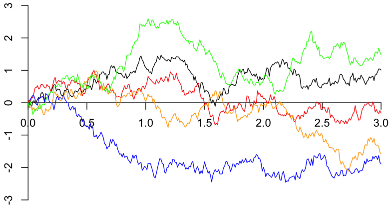

Figure 1 – Brownian motion

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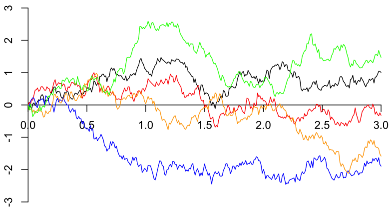

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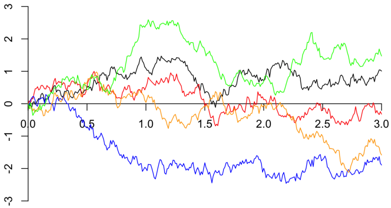

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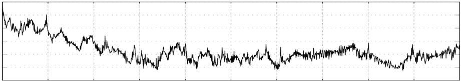

Figure 2 – LWS

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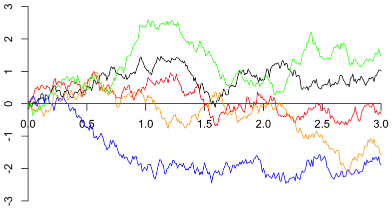

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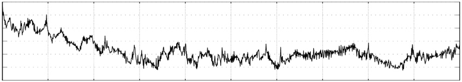

Figure 2 – LWS

Hölder regularity

Let $\alpha \geq 0, x_0 \in \mathbb{R}$ and $f \in L^{\infty}_{\text{loc}}(\mathbb{R})$. Then $f \in C^{\alpha}(x_0)$ if there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$|f(x) - P(x)| \leq C\,|x - x_0|^{\alpha} \quad \forall x \in B(x_0, R).$$

The Hölder exponent of f at x_0 is

$$h_f^{(+\infty)}(x_0) = h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^{\alpha}(x_0)\}.$$

Drawback : limited to locally bounded functions.

$\rightarrow$ We consider

$$f \in \bigcup_{\varepsilon > 0} C^{\varepsilon}(\mathbb{R}).$$

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Let $p \geq 1, \alpha \geq \frac{-1}{p}, x_0 \in \mathbb{R}$ and $f \in L^p_{\text{loc}}(\mathbb{R})$. Then $f \in T^p_\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$\left(\frac{1}{r} \int_{B(x_0,r)} |f(x) - P(x)|^p \; dx \right)^{\frac{1}{p}} \leq C r^\alpha \quad \forall r \leq R.$$

The p-exponent of f at x_0 is

$$h_f^{(p)}(x_0) = \sup \left\{ \alpha \geq \frac{-1}{p} : f \in T^p_\alpha(x_0) \right\}.$$

→ We consider every $p \geq 1$ such that

$$\eta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log \left(2^{-j} \right)} > 0.$$

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Wavelet series

Wavelet basis : orthonormal basis of $L^2(\mathbb{R})$ of the form

$$\left\{2^{\frac{j}{2}}\psi_{j,k} : j,k \in \mathbb{Z}\right\},$$

where

$$\psi_{j,k}(x) = \psi(2^j x - k).$$

To any function f, we can associate a sequence $(c_{j,k})_{j\in\mathbb{N},\, k\in\{0,\dots,2^j-1\}}$ such that

$$f = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}.$$

For every $j \in \mathbb{N}$ and every $k \in \{0,\dots,2^j-1\}$, let

$$\lambda_{j,k} = \left[k2^{-j}, (k+1)2^{-j}\right[$$

and for every $j \in \mathbb{N}$, let

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We write $\psi_{\lambda_{j,k}} = \psi_{j,k}$ and $c_{\lambda_{j,k}} = c_{j,k}$.

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Leaders

Fix $j \in \mathbb{N}$ and $\lambda \in \Lambda_j$.

