A stroll among binomial complexities
Some recent characterizations of families of words

Manon Stipulanti

Workshop "Words and Subwords"
ULiège (Belgium)
May 12, 2025

M. Rigo, M. S, M. A Whiteland, A note on aperiodic words sharing binomial complexities with the Thue-Morse word, in progress (2025+)

M. Rigo, M. S, M. A. Whiteland, Characterizations of families of morphisms and words via binomial complexities, *European J. Comb.* **118** (2024), 103932

M. Rigo, M. S., M. A. Whiteland, Binomial complexities of Parikh-collinear morphisms, DLT 2022, *Lect. Notes. in Comput. Sci.* **13257** (2022), 251-262

- infinite words in **bold**
- $|w|_a = \#$ letters a in w
- in a word
  factor = subsequence of consecutive letters
  (scattered) subword = subsequence of letters

Example: $|\texttt{reappear}|_\texttt{a} = 2 = |\texttt{reappear}|_\texttt{e}$

factor	subword
reappear	reappear
reappear	reappear
reappear	reappear
reappear	reappear

- length-n factors of $\mathbf{x}$: $\mathrm{Fac}_n(\mathbf{x})$

How to study combinatorial structure?

factor complexity $\mathsf{p_x} \colon \mathbb{N} \to \mathbb{N}, n \mapsto \#\mathrm{Fac}_n(\mathbf{x})$

Example: Fibonacci word $\mathbf{f} = 0100101001 \cdots$ (f.p. of $\phi \colon 0 \mapsto 01, 1 \mapsto 0$)

n	$\mathrm{Fac}_n(\mathbf{f})$	$\mathsf{p_f}(n)$
0	ε	1
1	$0, 1$	2
2	$00, 01, 10$	3
3	$001, 010, 100, 101$	4
4	$0010, 0100, 0101, 1001, 1010$	5

Theorem (Morse–Hedlund 1938)
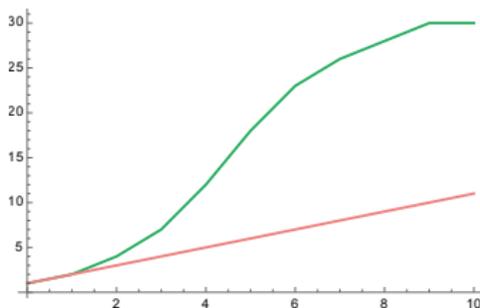
$\mathbf{x}$ with ℓ distinct letters
$\mathbf{x}$ ultimately periodic
  iff $\mathsf{p_x}$ bounded
  iff $\exists n \in \mathbb{N}$ s.t. $\mathsf{p_x}(n) < n + \ell - 1$

$\mathbf{x}$ Sturmian iff $\mathsf{p_x}(n) = n + 1$ $\forall n$

(binary, aperiodic, minimal factor complexity)

$\mathbf{x} = y\,01\,01\,01 \cdots$
$\mathbf{f} = 0100101001001 \cdots$

4

Variations

counting "different enough" factors

- with specific properties
  e.g. palindromes (Droubay–Pirillo 1999)
       privileged (Peltomäki 2013)

- extracted along specific subsequences
  e.g. arithmetical (Avgustinovich–Fon-Der-Flaass–Frid 2000)
       maximal pattern (Kamae–Zamboni 2002)

- with equivalence relations $u \sim v$
  $\mathbb{N} \to \mathbb{N}, n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\sim)$

today 2 equivalence relations

but many more

e.g. later in the week with Ollinger+Popoli

+e.g. list in [arXiv:2406.09302] by Allouche–Campbell–Li–Shallit–S.

