Simulations de processus d'Hermite et applications

Journées RT^2 ANAIS-MAIAGES 2025 : Aléatoire et Fractales – Vannes

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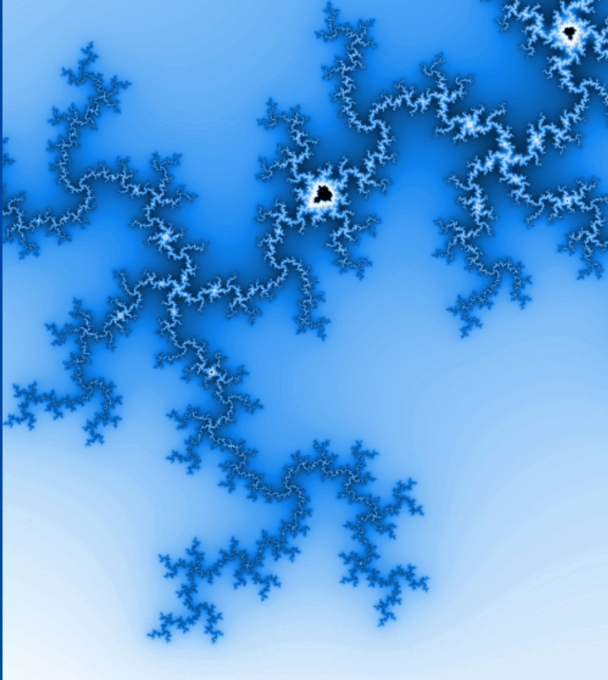
Mathématique



(A. Ayache, J. Hamonier, L. L. - 2024)

The random series

$$X_{\mathbf{h},J}^{(d)}(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_0^t \prod_{\ell=1}^d \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds, \tag{1}$$

is almost surely, for each $J \in \mathbb{N}$, uniformly convergent in t on each compact interval $I \subset \mathbb{R}_+$. Moreover, for all such I, there exists an almost surely finite random variable (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\begin{aligned}\|X_{\mathbf{h}}^{(d)} - X_{\mathbf{h},J}^{(d)}\|_{I,\infty} &= \left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{I,\infty} \\ &\leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}.\end{aligned}$$

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Hermite processes

Definition

Given $d \in \mathbb{N}$ and $H \in (1/2, 1)$, the Hermite process of order d and Hurst parameter H is defined as

$$\left\{ \frac{1}{c_H} \int_{\mathbb{R}^d} \left(\int_0^t \prod_{\ell=1}^{d} (s - x_\ell)_+^{\frac{H-1}{d} - \frac{1}{2}} \, ds \right) dB(x_1) \dots dB(x_d) \right\}_{t \geq 0}$$

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- $d = 1$, it is the Fractional Brownian Motion;
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Generalized Hermite processes

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Given $d \in \mathbb{N}$ and $\mathbf{h} := (h_1, \dots, h_d)$ whose coordinates satisfy $h_1, \cdots, h_d \in (1/2, 1)$ and $\sum_{\ell=1}^{d} h_\ell > d - \frac{1}{2}$, the generalized Hermite process is defined by

$$\left\{ \{X_{\mathbf{h}}^{(d)}(t)\}_{t\geq 0} := \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} \int_{\mathbb{R}^d} \left(\int_0^t \prod_{\ell=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2} \, ds \right) dB(x_1) \dots dB(x_d) \right\}_{t\geq 0}$$

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It has stationary increments, has long-range dependence and is H-self-similar for

$$H := \sum_{\ell=1}^d h_\ell - d + 1.$$

Brownian motion in the Schauder basis

Consider the triangle function

$$\Lambda(x) := \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

and, for all $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j - 1\}$, $\Lambda_{j,k}(x) = 2^{-j/2}\Lambda(2^j x - k)$.

If $\{\varepsilon\} \cup \{\varepsilon_{j,k}\}_{\in \mathbb{N}, k \in \{0,\ldots,2^j-1\}}$ is a family of i.i.d. $\mathcal{N}(0,1)$ random variables,

$$\left\{ B_t := \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \varepsilon_{j,k} \Lambda_{j,k}(t) + \varepsilon t \right\}_{t \in [0,1]} \tag{2}$$

is a Brownian motion on $[0,1]$.

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$d \geq 3$

Pipiras, in his paper from 2004, raised the question of providing a wavelet-type expansion for any Hermite processes.

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Here, ϕ and ψ stand for, respectively, the univariate scaling function and mother wavelet associated with a Meyer orthonormal wavelet basis of $L^2(\mathbb{R})$. We recall that ϕ and ψ belong to the Schwartz class $\mathscr{S}(\mathbb{R})$ and

$$\operatorname{supp}\widehat{\phi} \subseteq \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \quad \text{and} \quad \operatorname{supp}\widehat{\psi} \subseteq \left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right] \setminus \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right).$$

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For all $h \in [1/2, +\infty)$, the fractional primitive of ψ of order $h-1/2$ is the function $\psi_h \in \mathscr{S}(\mathbb{R})$ defined through its Fourier transform by:

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and

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the "ε" random variables are defined by

$$\varepsilon_{\mathbf{j},\mathbf{k}} := \int_{\mathbb{R}^d}' \psi_{\mathbf{j},\mathbf{k}}(x_1, \ldots, x_d) \, dB(x_1) \ldots dB(x_d).$$

with

$$\psi_{\mathbf{j},\mathbf{k}} = \bigotimes_{\ell=1}^{d} 2^{\frac{j_\ell}{2}} \psi(2^{j_\ell} \bullet - k_\ell).$$

