# Local stability of kidney exchanges

Élise Vandomme
Joint work with Yves Crama & Marie Baratto

HEC Liege Research Day 2025

# Joint work with

Yves Crama

Professor
HEC Liege
Liege University

Marie Baratto

PostDoc
at Rotterdam School of Management
Erasmus University

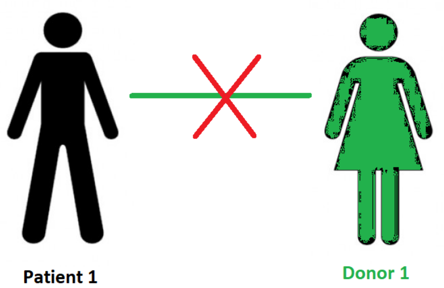
# Kidney Exchange Programs (KEP)

Patient with a serious kidney disease may resort to:

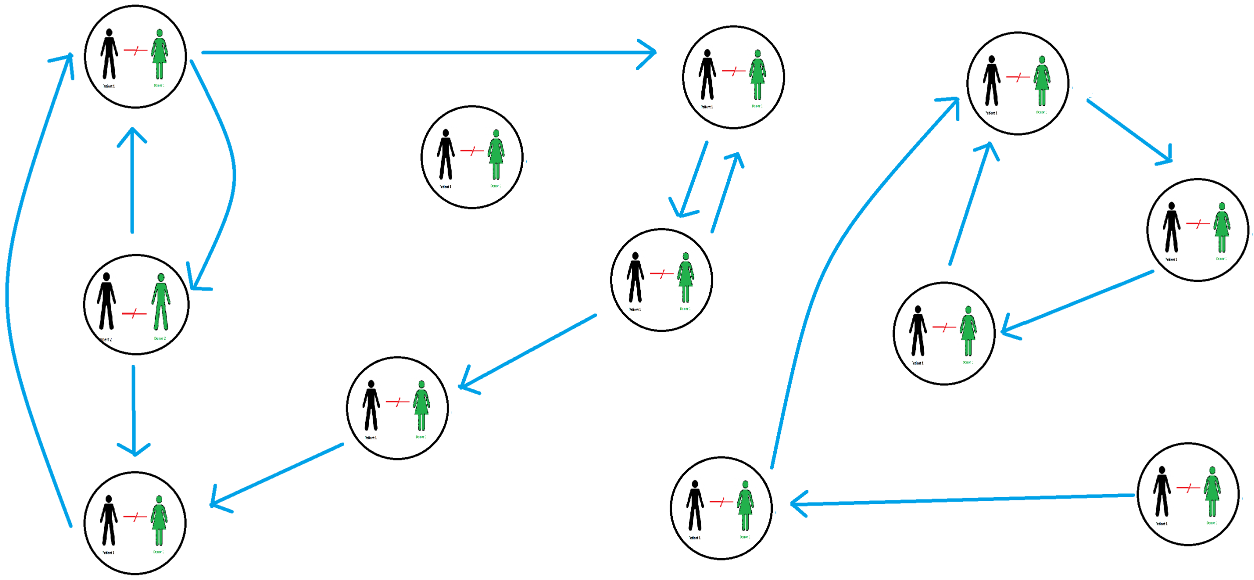
- Dialysis
- Transplant from a deceased donor
- **Transplant from a willing donor**