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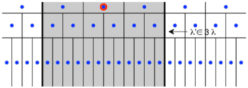
$$l_\lambda^{(+\infty)} = l_\lambda = \sup_{j' \geq j} \sup_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|$$



p-Leaders :

$$l_\lambda^{(p)} = \left(\sum_{j' \geq j} \sum_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \right)^{\frac{1}{p}}$$

For every $p \in [1, +\infty[$ such that $\eta_f(p) > 0$ and for $p = +\infty$ if $f \in \bigcup_{\varepsilon > 0} C^\varepsilon$,

$$h_f^{(p)}(x_0) = \liminf_{j \to +\infty} \frac{\log\left(l^{(p)}_{\lambda_j(x_0)}\right)}{\log\left(2^{-j}\right)}.$$

$\rightarrow$ Allows to define p-exponents when $p \in]0, 1[$ and $\eta_f(p) > 0$.

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p-Spectrum :

$$\mathscr{D}_f^{(p)} : h \in \left[\frac{-1}{p}, +\infty\right] \mapsto \dim_{\mathcal{H}} \left\{x \in [0,1] : h_f^{(p)}(x) = h\right\}$$

Multifractal formalism : computable formula which

- gives an upper bound to the spectrum
- gives the exact spectrum for a large class of functions
- is generically the exact spectrum in a well-chosen functional space

Spectra and formalisms

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Upper bound for the Hölder spectrum

If $f \in \bigcup_{\varepsilon>0} C^{\varepsilon}$, then for every $h \geq 0$,

$$\mathscr{D}_f^{(+\infty)}(h) \leq h \sup_{\alpha \in]0,h]} \frac{\rho(\alpha)}{\alpha},$$

where ρ is defined such that at infinitely many scales j,

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(J.-M. Aubry and S. Jaffard, *Random Wavelet Series*, 2002)

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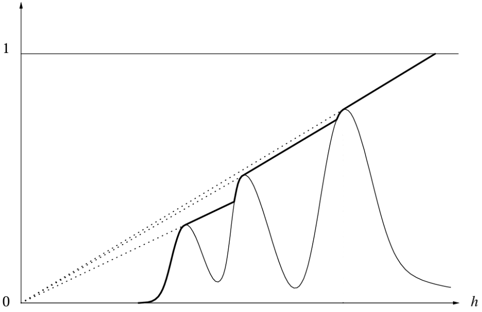

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RWS and LWS

RWS : let

$$f=\sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k},$$

where the 2^j random variables $\frac{-\log_2|c_{j,k}|}{j}$ are drawn independently according to a given probability law $\boldsymbol{\rho_j}$, hence

$$\mathbb{P}\left(|c_{j,k}|\geq 2^{-\alpha j}\right)=\boldsymbol{\rho_j}(]-\infty,\alpha]).$$

LWS : let $\alpha\in\mathbb{R}$, $\eta\in\,]0,1[$ and

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Let f be a RWS (satisfying a condition that ensures that f is uniformly Hölder). A.s. for every $h \geq 0$,

$$\mathscr{D}_f^{(+\infty)}(h) = \begin{cases} h \sup_{\alpha \in]0,h]} \frac{\rho(\alpha)}{\alpha} & \text{if } h \in [h_{\min}, h_{\max}] \\ -\infty & \text{otherwise} \end{cases} .$$

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A.s. for every $p < \begin{cases} \frac{\eta-1}{\alpha} & \text{if } \alpha < 0 \\ +\infty & \text{otherwise} \end{cases}$ and every $h \geq \frac{-1}{p}$,

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$$\left(h+\frac{1}{p}\right)\sup_{\alpha\in\left]\frac{-1}{p},h\right]}\frac{\rho(\alpha)}{\alpha+\frac{1}{p}}$$

Aim :