First variation (Erdős 1958)

- abelian equivalence relation: $u \sim_{\mathrm{ab}} v$ if $|u|_a = |v|_a \ \forall a \in A$

  <u>Example</u>: `evil` $\sim_{\mathrm{ab}}$ `live` $\sim_{\mathrm{ab}}$ `veil` $\sim_{\mathrm{ab}}$ `vile`

- abelian complexity $\mathsf{a_x} \colon \mathbb{N} \to \mathbb{N}, n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\!\sim_{\mathrm{ab}})$

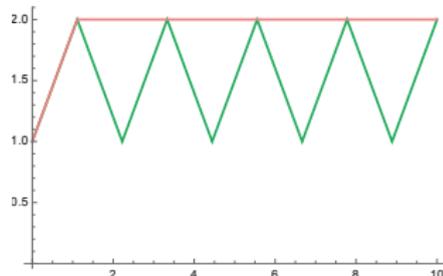 <u>Example</u>: Fibonacci word $\mathbf{f} = 0100101001 \cdots$

n	$\mathrm{Fac}_n(\mathbf{f})$	$\mathsf{p_f}(n)$	$\{\cdot\}_{\sim_{\mathrm{ab}}}$	$\mathsf{a_f}(n)$
0	ε	1	$\{\varepsilon\}$	1
1	$0, 1$	2	$\{0\}$ $\{1\}$	2
2	$00, 01, 10$	3	$\{00\}$ $\{01, 10\}$	2
3	$001, 010, 100, 101$	4	$\{001, 010, 100\}$ $\{101\}$	2

Theorem (Coven–Hedlund 1973)

$\mathbf{x}$ purely periodic iff $\exists n \in \mathbb{N}$ s.t. $\mathsf{a_x}(n) = 1$

$\mathbf{x}$ Sturmian iff $\mathbf{x}$ binary aperiodic
and $\mathsf{a_x}(n) = 2 \ \forall n \geq 1$

$\mathbf{x} = 01\,01\,01\cdots$
$\mathbf{f} = 0100101001001\cdots$

+see survey by Fici–Puzynina 2023

Second variation

- binomial coefficient

$$\binom{u}{v} = \# \text{ occurrences of } v \text{ as a subword of } u$$

Example: $\binom{101001}{101} = 6$

101001 101001 101001
101001 101001 101001

Remark: $\binom{a^n}{a^k} = \binom{n}{k}$

- binomial equivalence relation

$$k \geq 1 \qquad u \sim_k v \text{ if } \binom{u}{x} = \binom{v}{x} \ \forall x \text{ with } |x| \leq k$$

Example: $0110 \sim_2 1001$ but $0110 \not\sim_3 1001$

x	0	1	00	01	10	11	000	001
$\binom{0110}{x}$	2	2	1	2	2	1	0	0
$\binom{1001}{x}$	2	2	1	2	2	1	0	1

Remark: $\sim_1 = \sim_{ab}$

$\sim_{k+1}$ refines $\sim_k$

Binomial coefficients (notably) appear in...

- Chapter 6 in Lothaire's book   Sakarovitch–Simon 1983
- reconstruction problem

  Given n, what is the smallest k s.t. each length-n word is uniquely determined by all its length-k subwords (with multiplicities)?   Still open but bounds and variations

  Kalashnik 1973   Krasikov–Roditty 1997   Levenshtein 2001   Dudik–Schulman 2003
  Fleischmann–Lejeune–Manea–Nowotka–Rigo 2021   Richomme–Rosenfeld 2023   etc.

- piecewise testable languages   Simon 1975

  regular languages defined by the presence/absence of given subwords

- p-group languages   Eilenberg 1976   Renard–Rigo–Whiteland 2024

  L language and p prime, $\exists\, G$ p-group and $\alpha\colon A^* \to G$ morphism s.t. $L = \alpha^{-1}(G)$ iff

  $L =$ finite Boolean combination of $L_{v,r,p} = \left\{ u \in A^* \colon \binom{u}{v} \equiv r \pmod p \right\}$

- Parikh matrices   Mateescu–Salomaa[2]–Yu 2001   Şerbănuţă 2004
- avoidance of binomial powers   Rao–Rigo–Salimov 2015
- generalized Pascal's triangles   Leroy–Rigo–S. 2016-2017-2018   S. 2019
- gapped binomial coefficients

  Golm–Nahvi–Gabrys–Milenkovic 2022   Rigo–S.–Whiteland 2023

- subword entropy   Fang 2024
- multidimensional binomial coefficients   Golafshan–Rigo 2025

