# Positionality of Dumont-Thomas numeration systems

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Joint work with Sébastien Labbé and Manon Stipulanti

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# Numeration systems

## Definition

*Numeration system* over the *domain* $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}\}$:

- *representation map* $\mathsf{rep}\colon \mathbb{D} \to A^*$
- *evaluation map* $\mathsf{val}\colon L \to \mathbb{D}$, where $\mathsf{rep}(\mathbb{D}) \subseteq L \subseteq A^*$

such that $\mathsf{val} \circ \mathsf{rep} = id_{\mathbb{D}}$.

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Common ways of defining numeration systems:

- *Positional* numeration systems: given $(U_n)_{n\in\mathbb{N}}$, set $\mathsf{val}_U\colon A^* \to \mathbb{N} : w_{\ell-1} \cdots w_0 \mapsto \sum_{i=0}^{\ell-1} w_i U_i$.
- *Abstract* numeration systems: if $A$ is ordered, given a regular language $L \subseteq A^*$, $\mathsf{val}_L$ is the unique increasing bijection between $(L, \preccurlyeq_{\mathrm{rad}})$ and $(\mathbb{N}, \leq)$.

# Positionality (1)

Abstract numeration systems may or may not be also definable in a positional way.

## Example

The abstract numeration system defined from $L = \{\mathtt{0}, \ldots, \mathtt{9}\}^* \setminus \mathtt{0}\{\mathtt{0}, \ldots, \mathtt{9}\}^*$ is equal to the usual decimal system, corresponding to $U_i = 10^i$.

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## Example

The abstract numeration system defined from $L = \mathtt{1}^*\mathtt{2}^*$ verifies $\mathrm{rep}(3) = \mathtt{11}$ and $\mathrm{rep}(5) = \mathtt{22}$, so it cannot be defined as a positional numeration system.

# Positionality (2)

## Definition

A numeration system over $\mathbb{N}$ is *positional* if the underlying alphabet $A$ is a set of consecutive integers $\{0, 1, \dots, c\}$ for some $c \in \mathbb{N}$ and if there exists a sequence $(U_i)_{i \geq 0} \in \mathbb{N}^{\mathbb{N}}$ such that the evaluation map is of the form
$\mathrm{val}\colon A^* \to \mathbb{N}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-1} w_i U_i$.

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A numeration system over $\mathbb{Z}$ is *positional* if the underlying alphabet $A$ is a set of consecutive integers $\{0, 1, \ldots, c\}$ for some $c \in \mathbb{N}$ and if there exist sequences $(U_i)_{i\geq 0}, (V_i)_{i\geq 0} \in \mathbb{N}^{\mathbb{N}}$ such that the evaluation map is of the form
$\mathrm{val}\colon A^* \to \mathbb{Z}, w_{k-1}\cdots w_0 \mapsto \sum_{i=0}^{k-2} w_i U_i - w_{k-1} V_{k-1}$.

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## Example

The two's complement numeration system is defined by the weights $U_n = V_n = 2^n$. For instance, we have
$\mathrm{val}(\mathtt{1011}) = -8 + 2 + 1 = -5$.

# Substitutions

*Substitution*: map $\mu\colon A^* \to B^*$ such that $\mu(uv) = \mu(u)\mu(v)$ for all $u, v \in A^*$, that has a growing letter and is non-erasing.
May be extended to two-sided words:

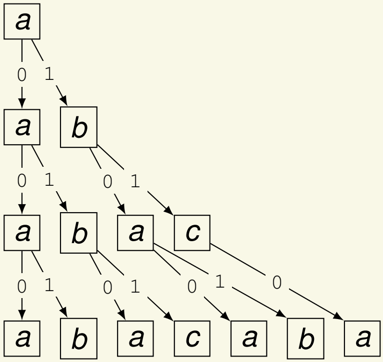
$$\mu(\cdots u_{-2}u_{-1}|u_0u_1u_2\cdots) = \cdots\mu(u_{-2})\mu(u_{-1})|\mu(u_0)\mu(u_1)\mu(u_2)\cdots$$

*Periodic point* of $\mu$: $u \in A^{\mathbb{D}}$ such that $\mu^p(u) = u$. $p$ is the *period*.
If $p = 1$: *fixed point*. *Seed* of a two-sided periodic point: $u_{-1}|u_0$.
*Tree representation*: given a fixed point with seed $a$, build a tree $\mathcal{T}_{\mu,a}$ with root $a$ and such that if $\mu(x) = y_0\cdots y_{\ell-1}$, the node $x$ has $\ell$ children labelled $y_0, \ldots, y_{\ell-1}$ with edges labelled $\mathtt{0}, \ldots, \ell\mathtt{-1}$.