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Definition

The *fractional scaling function* of order $\delta \in (-\infty, +\infty)$ of a univariate Meyer scaling function ϕ is the function $\Phi_\Delta^{(\delta)} \in \mathscr{S}(\mathbb{R})$ defined through its Fourier transform by:

$$\widehat{\Phi}_\Delta^{(\delta)}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^\delta \widehat{\phi}(\xi) \ \forall \xi \neq 0 \text{ and } \widehat{\Phi}_\Delta^{(\delta)}(0) = 1.$$

$\widehat{\Phi}_\Delta^{(\delta)} \in \mathscr{S}(\mathbb{R})$

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If

$$\mu_{J,\mathbf{k}} := 2^{J\frac{d}{2}} \int'_{\mathbb{R}^d} \phi(2^J x_1 - k_1) \cdots \phi(2^J x_d - k_d)\, dB(x_1) \ldots dB(x_d),$$

we have

$$\sigma^{(\mathbf{h})}_{J,\mathbf{k}} := \sum_{\mathbf{p} \in \mathbb{N}_0^d} \Big(\prod_{l=1}^{d} \gamma_{p_l}^{(h_l - 1/2)} \Big) \mu_{J,\mathbf{k}-\mathbf{p}}$$

where, for each $p \in \mathbb{N}$,

$$\gamma_p^{(\delta)} = \frac{\delta\, \Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}.$$

Let us consider, for $J \in \mathbb{Z}$, the sequence $(g^{\phi}_{J,k})_{k\in\mathbb{Z}}$ of i.i.d. $\mathcal{N}(0,1)$ Gaussian random variable defined, for all $k \in \mathbb{Z}$, by

$$g^{\phi}_{J,k} := 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k)\, dB(x).$$

For such a J and $\delta \in (-1/2, 1/2)$, the usual Gaussian FARIMA $(0,\delta,0)$ sequence $(Z^{(\delta)}_{J,\ell})_{\ell\in\mathbb{Z}}$ associated to $(g^{\phi}_{J,k})_{k\in\mathbb{Z}}$ is given, for all $\ell \in \mathbb{Z}$, by

$$Z^{(\delta)}_{J,\ell} := g^{\phi}_{J,\ell} + \sum_{p=1}^{+\infty} \gamma^{(\delta)}_p\, g^{\phi}_{J,\ell-p}.$$

We have

$$\sigma^{(\mathbf{h})}_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P\in\mathscr{P}^{(d)}_m} \prod_{r=1}^{m} \mathbb{E}[Z^{(h_{\ell_r}-1/2)}_{J,k_{\ell_r}} Z^{(h_{\ell'_r}-1/2)}_{J,k_{\ell'_r}}] \prod_{s=m+1}^{d-m} Z^{(h_{\ell''_s}-1/2)}_{J,k_{\ell''_s}}, \tag{3}$$

where $\mathscr{P}^{(d)}_m$ is the finite set of all partitions of $\{1,\ldots,d\}$ with m (non ordered) pairs and $d-2m$ singletons and the indices ℓ_r, ℓ'_r and ℓ''_s are such that

$$P = \Big\{\{\ell_1,\ell'_1\},\ldots,\{\ell_m,\ell'_m\},\{\ell''_{m+1}\},\ldots,\{\ell''_{d-m}\}\Big\}.$$

In the particular case $d = 2$, $\sigma^{(\mathbf{h})}_{J,\mathbf{k}} = \sigma^{(h_1,h_2)}_{J,k_1,k_2}$ reduces to

$$\sigma^{(h_1,h_2)}_{J,k_1,k_2} = Z^{(h_1-1/2)}_{J,k_1} Z^{(h_2-1/2)}_{J,k_2} - \mathbb{E}[Z^{(h_1-1/2)}_{J,k_1} Z^{(h_2-1/2)}_{J,k_2}].$$

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Key ideas of the proof of the range of convergence

Objective: bounding

$$\left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{[0,T],\infty}$$

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$$\left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{[0,T],\infty}$$

General idea : dyadic intervals that are far away from $[0,T]$ should contribute less

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General idea : dyadic intervals that are far away from $[0,T]$ should contribute less

We consider, for all $1 \leq n \leq d$,

$$\aleph_{n,J} := \left\{ \mathbf{j}\in\mathbb{Z}^d : j_n \geq J \text{ and } \max_{1\leq\ell\leq d} j_\ell = j_n \right\} \text{ and } \beth^{j_n}_{n,T} := \left\{ \mathbf{k}\in\mathbb{Z}^d : k_n \in \mathbb{Z}; \exists \ell \in [[1,d]]\setminus\{n\}, \quad |k_\ell| > 2^{j_n+1}T \right\}$$

and

$$\left\| \sum_{\mathbf{j}\in\aleph_{n,J}} \sum_{\mathbf{k}\in\beth^{j_n}_{n,T}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds \right\|_{[0,T],\infty}$$