Patient might not be compatible with the donor: e.g.,

- Blood incompatibility
- Tissue type incompatibility



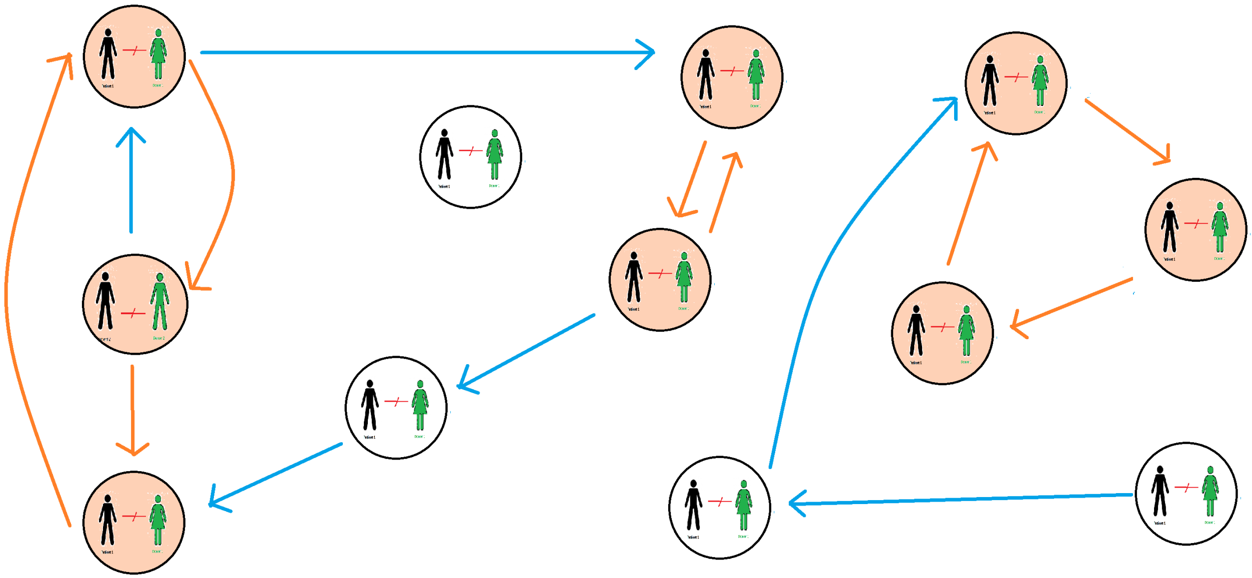
# KEP - Compatibility graph



G=(V,A) where:

- $V$ set of vertices, consisting of all patient-donor pairs.
- $A$, the set of arcs, designating compatibilities between the vertices. Two vertices $i$ and $j$ are connected by arc $(i, j)$ if the donor in pair $i$ is compatible with the patient in pair $j$.

# KEP - Possible exchanges



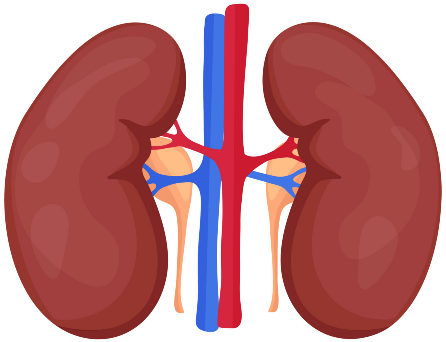
## Definition

An exchange is a set of disjoint cycles in the directed graph such that every cycle length does not exceed a given limit $K$.

- Aim: to **maximize the number of patients transplanted**
- When $K = 2$, an exchange is a matching.

# Kidney Exchange Programs

What is the link between



and

L'ART D'ENTREPRENDRE
HEC LIÈGE • 2025

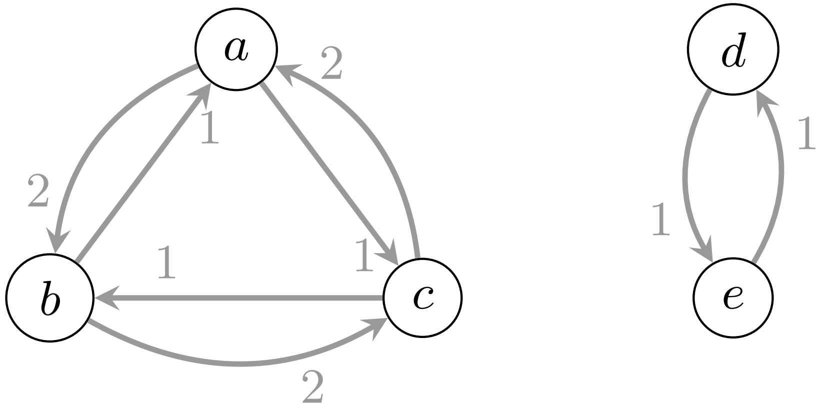
?