- For every f, every $p>0$ such that $\eta_f(p)>0$ and every $h\geq\frac{-1}{p}$,
$$\mathscr{D}_f^{(p)}(h)\leq\left(h+\frac{1}{p}\right)\sup_{\alpha\in\left]\frac{-1}{p},h\right]}\frac{\rho(\alpha)}{\alpha+\frac{1}{p}}.$$
- For every RWS f and every $p>0$ such that $\eta_f(p)>0$, a.s. for every $h\geq\frac{-1}{p}$,
$$\mathscr{D}_f^{(p)}(h)=\begin{cases}\left(h+\frac{1}{p}\right)\sup_{\alpha\in\left]\frac{-1}{p},h\right]}\frac{\rho(\alpha)}{\alpha+\frac{1}{p}} & \text{if } h\in\left[h_{\min},h_{\max}^{(p)}\right]\\ -\infty & \text{otherwise}\end{cases}.$$

Variation 1 : η replaced with $(\eta_j)_j$ such that $\limsup_{j\to+\infty} \eta_j = \eta$

$\rightarrow$ no change in the spectrum

Variation 2 : two exponents α_1 and α_2, each one with its own lacunarity η_1 or η_2

$\rightarrow$

$$p_0 = \min\left(p_0(\alpha_1), p_0(\alpha_2)\right), \quad \text{with } p_0(\alpha_i) = \begin{cases} \frac{\eta_i - 1}{\alpha_i} & \text{if } \alpha_i < 0 \\ +\infty & \text{otherwise} \end{cases},$$

and for every $p < p_0$,

$$\mathscr{D}^{(p)}(h) = \left(h + \frac{1}{p}\right) \sup_{i\in\{1,2\}\,:\,\alpha_i \le h} \frac{\eta_i}{\alpha_i + \frac{1}{p}}$$

if

$$h \in \left[\min(\alpha_1,\alpha_2), \min\left(\frac{\alpha_1}{\eta_1} + \left(\frac{1}{\eta_1} - 1\right)\cdot\frac{1}{p}, \frac{\alpha_2}{\eta_2} + \left(\frac{1}{\eta_2} - 1\right)\cdot\frac{1}{p}\right)\right]$$

Variation 1 : η replaced with $(\eta_j)_j$ such that $\limsup_{j\to+\infty} \eta_j = \eta$

$\rightarrow$ no change in the spectrum

Variation 2 : two exponents α_1 and α_2, each one with its own lacunarity η_1 or η_2

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p-Spectrum of some variations of LWS

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Proof

Classical case :

$$2^{-\alpha j_0(\lambda)} 2^{-\frac{j_0(\lambda)-j}{p}} \leq l_\lambda^{(p)} \leq C\, 2^{-\alpha j_0(\lambda)} 2^{-\frac{j_0(\lambda)-j}{p}}\, j^{\frac{2}{p}}$$

(P. Abry et al., *Multifractal analysis based on p-exponents and lacunarity exponents*, 2015)

Proof

Classical case :

$$2^{-\alpha j_0(\lambda)} 2^{-\frac{j_0(\lambda)-j}{p}} \leq l_\lambda^{(p)} \leq C\, 2^{-\alpha j_0(\lambda)} 2^{-\frac{j_0(\lambda)-j}{p}}\; j^{\frac{2}{p}}$$

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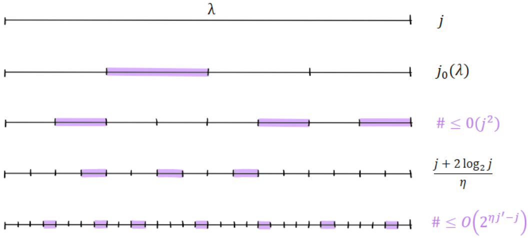
$$l_\lambda^{(p)} = \left(\sum_{j' \geq j_0(\lambda)} \sum_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \right)^{\frac{1}{p}}$$



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Proof

Classical case :

$$l_\lambda^{(p)} \sim 2^{-\left(\alpha+\frac{1}{p}\right) j_0(\lambda)} 2^{\frac{j}{p}} \quad \text{and} \quad j_0(\lambda) \lesssim \frac{j}{\eta}$$