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Let a be a fixed real number satisfying $1/2 < a < 1$. For all $(j,k)\in\mathbb{Z}_+\times\mathbb{Z}$, let us denote by $B_{j,k}$ the interval

$$B_{j,k} := [k2^{-j} - 2^{-ja}, k2^{-j} + 2^{-ja}]. \tag{4}$$

For all $j\in\mathbb{N}$ and for all $t\in\mathbb{R}_+$, we consider the three disjoint subsets of $\mathbb{Z}$:

$$D_j^1(t) := \{k\in\mathbb{Z} : B_{j,k}\subseteq[0,t]\}, \tag{5}$$

$$D_j^2(t) := \{k\in\mathbb{Z}\setminus D_j^1(t) : B_{j,k}\cap[0,t]\neq\emptyset\}, \tag{6}$$

$$D_j^3(t) := \{k\in\mathbb{Z} : B_{j,k}\cap[0,t]=\emptyset\}. \tag{7}$$

Key ideas of the proof of the range of convergence

Objective: bounding

$$\left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{[0,T],\infty}$$

We consider for all $1 \leq n \leq d$ the three series

$$\left\| \sum_{\mathbf{j}\in\aleph_{n,J}} \sum_{\mathbf{k}:k_n\in D^m_{j_n}(t)} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{[0,T],\infty}$$

with $m \in \{1,2,3\}$

Key ideas of the proof of the range of convergence

Objective: bounding

$$\left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max\limits_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds \right\|_{[0,T],\infty}$$

CONCLUSION :

$$\left\| \sum_{\mathbf{j}\in\aleph_{n,J}} \sum_{\mathbf{k}:k_n\in D^1_{j_n}(t)} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds \right\|_{[0,T],\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}$$

and the convergence to 0 of the others terms is faster.

(A. Ayache, J. Hamonier, L. L. - 2024)

The random series

$$X^{(d)}_{\mathbf{h},J}(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sigma^{(\mathbf{h})}_{J,\mathbf{k}} \int_0^t \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds, \tag{4}$$

is almost surely, for each $J\in\mathbb{N}$, uniformly convergent in t on each compact interval $I\subset\mathbb{R}_+$. Moreover, for all such I, there exists an almost surely finite random variable (depending on I) for which one has, almost surely, for each $J\in\mathbb{N}$,

$$\begin{aligned}\|X^{(d)}_{\mathbf{h}} - X^{(d)}_{\mathbf{h},J}\|_{I,\infty} &= \left\| \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[[1,d]]} j_\ell \geq J}} 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \varepsilon_{\mathbf{j},\mathbf{k}} \int_0^t \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds \right\|_{I,\infty} \\ &\leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}.\end{aligned}$$

Simulation process

Definition

For all $J \in \mathbb{N}$, the *simulation process at scale* J of the generalized Hermite process $\{X_{\mathbf{h}}^{(d)}(t)\}_{t\in\mathbb{R}_+}$ is the process defined, for all $t \in \mathbb{R}_+$, by

$$S_{\mathbf{h},J}^{(d)}(t) = 2^{-J(h_1+\ldots+h_d-d+1)} \sum_{\mathbf{k}\in \mathscr{J}_J^1(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(s-k_\ell)\, ds. \tag{5}$$

with

$$\mathscr{J}_J^1(t) := \{\mathbf{k} \in (D_J^1(t))^d : \max_{\ell,\ell'\in[[1,d]]} |k_\ell - k_{\ell'}| \leq 2^{\varepsilon J}\}.$$

$$D_J^1(t) = \{k \in \mathbb{Z} : [k2^{-J} - 2^{-Ja}, k2^{-J} + 2^{-Ja}] \subseteq [0,t]\}$$

Convergence of the simulation process

A. Ayache, J. Hamonier, L.L.-2025+

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X_{\mathbf{h}}^{(d)} - S_{\mathbf{h},J}^{(d)}\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{6}$$

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For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\| X^{(d)}_{\mathbf{h}} - S^{(d)}_{\mathbf{h},J} \|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{6}$$

The same result holds true for the process obtained by linear interpolation

$$\widetilde{S}^{(d)}_{\mathbf{h},J}(t) := \frac{s_{m_0,J}}{|\lambda^{(a)}_{0,J}|} t \mathbb{1}_{\lambda^{(a)}_{0,J}}(t) \\ + \sum_{m \in \mathscr{I}_J} \left(2^J \left(s_{m+1,J} - s_{m,J}\right)(t - (m2^{-J} + 2^{-aJ})) + s_{m,J}\right) \mathbb{1}_{\lambda_{m,J}}(t),$$

Ideas of the proof

The convergence to zero of

$$\mathscr{G}_J^2 := \sup_{t\in[0,T]} \left\{ 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathscr{J}_J^2(t)} |\sigma_{J,\mathbf{k}}^{(\mathbf{h})}| \left| \int_0^t \prod_{\ell=1}^d \Phi_\Delta^{(h_\ell-1/2)}(2^J s-k_\ell)\, ds \right| \right\}$$

with

$$\mathscr{J}_J^2(t) := \{\mathbf{k}\in\mathbb{Z}^d : \exists\, n \text{ such that } k_n \in D_J^2(t)\}$$

and

$$\mathscr{G}_J^3 := \sup_{t\in[0,T]} \left\{ 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathscr{J}_J^3(t)} |\sigma_{J,\mathbf{k}}^{(\mathbf{h})}| \left| \int_0^t \prod_{\ell=1}^d \Phi_\Delta^{(h_\ell-1/2)}(2^J s-k_\ell)\, ds \right| \right\}$$

with

$$\mathscr{J}_J^3(t) := \{\mathbf{k}\in\mathbb{Z}^d : \exists\, n \text{ such that } k_n \in D_J^3(t)\}$$

is faster than the expected rate of convergence.

Ideas of the proof

The convergence to zero of

$$\mathscr{G}_J^1 := \sup_{t\in[0,T]} \left\{ 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in(D_J^1(t))^d} |\sigma_{J,\mathbf{k}}^{(\mathbf{h})}| \left| \int_{\mathbb{R}\setminus[0,t]} \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \right| \right\}$$

is faster than the expected rate of convergence.