Binomial complexities

Definition (Rigo–Salimov 2015)

Let $k \geq 1$.

k-binomial complexity of $\mathbf{x}$: $\mathsf{b}_{\mathbf{x}}^{(k)} \colon \mathbb{N} \to \mathbb{N}, \ n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\sim_k)$

Example: Thue–Morse $\mathbf{t} = 011010011001\cdots$ (f.p. of $\varphi \colon 0 \mapsto 01, 1 \mapsto 10$)

n	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{a}_{\mathbf{x}}(n) = \mathsf{b}_{\mathbf{t}}^{(1)}(n)$	1	2	3	2	3	2	3	2	3	2	3
$\mathsf{b}_{\mathbf{t}}^{(2)}(n)$	1	2	4	6	9	8	8	8	9	8	8
$\mathsf{p}_{\mathbf{t}}(n)$	1	2	4	6	10	12	16	20	22	24	28

Observation: $\sim_{k+1}$ refines $\sim_k$ so

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$
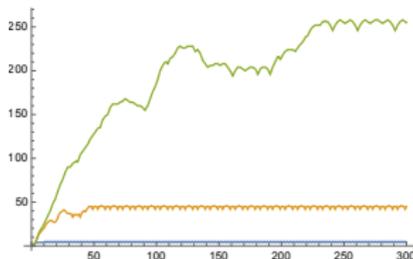
A little bit of history since 2015

Sturmian word $\mathbf{s}$	$b_{\mathbf{s}}^{(k)} = p_{\mathbf{s}} \ \forall k \geq 2$	Rigo–Salimov 2015
Tribonacci word $\mathbf{z}$ (f.p. of $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$)	$b_{\mathbf{z}}^{(k)} = p_{\mathbf{z}} \ \forall k \geq 2$	Lejeune–Rosenfeld–Rigo 2020 Lejeune's PhD thesis 2021
Thue–Morse word $\mathbf{t}$ + wider class of words	$b_{\mathbf{t}}^{(k)}$ bounded $\forall k \geq 1$	Rigo–Salimov 2015
Thue–Morse word $\mathbf{t}$	precise values of $b_{\mathbf{t}}^{(k)}$ $\forall k \geq 1$	Lejeune–Leroy–Rigo 2020 Lejeune's PhD thesis 2021
generalized TM words	precise values of $b^{(k)}$ $\forall k \geq 1$	Chen–Wen 2019 ($k = 1$) Lü–Chen–Wen–Wu 2024 ($k = 2$) Golafshan–Rigo–Whiteland 2025+
hypercubic billiard words & 1-balanced words	$b^{(k)} = p \ \forall k \geq 2$	Andrieu–Vivion 2025+

Motivation and goal

complexity	p	a	$b^{(k)}$
theory general behavior properties etc.	rich	rich	not much is known

What do we want?

- possible behavior of binomial complexities (growth)
- deduce structure of words from their binomial complexities
- understand binomial complexities of large classes of words
- find words attaining lowest complexities

today

- 3 characterizations
- 1 (or 2) question(s)

Part I

Sturmian words

Theorem (Rigo–Salimov 2015)

$\mathbf{s}$ Sturmian $\implies \forall k \geq 2$, $b_{\mathbf{s}}^{(k)}(n) = p_{\mathbf{s}}(n) = n + 1 \quad \forall n$

Corollary (Fici–Puzynina 2023)

$\quad\quad \mathbf{x}$ Sturmian

$\Leftrightarrow b_{\mathbf{x}}^{(1)}(n) = 2$ & $\exists k \geq 2$ s.t. $b_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

$\Leftrightarrow b_{\mathbf{x}}^{(1)}(n) = 2$ & $\forall k \geq 2$ $b_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

stronger version

Proposition

$\mathbf{x}$ binary s.t. $\exists k \geq 2$ with $b_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n \implies \mathbf{x}$ Sturmian

Proof by mixing results of Morse–Hedlund (1938)   Coven–Hedlund (1973)