# From Kidneys to Corporations


**Kidney Exchange** **Corporate Partnership**

- KEP: Stable matchings among incompatible donor–patient pairs
- CEOs: Stable partnerships among companies
- **Shared goal**: Maximize number of matches
  while ensuring stable configurations

# Instances

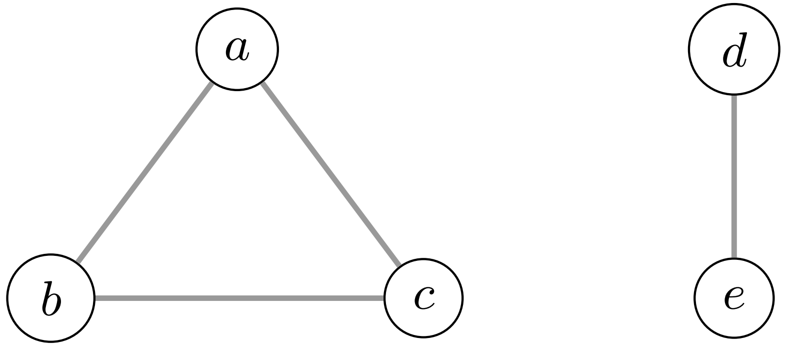



## Definition

An instance consists of

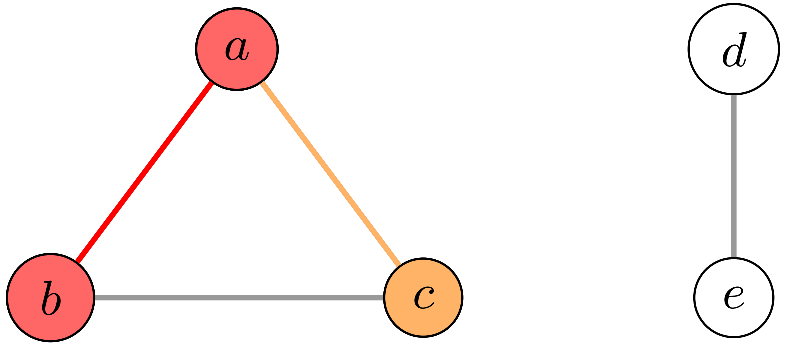
- a preference table
- an undirected graph

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |
| $d$ | $e$ | |
| $e$ | $d$ | |



# Stable Matching

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |
| $d$ | $e$ | |
| $e$ | $d$ | |



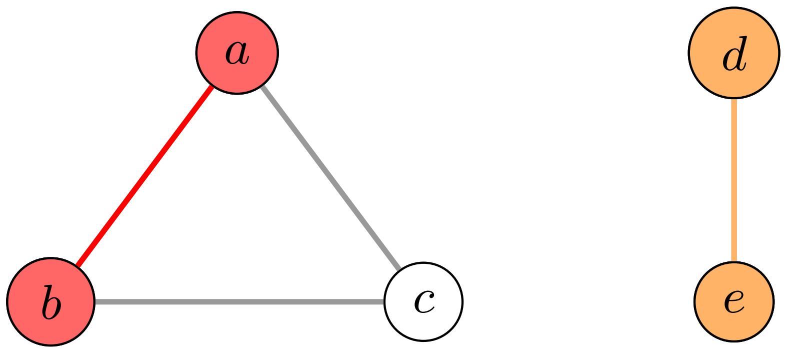

## Definition

Given an undirected graph $G = (V, E)$,

- a matching is a collection of disjoint edges
- a matching $M$ is stable if there are no blocking pairs
- a blocking pair for $M$ is an edge $xy \in E \setminus M$ such that
  - either $x$ prefers $y$ to its mate $M(x)$ or $x$ is not matched in $M$, and
  - either $y$ prefers $x$ to its mate $M(y)$ or $y$ is not matched in $M$.