Ideas of the proof

The convergence to zero of

$$\mathcal{G}_J := \sup_{t\in[0,T]} \left\{ 2^{-J(h_1+\ldots+h_d-d+1)} \sum_{\mathbf{k}\in(D_J^1(t))^d \setminus \mathscr{J}_J^1(t)} |\sigma_{J,\mathbf{k}}^{(\mathbf{h})}| \left| \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(s-k_\ell)\, ds \right| \right\}$$

is faster than the expected rate of convergence.

Interesting fact

Successive applications of the Parseval-Plancherel identity and the equality $\widehat{fg} = \widehat{f} \star \widehat{g}$, for all $f, g \in L^2$, give

$$\int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell - 1/2)}(s - k_\ell)\, ds = \int_{\mathbb{R}^{d-1}} e^{i(k_2-k_1)\xi} e^{i(k_3-k_2)\eta_3} \cdots e^{i(k_d-k_2)\eta_d} F(\xi, \eta_3, \dots, \eta_d)\, d\xi\, d\eta_3 \cdots d\eta_d$$

where we put

$$F : (\xi, \eta_3, \dots, \eta_d) \mapsto (2\pi)^{-d}\, \widehat{\Phi_{\Delta}^{(h_1-1/2)}}(\xi) \overline{\widehat{\Phi_{\Delta}^{(h_2-1/2)}}}(\xi - (\eta_3 + \cdots + \eta_d)) \prod_{\ell=3}^{d} \overline{\widehat{\Phi_{\Delta}^{(h_\ell-1/2)}}}(\eta_\ell).$$