# Substitutions as trees

$$\mu: \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$

Fixed point
starting with $a$.
Initial segment of $\mathcal{T}_{\mu,a}$:



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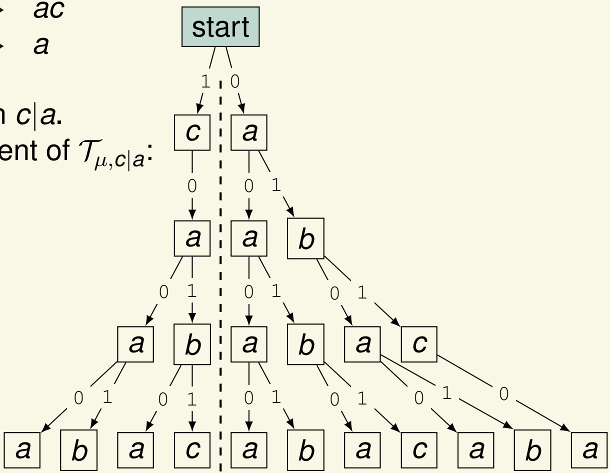

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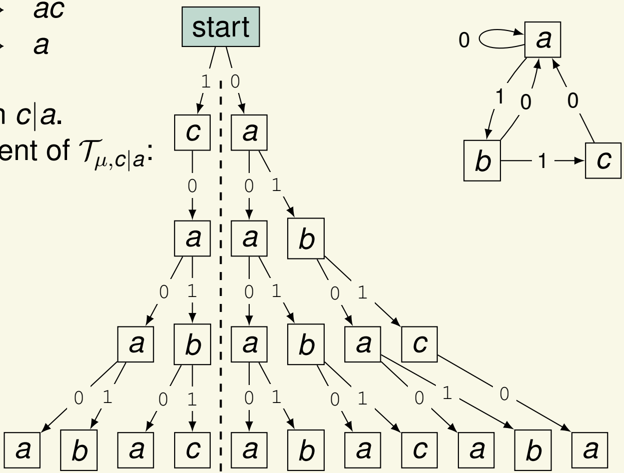
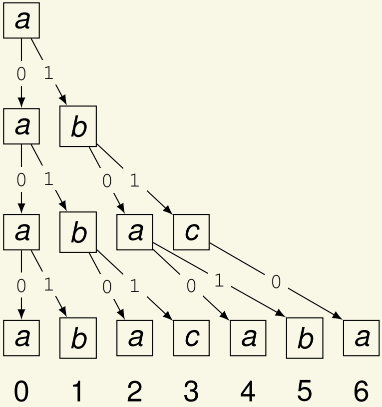


Idea (Dumont, Thomas, 1989): organize the tree in columns,



# Dumont-Thomas numeration systems

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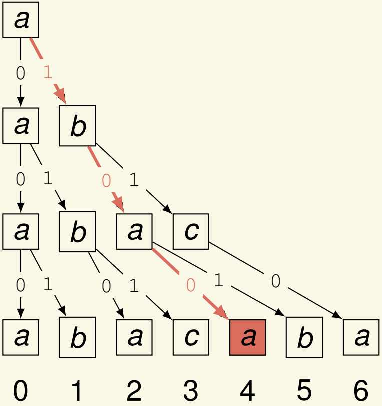


$\mathrm{rep}(4) = \texttt{100}$

# Dumont-Thomas numeration systems

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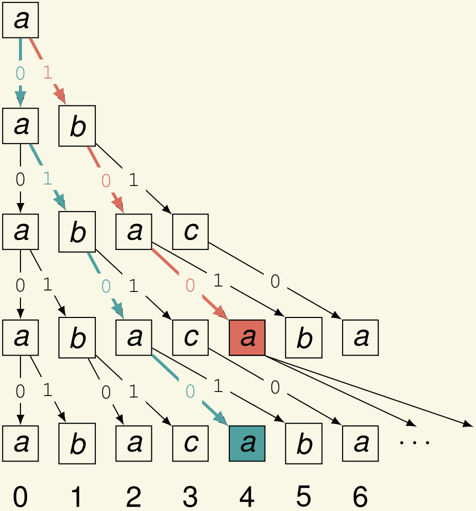