# Stable Matching

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |
| $d$ | $e$ | |
| $e$ | $d$ | |



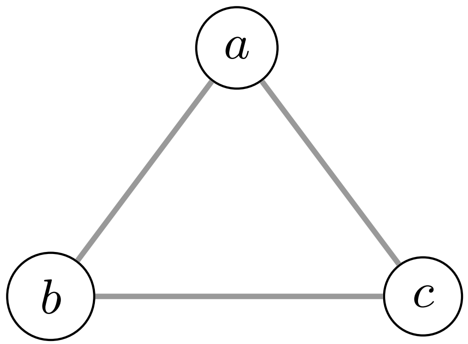

## Definition

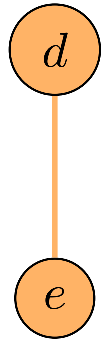
Given an undirected graph $G = (V, E)$,

- a matching is a collection of disjoint edges
- a matching $M$ is stable if there are no blocking pairs
- a blocking pair for $M$ is an edge $xy \in E \setminus M$ such that
  - either $x$ prefers $y$ to its mate $M(x)$ or $x$ is not matched in $M$, and
  - either $y$ prefers $x$ to its mate $M(y)$ or $y$ is not matched in $M$.

# Locally Stable Matching

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |
| $d$ | $e$ | |
| $e$ | $d$ | |



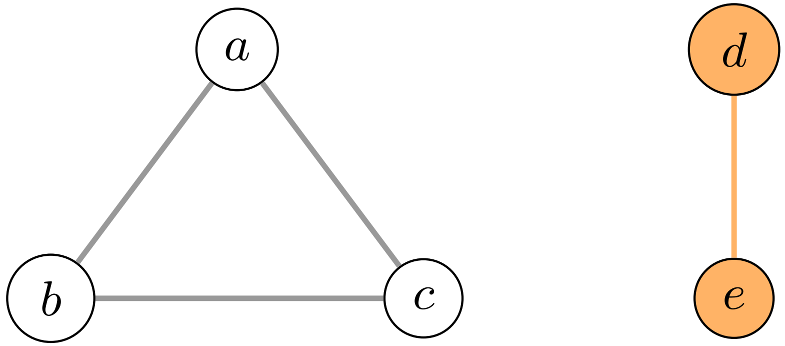

## Definition (Baratto–Crama–Pedroso–Viana, 2025)

Given an undirected graph $G = (V, E)$,

- a matching $M$ is locally stable if there are no locally blocking pairs
- a locally blocking pair for $M$ is an edge $xy \in E \setminus M$ such that
  - $xy$ is blocking for $M$, and
  - $x$ or $y$ is matched in $M$

# Locally Stable Matching

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |
| $d$ | $e$ | |
| $e$ | $d$ | |




## Definition (Baratto–Crama–Pedroso–Viana, 2025)

Given an undirected graph $G = (V, E)$,

- a matching $M$ is locally stable if there are no locally blocking pairs
- a locally blocking pair for $M$ is an edge $xy \in E \setminus M$ such that
  - $xy$ is blocking for $M$, and
  - $x$ or $y$ is matched in $M$

All stable matchings are locally stable ones.
Empty matchings are locally stable ones.

## Stability vs Local Stability

A stable matching is maximum if it has the largest possible size among all stable matchings. And similarly for maximum locally stable matchings.

- **Stability**: Stable Roommates Problem (SRP)
  What is the maximum size of a stable matching?
  (72 don't have a stable matching out of 600 tested : 12% )

- **Local Stability**: Locally Stable Roommates Problem (L-SRP)
  What is the maximum size of a locally stable matching?
  (1 out of 600 tested has a solution of cardinality zero : 0.2% )

  - For 50 instances with $|V| \approx 200$,
    45 out of 50 have a stable exchange;
    50 out of 50 have a non-trivial locally stable exchange;
    45 instances max. stable exchange = max. locally stable exchange

**Our contribution:**

- (L-SRP) Computing a maximum locally stable matching is polynomial.

## Structural results

### Proposition

If $M$ is a stable matching and $M'$ is a locally stable matching, then $V(M') \subseteq V(M)$ and $|M'| \leq |M|$.