Algorithm for the simulation of the Hermite parameter of order 3

Hermite3(J)

```
result[0]=sigmaJ(farima,IJ[0],IJ[0],IJ[0],H/2) × integralmatrix[0,0]
for index=1 to length(IJ)
    element=epaissi(IJ[index])
    part=sigmaJ(farima,element[0],element[0],element[0],H/2) × integralmatrix[0,0]
    for i=1 to length(element)
        part=part+3 × sigmaJ(farima,element[0],element[0],element[i],H/2) × integralmatrix[0,i]
        part=part+3 × sigmaJ(farima,element[0],element[i],element[i],H/2) × integralmatrix[0,i]
        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j],H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

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```

`farima` contains a simulation of the random variables $(Z_{J,\ell}^{(\frac{2H+1}{6})})_{\ell \in \mathscr{I}_J(T)}$

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        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j], H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

$\mathtt{sigmaJ}(\mathtt{farima}, k_1, k_2, k_3, \frac{H}{2})$ is computing

$$Z_{J,k_1}^{(\frac{2H+1}{6})} Z_{J,k_2}^{(\frac{2H+1}{6})} Z_{J,k_3}^{(\frac{2H+1}{6})} - \mathbb{E}[Z_{J,k_1}^{(\frac{2H+1}{6})} Z_{J,k_2}^{(\frac{2H+1}{6})}] Z_{J,k_3}^{(\frac{2H+1}{6})} - \mathbb{E}[Z_{J,k_1}^{(\frac{2H+1}{6})} Z_{J,k_3}^{(\frac{2H+1}{6})}] Z_{J,k_2}^{(\frac{2H+1}{6})} - \mathbb{E}[Z_{J,k_2}^{(\frac{2H+1}{6})} Z_{J,k_3}^{(\frac{2H+1}{6})}] Z_{J,k_1}^{(\frac{2H+1}{6})}$$

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        part=part+3 × sigmaJ(farima,element[0],element[i],element[i],H/2) × integralmatrix[0,i]
        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j],H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

IJ is a vector containing $\mathbb{N}\cap(2^{J(1-a)}-1,+\infty)$ with IJ[0]$= m_0$

Algorithm for the simulation of the Hermite parameter of order 3

Hermite3(J)

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        part=part+3 × sigmaJ(farima,element[0],element[i],element[i],H/2) × integralmatrix[0,i]
        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j],H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

For all $m \in I_J(T)$, `epaissi(m)` is a vector of size $\min\{\lfloor 2^{\varepsilon J} \rfloor, m - m_0\} + 1$ such that, for all $0 \leq i \leq \min\{\lfloor 2^{\varepsilon J} \rfloor, m - m_0\}$, `epaissi(m)[`$i$`]` $= m - i$.

Algorithm for the simulation of the Hermite parameter of order 3

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        part=part+3 × sigmaJ(farima,element[0],element[i],element[i],H/2) × integralmatrix[0,i]
        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j],H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

integralmatrix is a matrix of dimension $\lfloor 2^{\varepsilon J} \rfloor + 1$ such that, for all $0 \leq k, \ell \leq \lfloor 2^{\varepsilon J} \rfloor$, integralmatrix[k,l] is the value of

$$\iint_{[-\frac{4\pi}{3},\frac{4\pi}{3}]^2} e^{i\xi k} e^{i\eta \ell} \widehat{\phi}(\xi)\widehat{\phi}(\xi-\eta)\widehat{\phi}(\eta) \left(\frac{\sin(\xi/2)}{\xi/2} \frac{\sin((\xi-\eta)/2)}{(\xi-\eta)/2} \frac{\sin(\eta/2)}{\eta/2} \right)^{\frac{2H+1}{6}} d\xi d\eta$$

Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

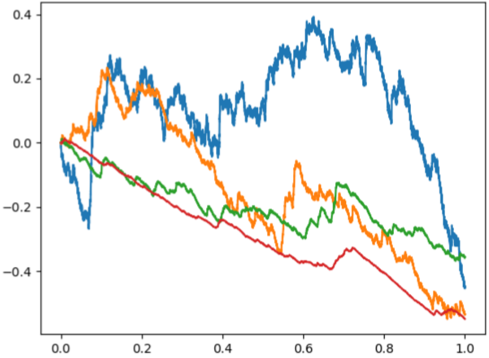

Figure: Paths of the Rosenblatt process of Hurst parameter $0,6$ (blue), $0,7$ (orange), $0,8$ (green) and $0,9$ (red).

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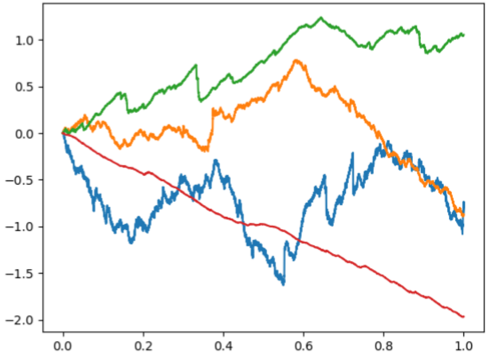

Figure: Paths of the Hermite process of order 3 of Hurst parameter $0,6$ (blue), $0,7$ (orange), $0,8$ (green) and $0,9$ (red).

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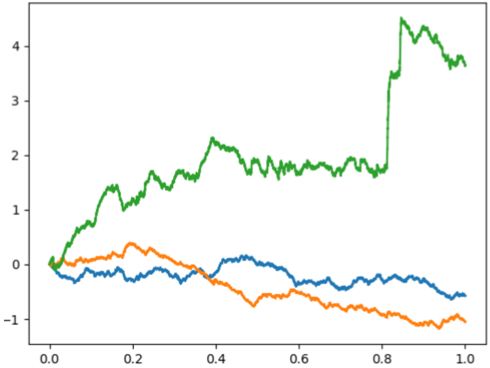

Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter $0{,}6$.

Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

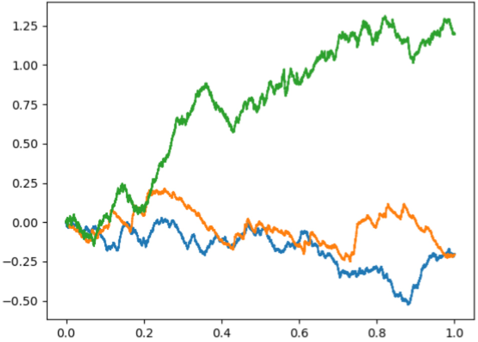

Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter $0,7$.

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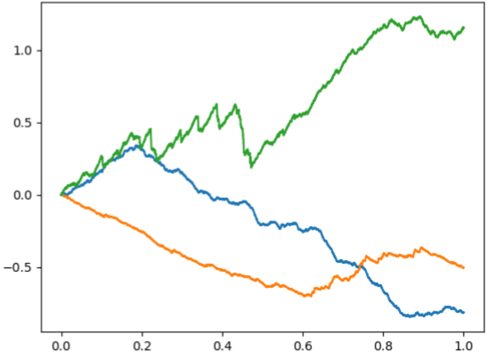

Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter $0{,}8$.

Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

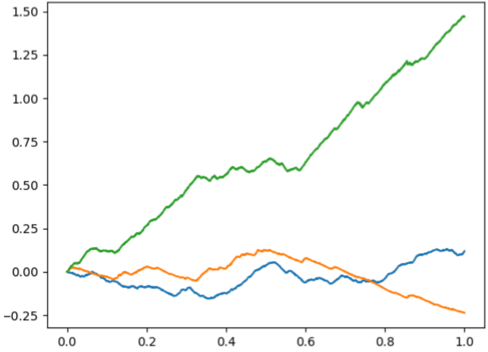

Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter $0{,}9$.

What about generalized processes ?

Hermite3(J)

```
result[0]=sigmaJ(farima,IJ[0],IJ[0],IJ[0],H/2) × integralmatrix[0,0]
for index=1 to length(IJ)
    element=epaissi(IJ[index])
    part=sigmaJ(farima,element[0],element[0],element[0],H/2) × integralmatrix[0,0]
    for i=1 to length(element)
        part=part+3 × sigmaJ(farima,element[0],element[0],element[i],H/2) × integralmatrix[0,i]
        part=part+3 × sigmaJ(farima,element[0],element[i],element[i],H/2) × integralmatrix[0,i]
        for j=i+1 to length(element)
            part=part+6 × sigmaJ(farima,element[0],element[i],element[j],H/2) × integralmatrix[i,j-i]
    result[index]=result[index-1]+part
```

What about generalized processes ?

GenHermite3(J)