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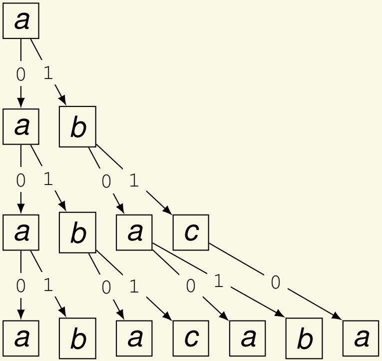
# Generalized Dumont-Thomas numerations (1)

Motivation: use Dumont-Thomas numeration systems as a way to generate Wang tilings



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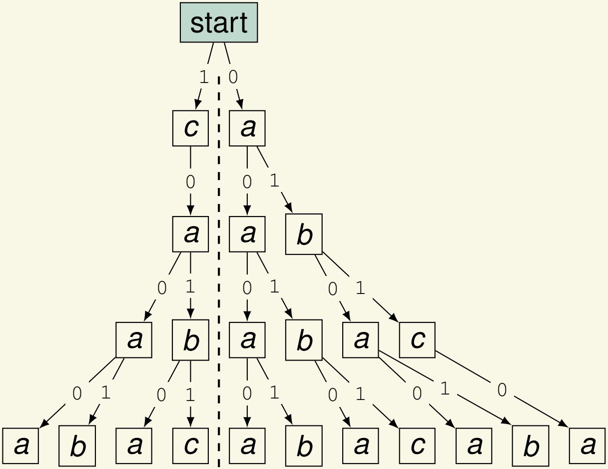
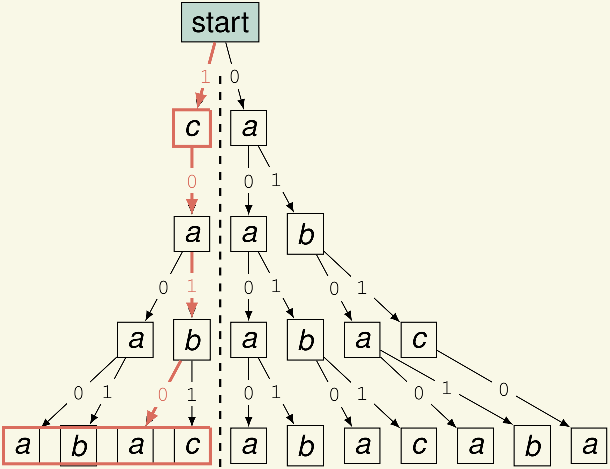


# Generalized Dumont-Thomas numerations (1)

Motivation: use Dumont-Thomas numeration systems as a way to generate Wang tilings ⇝ we must be able to deal with negative numbers ⇝ we must be able to deal with periodic points of substitutions.




| $n$ | rep($n$) |
|---|---|
| 7 | 0001000 |
| 6 | 0110 |
| 5 | 0101 |
| 4 | 0100 |
| 3 | 0011 |
| 2 | 0010 |
| 1 | 0001 |
| 0 | 0 |
| -1 | 1 |
| -2 | 1010 |
| -3 | 1001 |
| -4 | 1000 |
| -5 | 1010110 |

# Generalized Dumont-Thomas numerations (2)

**Definition (Labbé, Lepšová, 2024; K., Labbé, Stipulanti, 2025)**

Given a substitution $\mu$, a periodic point $u$ of period $p$ of $\mu$ with seed $b|a$ and some $r \in \{0, \ldots, p-1\}$, the *complement Dumont-Thomas numeration system associated with* $\mu$*, u and r* is defined by its representation map, such that $\mathrm{rep}(n)$ is the label of a shortest path of length congruent to $r \bmod p$, going from the root to a node in column $n$ in the tree $\mathcal{T}_{\mu,b|a}$.

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**Question**

Given $\mu, u, r$, can we decide the positionality of the associated Dumont-Thomas numeration system?