### Proposition

If a graph has a stable matching, then

- all its stable matchings cover the same set of vertices, and
- all its stable matchings are maximum locally stable.

A locally stable matching is maximal if it is not included (edge-wise) in any other locally stable matching.

### Proposition

All maximal locally stable matchings cover the same set of vertices and hence, they have the same size.

### Theorem

If a graph has a stable matching, then all its maximal locally stable matchings are stable.

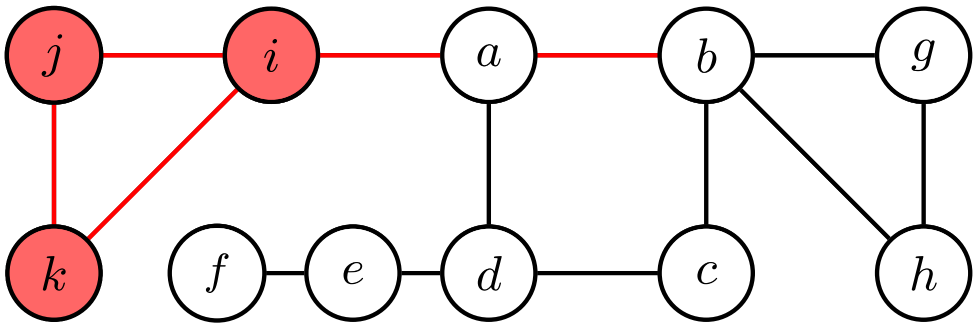
# Algorithm

**Idea:** Perform repeatedly transformations so that each solution of the initial instance is still a solution of the modified instance.

- Identify vertices that can never be covered by a maximal locally stable matching
  - Mark them as rejected
- Deduce all the possible consequences for edges to be rejected

$T_0$ :

| | | | | |
|---|---|---|---|---|
| $a$ | $d$ | $i$ | $b$ | |
| $b$ | $a$ | $g$ | $h$ | $c$ |
| $c$ | $b$ | $d$ | | |
| $d$ | $c$ | $a$ | $e$ | |
| $e$ | $f$ | $d$ | | |
| $f$ | $e$ | | | |
| $g$ | $h$ | $b$ | | |
| $h$ | $b$ | $g$ | | |
| $i$ | $j$ | $k$ | $a$ | |
| $j$ | $k$ | $i$ | | |
| $k$ | $i$ | $j$ | | |



# Algorithm

**function** MAXLSTABLE($T$: a preference table)
  Build $G = (V, E)$, the graph associated with $T$
  $V^r \leftarrow \emptyset$
  $E^r \leftarrow \emptyset$
  } Initialization
  **repeat**
    $G^- \leftarrow (V \setminus V^r, E \setminus E^r)$
    $\pi \leftarrow$ stable partition of $G^-$
    $\mathcal{O} \leftarrow$ union of odd parties of $\pi$
    $V^r \leftarrow V^r \cup \mathcal{O}$
    } Identify vertices to reject
    $\Delta \leftarrow \{xy \in E \setminus E^r : x \in \mathcal{O}$ or $\exists\, z \in \mathcal{O}$ s.t. $x$ prefers $z$ to $y\}$
    $E^r \leftarrow E^r \cup \Delta$
    } Deduce edges to reject
  **until** $\mathcal{O} = \emptyset$
  } $G^-$ has a perfect stable matching given by $\pi$
  $M \leftarrow$ perfect stable matching of $G^-$
  **return** $M$
**end function**

- Overall time complexity : $\mathcal{O}(|V||E|)$

## Conclusion

- Local stability is a natural extension of the classical stability concept for the roommates problem.
- We described an algorithm which identifies a maximum locally stable matching in polynomial time.
  - We can deduce from it an efficient algorithm to enumerate all of them
- Can we find a succinct certificate as it is done for the classical stable roommates problem?

# Want to know more?

Check our preprint:

- Vandomme, E., Crama, Y., & Baratto, M. (2025).
  *Locally stable matchings for the roommates problem.*
  ORBi-University of Liège. `https://hdl.handle.net/2268/330279`