For $\delta \leq 1$,

$$E_\delta = \limsup_{j \to +\infty} \bigcup_{k \,:\, c_{j,k} = 2^{-\alpha j}} B(k2^{-j}, 2^{-\delta j})$$

- Regularity :

$$j_0(\lambda) \sim \frac{j}{\delta}$$

  and

$$h = \frac{\alpha + \frac{1}{p}}{\delta} - \frac{1}{p} \Leftrightarrow \delta = \frac{h + \frac{1}{p}}{\alpha + \frac{1}{p}}$$

- Hausdorff dimension : $\frac{\eta}{\delta}$

(P. Abry et al., *Multifractal analysis based on p-exponents and lacunarity exponents*, 2015)

Variation 2 :

$$l_\lambda^{(p)} \sim 2^{\frac{j}{p}} \max\left(2^{-\left(\alpha_1+\frac{1}{p}\right)j_1(\lambda)}, 2^{-\left(\alpha_2+\frac{1}{p}\right)j_2(\lambda)}\right)$$
$$\sim 2^{\frac{j}{p}} 2^{-\min\left(\left(\alpha_1+\frac{1}{p}\right)j_1(\lambda), \left(\alpha_2+\frac{1}{p}\right)j_2(\lambda)\right)}$$

$\to j_1(\lambda) \sim \frac{j}{\delta_1}$ and $j_2(\lambda) \sim \frac{j}{\delta_2}$ create a competition between $\frac{\alpha_1+\frac{1}{p}}{\delta_1}$ and $\frac{\alpha_2+\frac{1}{p}}{\delta_2}$, where the worst regularity, i.e.

$$\min\left(\frac{\alpha_1+\frac{1}{p}}{\delta_1}, \frac{\alpha_2+\frac{1}{p}}{\delta_2}\right),$$

wins.

Proof (continuation and end)

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Upper bound for the p-spectrum

For every $h \geq \frac{-1}{p}$,

$$\mathscr{D}_f^{(p)}(h) \leq \widetilde{\rho}^{(p),*}(h) \leq \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left]\frac{-1}{p}, h\right]} \frac{\rho(\alpha)}{\alpha + \frac{1}{p}},$$

where there are about $2^{\widetilde{\rho}^{(p)}(h)j}$ p-leaders of order 2^{-hj} at infinitely many scales j.

For $\lambda \in \Lambda_j$,

$$l_\lambda^{(p)} \sim 2^{-hj} \Rightarrow \exists j'(\lambda) \in [j, Cj] \text{ s.t. } \sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'(\lambda)-j)} \sim 2^{-hpj}$$

$$\not\Rightarrow \exists \lambda' \subseteq \lambda \text{ s.t. } |c_{\lambda'}| \sim 2^{-\left(h+\frac{1}{p}\right)j} 2^{\frac{j'(\lambda)}{p}}$$

For every $\lambda' \in \Lambda_{j'(\lambda)}$ s.t. $\lambda' \subseteq \lambda$, there exists $\alpha = \alpha(\lambda') \in \left[\alpha_0 + \frac{1}{p}, h + \frac{1}{p}\right]$ for which

$$|c_{\lambda'}| = 2^{-\alpha j'(\lambda)} 2^{\frac{j'(\lambda)}{p}}.$$

Discretization :

- $j'(\lambda) \sim A(\lambda)j$
- $\alpha \sim l\beta$

Upper bound for the p-spectrum

For every $h \geq \frac{-1}{p}$,

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Discretization :

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Upper bound for the p-spectrum (continuation and end)

$$\sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'(\lambda)-j)}$$

$$\sim \sum_l \# \left\{ \lambda' \subseteq \lambda : |c_{\lambda'}| \sim 2^{-l\beta j'(\lambda)} 2^{\frac{j'(\lambda)}{p}} \right\} \cdot 2^{-l\beta p j'(\lambda)} 2^{j'(\lambda)} 2^{-(j'(\lambda)-j)}$$

$$\sim \sum_l 2^{r_\lambda(l) j'(\lambda) - j} 2^{-l\beta p j'(\lambda)} 2^j$$

$$\sim \sum_l 2^{r_\lambda(l) A(\lambda) j} 2^{-l\beta p A(\lambda) j}$$

$$\sim 2^{r_\lambda(l_0(\lambda)) A(\lambda) j} 2^{-l_0(\lambda)\beta p A(\lambda) j}$$