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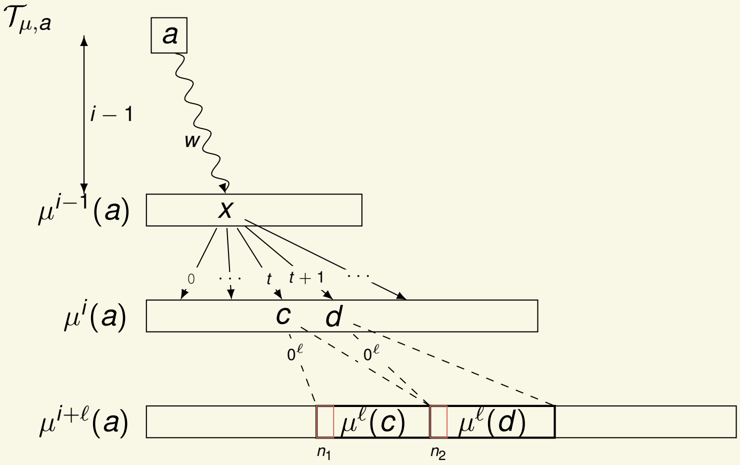
## Question

Given $\mu, u, r$, can we decide the positionality of the associated Dumont-Thomas numeration system?

If we consider $\mu\colon a \mapsto aab,\ b \mapsto a$ and $\rho\colon a \mapsto abb,\ b \mapsto ab$, both with seed $b|a$, the system associated with $\mu$ is positional but the one associated with $\rho$ is not.

# Sketch of the argument (1)

Assume the system is positional and let us compare the values of the words $wt\,0^\ell$ and $w(t+1)\,0^\ell$.



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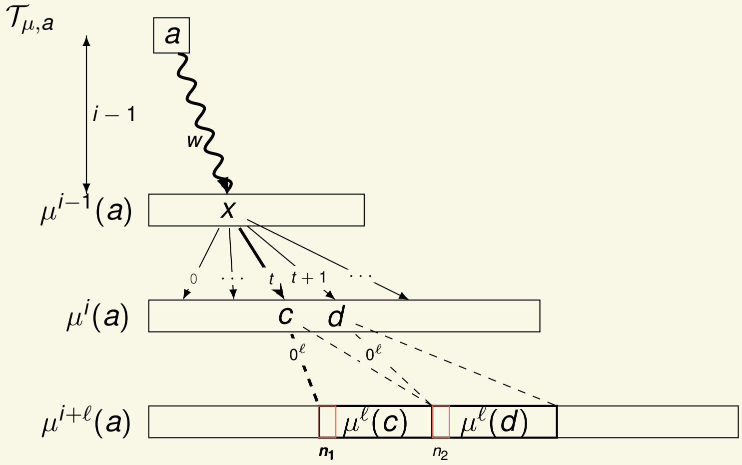


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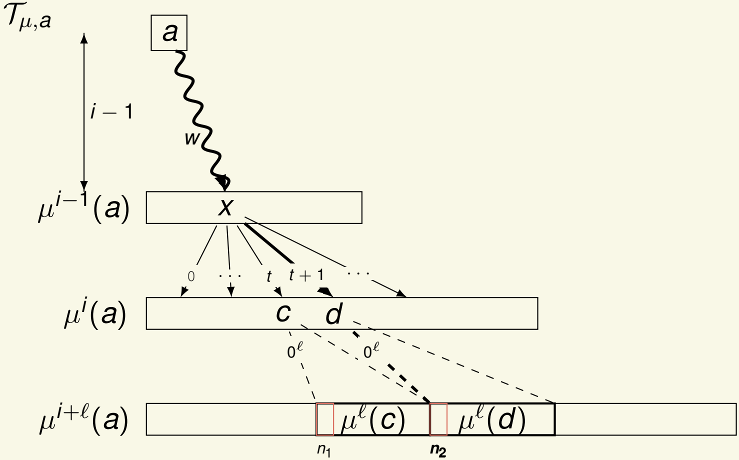