```
result[0]=sigmaJ(f1,f2,f3,IJ[0],IJ[0],IJ[0],h) × integralmatrix[0,0]
for index=1 to length(IJ)
    element=epaissi(IJ[index])
    part=sigmaJ(f1,f2,f3,element[0],element[0],element[0],h) × integralmatrix[0,0]
    for i=1 to length(element)
        part=part+ sigmaJ(f1,f2,f3,element[0],element[0],element[i],h) × integralmatrix[0,-i]
        part=part+ sigmaJ(f1,f2,f3,element[0],element[i],element[0],h) × integralmatrix[-i,i]
        part=part+ sigmaJ(f1,f2,f3,element[i],element[0],element[0],h) × integralmatrix[i,0]
        part=part+ sigmaJ(f1,f2,f3,element[0],element[i],element[i],h) × integralmatrix[-i,0]
        part=part+ sigmaJ(f1,f2,f3,element[i],element[0],element[i],h) × integralmatrix[i,-i]
        part=part+ sigmaJ(f1,f2,f3,element[i],element[i],element[0],h) × integralmatrix[0,i]
        for j=1 to length(element)
            if j ≠ i
                part=part+ sigmaJ(f1,f2,f3,element[0],element[i],element[j],h) × integralmatrix[-i,j-i]
                part=part+ sigmaJ(f1,f2,f3,element[0],element[j],element[i],h) × integralmatrix[-j,j-i]
                part=part+ sigmaJ(f1,f2,f3,element[i],element[0],element[j],h) × integralmatrix[i,-j]
                part=part+ sigmaJ(f1,f2,f3,element[j],element[0],element[i],h) × integralmatrix[j,-i]
                part=part+ sigmaJ(f1,f2,f3,element[i],element[j],element[0],h) × integralmatrix[i-j,j]
                part=part+ sigmaJ(f1,f2,f3,element[j],element[i],element[0],h) × integralmatrix[j-i,i]
    result[index]=result[index-1]+part
```

What about generalized processes ?

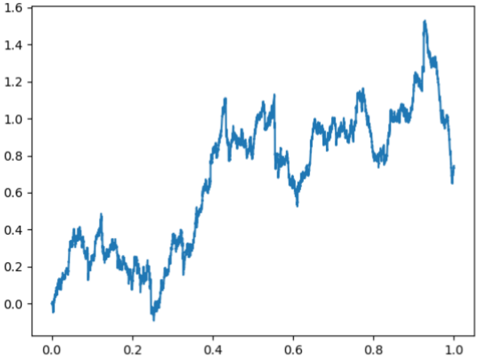

Figure: Sample path of the generalized Hermite of order 3 with index $\mathbf{h} = (0,8, 0,85, 0,9)$.

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- the existence of long-range dependence;
- the Hölder regularity.

$\Rightarrow$ Need of statistical estimator.

Two standard strategies

- Estimator based on quadratic variation

$$S_N = 2^{-N} \sum_{k=0}^{2^N-1} (X(\frac{k+1}{2^N}) - X(\frac{k}{2^N}))^2$$

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$$\widehat{H}_N^{\Psi} := -\frac{1}{2}\frac{d\left(\sum_{M=1}^{d}\log\widehat{S}_{N,M}\log M\right) - \left(\sum_{M=1}^{d}\log\widehat{S}_{N,M}\right)\left(\sum_{M=1}^{d}\log M\right)}{d\left(\sum_{M=1}^{d}(\log M)^2\right) - \left(\sum_{M=1}^{d}\log M\right)^2} - \frac{1}{2} \to H \text{ a.s.}$$

Major drawback

Asymptotic distribution of these estimators

- $d = 1$ and $H \leq \frac{3}{4}$ these estimators are asymptotically normal.

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- if $d > 1$ these estimators are NOT asymptotically normal.

Distance between random variables

General form

$$d_{\mathscr{A}}(F,G) = \sup\{|\mathbb{E}[h(F)] - \mathbb{E}[h(G)]| : h \in \mathscr{A}\}$$

for some appropriate class $\mathscr{A}$ of Borel-measurable complex valued functions.

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Example

- If $\mathscr{A}$ is the set of functions of the form $\mathbb{1}_{(-\infty,z_1]\times\cdots\times(-\infty,z_d]}$, with $z \in \mathbb{R}^d$, we get the so-called Kolmogorov distance d_{Kol};

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- If $\mathscr{A}$ is the set of Lipschitz functions with Lipschitz norm less than or equal to 1, we get the so-called Wasserstein distance d_{W}.

Malliavin calculus in a nutshell

Smooth random variables

The set $\mathscr{S}$ of random variables of the form

$$F = f(I_1(h_1), \dots, I_1(h_d))$$

with $d \in \mathbb{N}^*$, $h_1, \dots, h_d \in L^2(\mathbb{R})$ and $f : \mathbb{R}^d \to \mathbb{R}$ a C^∞-function with all partial derivatives having at most polynomial growth.

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$$DF = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(I_1(h_1), \ldots, I_1(h_d))h_j$$

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"Sobolev space"

For all $q \in [1,\infty)$, $\mathbb{D}^{1,q}$ is the closure of $\mathscr{S}$ with respect to the norm