Each of the $2^{\widetilde{\rho}^{(p),*}(h)j}$ intervals $\lambda \in \Lambda_j$ for which $l_\lambda^{(p)} \sim 2^{-hj}$ comes from

$$2^{r_\lambda(l_0(\lambda))A(\lambda)j - j}$$

coefficients of order $2^{-l_0(\lambda)\beta A(\lambda) j} 2^{\frac{A(\lambda)j}{p}}$ at scale $A(\lambda)j$.

Moreover,

- $$r_\lambda(l_0(\lambda)) \sim \frac{l_0(\lambda)\beta p A(\lambda) - hp}{A(\lambda)}$$

Upper bound for the p-spectrum (continuation and end)

$$\sum_{\lambda'\subseteq\lambda} |c_{\lambda'}|^p\, 2^{-(j'(\lambda)-j)}$$

$$\sim \sum_l \#\left\{\lambda'\subseteq\lambda : |c_{\lambda'}| \sim 2^{-l\beta j'(\lambda)} 2^{\frac{j'(\lambda)}{p}}\right\}\cdot 2^{-l\beta p j'(\lambda)} 2^{j'(\lambda)} 2^{-(j'(\lambda)-j)}$$

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Upper bound for the p-spectrum (continuation and end)

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&\sim \sum_l 2^{r_\lambda(l) j'(\lambda) - j} 2^{-l\beta p j'(\lambda)} 2^j \\
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Upper bound for the p-spectrum (continuation and end)

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Moreover,

- $$r_\lambda(l_0(\lambda)) \sim \frac{l_0(\lambda)\beta p A(\lambda) - hp}{A(\lambda)}$$
- there exist A, l such that for infinitely many scales j and for each $\lambda \in \Lambda_j$ satisfying $l_\lambda^{(p)} \sim 2^{-hj}$, $A(\lambda) = A$ and $l_0(\lambda) = l$.

At those scales, there are about

$$2^{\widetilde{\rho}^{(p),*}(h)j} 2^{(l\beta p A - hp)j - j}$$

coefficients of order $2^{-l\beta Aj} 2^{\frac{Aj}{p}}$ at scale Aj.

Upper bound for the p-spectrum (continuation and end)

$$2^{-hpj} \sim 2^{r_\lambda(l_0(\lambda))A(\lambda)j} 2^{-l_0(\lambda)\beta p A(\lambda)j}$$

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Upper bound for the p-spectrum (continuation and end)

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Each of the $2^{\widetilde{\rho}^{(p),*}(h)j}$ intervals $\lambda \in \Lambda_j$ for which $l_\lambda^{(p)} \sim 2^{-hj}$ comes from

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Moreover,

- $$r_\lambda(l_0(\lambda)) \sim \frac{l_0(\lambda)\beta p A(\lambda) - hp}{A(\lambda)}$$
- there exist A, l such that for infinitely many scales j and for each $\lambda \in \Lambda_j$ satisfying $l_\lambda^{(p)} \sim 2^{-hj}$, $A(\lambda) = A$ and $l_0(\lambda) = l$.

At those scales, there are about

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Main result

Theorem :

- For every f, every $p > 0$ such that $\eta_f(p) > 0$ and every $h \geq \frac{-1}{p}$,

$$\mathscr{D}_f^{(p)}(h) \leq \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left]\frac{-1}{p}, h\right]} \frac{\rho(\alpha)}{\alpha + \frac{1}{p}}.$$

- For every RWS f and every $p > 0$ such that $\eta_f(p) > 0$, a.s. for every $h \geq \frac{-1}{p}$,

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To come : studying the genericity of the p-spectrum in S^ν spaces.

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