$\mathrm{val}(wt0^\ell) = n_1$, $\mathrm{val}(w(t+1)0^\ell) = n_2$,

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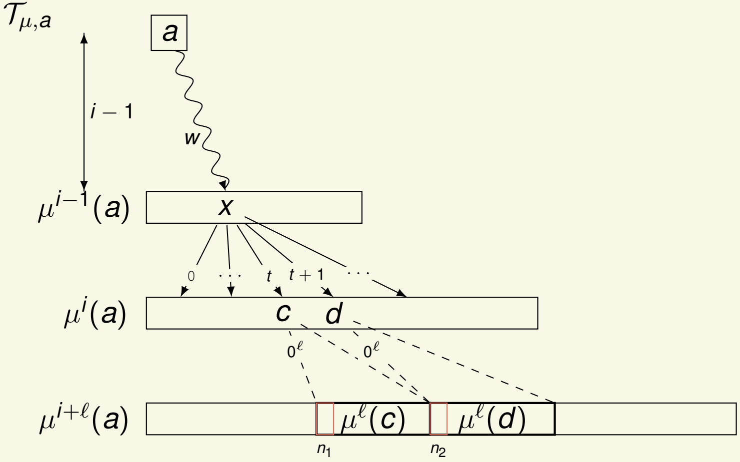


$\mathrm{val}(wt0^\ell) = n_1$, $\mathrm{val}(w(t+1)0^\ell) = n_2$, so $U_\ell = n_2 - n_1 = |\mu^\ell(c)|$.

# Sketch of the argument (2)

**Conjecture (K., Labbé, Stipulanti, 2024)**

If the Dumont–Thomas numeration system associated with $\mu$, $u$, and $r$ is positional, then $\left|\mu^\ell(c)\right| = U_\ell$ for every letter $c$ that has a younger sibling in $\mathcal{T}_{\mu,b|a}$.

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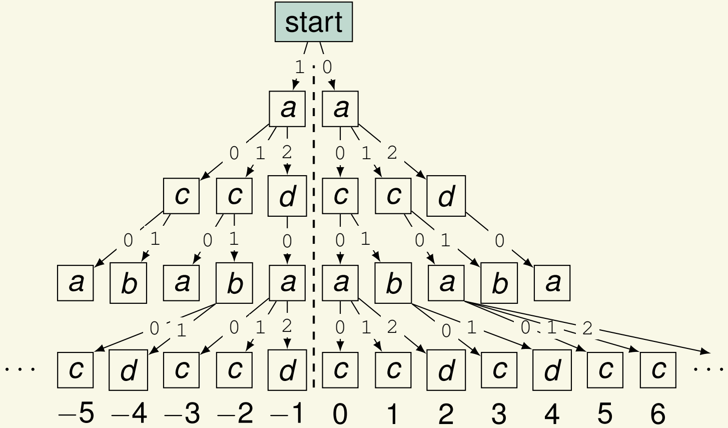
Other direction: If $\left|\mu^{\ell}(c)\right| = U_{\ell}$ for every letter $c$ that has a younger sibling in $\mathcal{T}_{\mu,b|a}$, then incrementing the digit at position $\ell$ in an expansion increases the value by $U_{\ell}$, so the system is indeed positional.

# Complications (1)

$\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$, seed $a|a$.

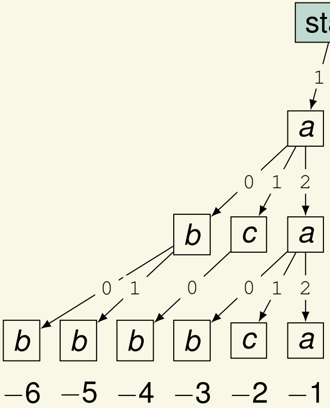



The system is positional for both values of $r$, despite the fact that $|\mu^\ell(a)| \neq |\mu^\ell(c)|$ for all odd $\ell$:

| $\ell$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\lvert\mu^\ell(a)\rvert$ | 1 | 3 | 5 | 13 | 21 |
| $\lvert\mu^\ell(c)\rvert$ | 1 | 2 | 5 | 8 | 21 |

# Complications (2)

$\mu$: $a \mapsto bca$, $b \mapsto bb$, $c \mapsto b$ with the seed $a|b$.




| $n$ | $\mathrm{rep}_{u,0}(n)$ |
|---:|---:|
| -1 | 1 |
| -2 | 11 |
| -3 | 10 |
| -4 | 110 |
| -5 | 101 |
| -6 | 100 |
| -7 | 1101 |
| weight | $\cdots 421$ |

Positional ($U_\ell = 2^\ell$, $V_0 = 1$, $V_\ell = 3.2^\ell$), despite the fact that $|\mu^\ell(b)| \neq |\mu^\ell(c)|$ for all $\ell \geq 1$.