$$\|\cdot\|_{1,q} : F \mapsto \mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|^q_{L^2(\mathbb{R})}].$$

One can extend the definition of D to $\mathbb{D}^{1,q}$.

Quantitative CLT using Stein-Malliavin calculus

Theorem (Nourdin, Peccati, Réveillac - 2010)

Let $m \geq 1$ be an integer number and consider a m-dimensional random vector $F = (F_1, \ldots, F_m)$. Assume $F_j \in \mathbb{D}^{1,4}$ for every $j = 1, \ldots m$. Let $C \in M_m(\mathbb{R})$ be a symmetric and positive definite matrix and let $Z \sim N(0, C)$. Then

$$d_W(F, Z) \leq C \sqrt{\sum_{j,k=1}^{m} \mathbf{E}\left[(C_{j,k} - \langle DF_j, D(-L)^{-1} F_k \rangle)^2\right]}. \tag{7}$$

Where we use the notation

$$(-L)^{-1}\left(\mathbb{E}[F] + \sum_{d=1}^{+\infty} I_d(f_d)\right) = \sum_{d=1}^{+\infty} \frac{1}{d} I_d(f_d).$$

General facts

Definition

Two square integrable random variables F and G admitting the chaos expansion $F=\sum_{n\geq 0} I_n(f_n), G=\sum_{n\geq 0} I_n(g_n)$, where $f_n, g_n \in L^2(\mathbb{R}^n)$ are symmetric for every $n\geq 0$, are strongly independent if for every $m,n\geq 0$, the random variables $I_n(f_n)$ and $I_m(g_m)$ are independent.

Lemma (Bourguin, Diez, Tudor)

Assume that F and G are two strongly independent random variables in $\mathbb{D}^{1,2}$. Then

1. $-\langle DF, D(-L)^{-1}G\rangle = \langle D(-L)^{-1}F, DG\rangle = 0.$
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We need to “create” independence!

Creating independence

Recalling

$$X_H^{(d)}(t) = \frac{1}{c_H}\int_{\mathbb{R}^d}\left(\int_0^t \prod_{\ell=1}^d (s-x_\ell)_+^{\frac{H-1}{d}-\frac{1}{2}}\,ds\right) dB(x_1)\dots dB(x_d)$$

and writing

$$f_s(x_1,\dots,x_d) = \frac{1}{c_H}\prod_{\ell=1}^d (s-x_\ell)_+^{\frac{H-1}{d}-\frac{1}{2}}$$ and I_d for the d-multiple stochastic integral

we get,

$$\Delta_{2^N,k} := X_H^{(d)}\left(\frac{k+1}{2^N}\right) - X_H^{(d)}\left(\frac{k}{2^N}\right) = I_d\left(\mathbb{1}^{\otimes d}_{(-\infty,2^{-N}(k+1))}\int_{\frac{k}{2^N}}^{\frac{k+1}{2^N}} f_s\,ds\right)$$

and, by Fubini-type argument, and using the compact support of the wavelet

$$c((M2^N)^{-1},k) = \sqrt{(M2^N)^{-1}}\, I_d\left(\mathbb{1}^{\otimes d}_{(-\infty,(M2^N)^{-1}(k+1))}\int_{\mathbb{R}}\Psi(x)\int_{(M2^N)^{-1}k}^{(M2^N)^{-1}(x+k)} f_s\,ds\,dx\right). \qquad (8)$$

Strategy – First step

Instead of considering all increments $(\Delta_{2^N,k})_{0\leq k\leq 2^N-1}$ or wavelet coefficients $(c((M2^N)^{-1},k))_{0\leq k\leq M2^N-1}$, we only work with a proportion of them

$$\Delta_{2^N,k2^{\lfloor N^\beta\rfloor}} \text{ and } c((M2^N)^{-1},kM2^{\lfloor N^\beta\rfloor}) \text{ for } k\in\mathscr{L}_N := \left[1,2^{N-[N^\beta]}\right] \text{ and } 0<\beta<1$$

Strategy – First step

$$\Delta_{2^N,k2^{\lfloor N^\beta \rfloor}} \text{ and } c((M2^N)^{-1}, kM2^{\lfloor N^\beta \rfloor}) \text{ for } k \in \mathscr{L}_N := \left[1, 2^{N-\lfloor N^\beta \rfloor}\right] \text{ and } 0 < \beta < 1$$

In this way, the random variables

$$\left(\widetilde{\Delta}_{2^N,k2^{\lfloor N^\beta \rfloor}} := I_d\left(\mathbb{1}^{\otimes d}_{(2^{-N}((k-1)2^{\lfloor N^\beta \rfloor}+1),2^{-N}(k2^{\lfloor N^\beta \rfloor}+1))} \int_{\frac{k}{2^N}}^{\frac{k+1}{2^N}} f_s ds\right)\right)_{k \in \mathscr{L}_N}$$

are independent as well as the random variables

$$\left(\widetilde{c}((M2^N)^{-1}, kM2^{\lfloor N^\beta \rfloor}) := \sqrt{(M2^N)^{-1}}\, I_d\left(\mathbb{1}^{\otimes d}_{(2^{-N}((k-1)M2^{\lfloor N^\beta \rfloor}+1),2^{-N}(k2^{\lfloor N^\beta \rfloor}+1))} \int_{\mathbb{R}} \Psi(x) \int_{(M2^N)^{-1}k}^{(M2^N)^{-1}(x+k)} f_s ds\, dx\right)\right)_{k \in \mathscr{L}_N}$$

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while

$$\mathbb{E}\left[\left(\widehat{\Delta}_{2^N,k2^{\lfloor N^\beta\rfloor}}\right)^2\right]\le C2^{-2NH}2^{\lfloor N^\beta\rfloor\frac{2H-2}{d}} \text{ and } \mathbb{E}\left[\left(\widehat{c}((M2^N)^{-1},kM2^{\lfloor N^\beta\rfloor})\right)^2\right]\le C2^{-N(2H+1)}2^{\lfloor N^\beta\rfloor\frac{2H-2}{d}}$$

Strategy – Second step

Applying the Stein-Malliavin approach with