# Complications (3)

In the first example, the letters *a* and *c* occur only at levels of a given parity in the tree, so the sketch can only be applied for half the values of $\ell$.

In the second example, the letter to the right of a *c* can never be part of a shortest path to a column, so the sketch cannot be applied to *c*.

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## Definition

For $j \in \{0, \ldots, p-1\}$, the set $E_j$ is the set of letters $c$ verifying one of the following conditions:

- There exists a node in $\mathcal{T}_{\mu,b|a}$ labelled by $c$ at a level congruent to $j \bmod p$, in a column other than $-2$, that has a younger sibling.
- The node at level $j$ and column $-2$ in $\mathcal{T}_{\mu,b|a}$ is labelled by $c$, it has a younger sibling labelled by $d$, and $\left|\mu^{p-j}(d)\right| > 1$.

# Main result

We let $\mu$ be a substitution, $u$ a periodic point with period $p$ and seed $b|a$, and $r \in \{0, \ldots, p-1\}$.

## Theorem (K., Labbé, Stipulanti 2025, WORDS)

The Dumont-Thomas numeration system associated with $\mu, u$ and $r$ is positional if and only if both of the following occur:

- The map $c \mapsto \left|\mu^\ell(c)\right|$ has a constant value $U_\ell$ over $E_j$ for all $\ell, j$ such that $\ell + j \equiv r \bmod p$.
- For $j \in \{0, \ldots, r-1\}$, if the node at level $j$ and column $-2$ in $\mathcal{T}_{\mu,b|a}$ is labelled by $c$ and has a younger sibling labelled by $d$ with $\left|\mu^{p-j}(d)\right| = 1$, then $\left|\mu^{r-j}(c)\right| = U_{r-j}$.

In this case, $(U_\ell)_\ell$ and $V_\ell = \left|\mu^\ell(b)\right|$ are the sequences of weights of the system.

# Corollaries (1)

In some cases, the complications outlined above do not occur. The first is the case of a primitive substitution.
We say that a letter is *non-final* in the substitution $\mu$ if it occurs in the image of any letter at any position other than the last one.

## Corollary

Let $\mu$ be a primitive substitution, $u$ be a periodic point of $\mu$ with seed $b|a$ and period $p$, and $r \in \{0, \ldots, p-1\}$. The Dumont-Thomas complement system associated with $\mu, u$ and $r$ is positional if and only if the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over the non-final letters in $\mu$ for every $\ell$.
In this case, the value of the constant is $U_\ell$, and $V_\ell = \left|\mu^\ell(b)\right|$.

# Corollaries (2)

The second case where no complication occurs is that of the original Dumont-Thomas numeration systems.

**Corollary**

Let $\mu$ be a substitution and $u$ be a right-infinite fixed point of $\mu$.
The Dumont-Thomas numeration system associated with $\mu$ and $u$ is positional if and only if the map $c \mapsto \left|\mu^{\ell}(c)\right|$ is constant over the non-final letters in $\mu$ for every $\ell$.
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## Example

Consider the substitution $\mu\colon a \mapsto abc$, $b \mapsto aac$, $c \mapsto a$. The non-final letters are $a$ and $b$, and we can show by induction that their images by $\mu^\ell$ have the same length for any $\ell$. Thus, the system is positional.

# Bertrand numeration systems

*Bertrand numeration systems* (Bertrand-Mathis, 1989; Charlier, Cisternino, Stipulanti, 2022) are a special case of greedy positional numeration systems, defined by either of the two properties:

- $w \in \mathsf{rep}(\mathbb{N}) \Leftrightarrow w\mathtt{0} \in \mathsf{rep}(\mathbb{N})$ for any nonempty $w$.
- The lexicographically largest words of each length in $\mathsf{rep}(\mathbb{N})$ are all prefixes of one another.

There are three kinds of Bertrand numeration systems:

- $U_\ell = \ell + 1$ (trivial).
- $U_\ell = d_1 U_{\ell-1} + d_2 U_{\ell-2} + \ldots + d_\ell U_0 + 1$ where $d_1 d_2 \cdots$ is the quasi-greedy Rényi representation of 1 in some base $\beta$ (canonical).
- $U_\ell = d_1 U_{\ell-1} + d_2 U_{\ell-2} + \ldots + d_\ell U_0 + 1$ where $d_1 d_2 \cdots$ is the greedy Rényi representation of 1 in some simple Parry base $\beta$ (non-canonical).

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The original Dumont-Thomas numeration systems have the above properties.

# Simplifying the morphism

Let us study the case where $\mu$ has a right-infinite fixed point $u$.

**Lemma**

*Let $\mu\colon A^* \to A^*$ be a substitution such that the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over the non-final letters in $\mu$ for every integer $\ell \geq 0$. Then there exist an alphabet $B \subseteq A$, a substitution $\nu\colon B^* \to B^*$ such that $\nu$ has only one non-final letter and a coding $\phi\colon A \to B$ such that $\nu$ and $\phi(u)$ define the same Dumont-Thomas numeration system as $\mu$ and $u$.*

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The substitution $\mu\colon a \mapsto abc,\ b \mapsto aac,\ c \mapsto a$ mentioned above, with the seed $a$, defines the same numeration system as $\nu\colon a \mapsto aac,\ c \mapsto a$ with the seed $a$.

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A substitution that has a fixed point and only one non-final letter is of the form

$$\mu : a_1 \mapsto a_1^{d_1} a_2, a_2 \mapsto a_1^{d_2} a_3, \ldots, a_n \mapsto a_1^{d_n} a_k$$

for some $n \geq 1$, $1 \leq k \leq n$, $d_1 > 0$ and $d_2, \ldots, d_n \geq 0$.

# Fabre substitutions

Another approach to Bertrand numeration systems (Fabre, 1995): If $\beta$ is a Parry number and the quasi-greedy Rényi representation of 1 is $(d_1 \cdots d_{k-1})(d_k \cdots d_n)^\omega$, define the substitution

$$\mu_\beta \colon 1 \mapsto 1^{d_1}2,\ 2 \mapsto 1^{d_2}3 \ldots, n \mapsto 1^{d_n}k.$$

For instance, for $\beta$ equal to the positive root of $x^3 - x^2 - x - 1 = 0$, the quasi-greedy representation of 1 is $(110)^\omega$ and we find

$$\mu_\beta \colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1.$$

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$$\mu_\beta \colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1.$$

## Theorem (Fabre, 1995)

*$\left|\mu_\beta^\ell(1)\right|$ is the $U_\ell$ defined for the canonical Bertrand numeration system for $\beta$. This numeration system is the Dumont-Thomas numeration system associated with $\mu_\beta$ and $\mu_\beta^\infty(1)$.*

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Consider $\mu\colon a \mapsto aab,\ b \mapsto aaaa$. The associated Dumont-Thomas system is positional, with weight sequence $1, 3, 10, 32, \ldots$. However, it is not equal to a Bertrand numeration system: $\mathrm{rep}(9) = \mathtt{23}$, but it should be $\mathtt{30}$.

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If such a system were Bertrand associated with some $\beta$, the quasi-greedy $\beta$-representation of 1 would be $(23)^\omega$, which breaks the Parry conditions.

# Equivalence between the two systems (2)

## Proposition (K., Labbé, Stipulanti 2025)

*Let $\mu$ be a substitution of the form*

$$\mu : a_1 \mapsto a_1^{d_1} a_2, a_2 \mapsto a_1^{d_2} a_3, \ldots, a_n \mapsto a_1^{d_n} a_k$$

*for some $n \geq 1$, $1 \leq k \leq n$, $d_1 > 0$ and $d_2, \ldots, d_n \geq 0$. Construct the word $d_1 d_2 \cdots = d_1 \cdots d_{k-1}(d_k \cdots d_n)^\omega$. The Dumont–Thomas numeration system associated with $\mu$ and the seed $a_1$ is equal to a Bertrand numeration system if and only if we have $d_i d_{i+1} \cdots \preccurlyeq_{lex} d_1 d_2 \cdots$ for each $i \geq 1$.*

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Thank you!