$$S_N = \frac{1}{|\mathscr{L}_N|} \sum_{k \in \mathscr{L}_N} (\Delta_{2^N, k2^{\lfloor N^\beta \rfloor}})^2$$

or

$$S_{N,M} \frac{1}{|\mathscr{L}_N|} \sum_{k \in \mathscr{L}_N} c((M2^N)^{-1}, kM2^{\lfloor N^\beta \rfloor})^2$$

lead to unsatisfactory result.

Strategy – Second step

SOLUTION : choose again a proportion of the coefficients : that is take $0 < \gamma < \beta < 1$, $\mathscr{L}_{N,\gamma} := \mathscr{L}_N \cap [1, 2^{[N^\gamma]}]$ and the variations

$$S_N = \frac{1}{|\mathscr{L}_{N,\gamma}|} \sum_{k \in \mathscr{L}_{N,\gamma}} (\Delta_{2^N, k2^{\lfloor N^\beta \rfloor}})^2$$

or

$$S_{N,M} \frac{1}{\mathscr{L}_{N,\gamma}|} \sum_{k \in \mathscr{L}_{N,\gamma}} c((M2^N)^{-1}, kM2^{\lfloor N^\beta \rfloor})^2$$

Conclusion

$$S_N = \frac{1}{|\mathscr{L}_{N,\gamma}|} \sum_{k \in \mathscr{L}_{N,\gamma}} (\Delta_{2^N, k2^{\lfloor N^\beta \rfloor}})^2$$

Theorem (A. Ayache, C.A. Tudor, 2024)

The estimator

$$\widehat{H}_N = -\frac{\log(S_N)}{2N\log(2)}$$

is strongly consistent and

$$2N(\log 2)\sqrt{|\mathscr{L}_{N,\gamma}|}(H - \widehat{H}_N) \to^{(d)} \mathcal{N}(0, \mathbb{E}[X_H^{(d)}(1)^4] - 1)$$

Conclusion

$$S_{N,M} = \frac{1}{|\mathscr{L}_{N,\gamma}|} \sum_{k\in\mathscr{L}_{N,\gamma}} c((M2^N)^{-1}, kM2^{\lfloor N^\beta \rfloor})^2$$

Theorem (L. L., C.A. Tudor, 2025+)

The estimator

$$-\frac{1}{2}\frac{d\left(\sum_{M=1}^d \log\widehat{S}_{N,M}\log M\right) - \left(\sum_{M=1}^d \log\widehat{S}_{N,M}\right)\left(\sum_{M=1}^d \log M\right)}{d\left(\sum_{M=1}^d (\log M)^2\right) - \left(\sum_{M=1}^d \log M\right)^2} - \frac{1}{2}$$

is strongly consistent and

$$\sqrt{|\mathscr{L}_{N,\gamma}|}(H - \widehat{H}_N) \xrightarrow[N\to\infty]{(d)} N(0,\sigma^2),$$

with $\sigma^2 = \frac{1}{4}(L_d^T L_d)^{-1} L_d K L_d^T (L_d^T L_d)^{-1}$, where K is a variance-covariance matrix and L_d is the matrix with $(L_d)_{M,1} = \log M$ and $(L_d)_{M,2} = 1$, for $M = 1,\dots,d$.

Numerical experiments

We need to compute

$$\widehat{S}_{N,M} = \frac{1}{|\mathscr{L}_{N,\gamma}|} \sum_{k \in \mathscr{L}_{N,\gamma}} \left(\sqrt{\frac{1}{M2^N}} \frac{1}{2^N} \sum_{\ell=1}^{2^N} \Psi\left(\frac{\ell}{2^N}\right) X_H^{(d)}\left(\frac{\ell 2^{-N} + Mk2^{[N^\beta]}}{M2^N}\right) \right)$$

using the simulation process

$$S_{\mathbf{h},J}^{(d)}(t) = 2^{-JH} \sum_{\mathbf{k} \in \mathscr{J}_J^1(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell - 1/2)}(s - k_\ell)\, ds.$$

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- Approximatively 2^J values are simulated so it seems reasonable to work with $d2^{2N} \approx 2^J$.
- $d \gg$ means that we use more terms in the the regression but also we use many "close" observations.
- increasing the value of the parameter γ improves the rate of convergence to the normal distribution but reduces the number of data used in the statistical inference.

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- take $\gamma < \beta$ such that $|N - [N^\beta] - [N^\gamma]| \leq 1$;

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- take $\gamma < \beta$ such that $|N - [N^\beta] - [N^\gamma]| \leq 1$;
- avoid the small values of d;
- $d2^{2N} \leq 2^{J+2}$.

Results

		0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
1	m	0.571	0.581	0.663	0.690	0.746	0.793	0.860	0.908	0.979
	s	0.110	0.109	0.095	0.108	0.117	0.106	0.087	0.103	0.107
2	m	0.616	0.676	0.691	0.742	0.751	0.813	0.834	0.906	0.937
	s	0.153	0.159	0.145	0.147	0.142	0.171	0.135	0.142	0.150
3	m	0.632	0.710	0.734	0.752	0.770	0.800	0.857	0.889	0.929
	s	0.193	0.208	0.186	0.182	0.177	0.138	0.143	0.199	0.123

Table: $J = 17$; $N = 7$ and $d = 32$

Results

		0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
1	m	0.551	0.600	0.650	0.700	0.749	0.803	0.851	0.905	0.962
	s	0.022	0.019	0.018	0.016	0.018	0.018	0.018	0.021	0.044
2	m	0.604	0.626	0.668	0.713	0.770	0.821	0.873	0.929	1.011
	s	0.030	0.032	0.038	0.038	0.042	0.039	0.045	0.056	0.541
3	m	0.500	0.515	0.557	0.603	0.668	0.714	0.768	0.843	0.899
	s	0.500	0.042	0.053	0.060	0.504	0.067	0.060	0.043	0.051

Table: Quadratic estimation $J = 17;\ N = 10$