Richomme-Séébold (2011)   Rigo–Salimov (2015)

Theorem (1st characterization)

$\mathbf{x}$ Sturmian iff $\exists k \geq 2$ with $b_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

Words with bounded binomial complexities

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$ (f.p. of $\varphi\colon 0 \mapsto 01, 1 \mapsto 10$)

$\mathbf{b_t}^{(k)}$ is bounded + precise values

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1 \quad \mathbf{b_t}^{(k)}(n) = \begin{cases} \mathbf{p_t}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$
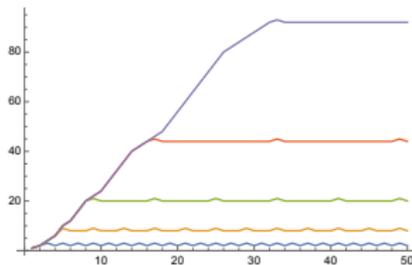
Observation:

$$\begin{pmatrix} |\varphi(0)|_0 \\ |\varphi(0)|_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} |\varphi(1)|_0 \\ |\varphi(1)|_1 \end{pmatrix}$$

Definition:

Parikh vector of w: $\Psi(w) = (|w|_a)_{a \in A}$

Parikh-constant: $\Psi(f(a)) = \Psi(f(b)) \; \forall a, b \in A$

14

Theorem (Rigo–Salimov 2015)

A fixed point of a Parikh-constant morphism has bounded $\mathsf{b}^{(k)}$ ($\forall k$).

Parikh-collinear: $\forall a, b \in A, \exists r_{a,b} \in \mathbb{Q}$ s.t. $\Psi(f(b)) = r_{a,b}\Psi(f(a))$

Example: $f : 0 \mapsto 000111, 1 \mapsto 0110$

$$\Psi(f(0)) = \begin{pmatrix} |f(0)|_0 \\ |f(0)|_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$
$$\Psi(f(1)) = \begin{pmatrix} |f(1)|_0 \\ |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \Psi(f(1)) = \frac{2}{3}\Psi(f(0))$$

Remark: Parikh-collinear iff rank-1 adjacency matrix

Example: $\begin{pmatrix} |f(0)|_0 & |f(1)|_0 \\ |f(0)|_1 & |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

Characterizations in terms of a and $\mathsf{b}^{(k)}$

Theorem (Cassaigne–Richomme–Saari–Zamboni 2011)

f Parikh-collinear
iff f maps all infinite words to words with bounded $\mathsf{a} = \mathsf{b}^{(1)}$

$$\mathsf{b}^{(1)}_{\mathbf{x}}$$
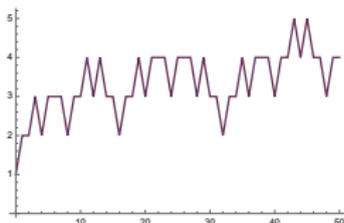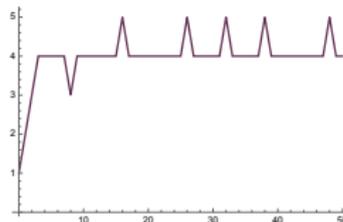
$$\mathsf{b}^{(1)}_{f(\mathbf{x})} \text{ with } f\colon 0 \mapsto 000111, 1 \mapsto 0110$$

generalization

Theorem (2nd characterization)

f Parikh-collinear

$\Leftrightarrow \forall k$ f maps words with bounded $\mathsf{b}^{(k)}$ to words with bounded $\mathsf{b}^{(k+1)}$

$\Leftrightarrow \exists k$: f maps words with bounded $\mathsf{b}^{(k)}$ to words with bounded $\mathsf{b}^{(k+1)}$

Proof using the previous theorem, another characterization of Parikh-collinear morphisms
by Rigo–S–Whiteland, and technical lemmas

Corollary

A f.p. $\mathbf{x}$ of a Parikh-collinear morphism has bounded $\mathsf{b}_{\mathbf{x}}^{(k)}$ ($\forall k$).

<u>Proof</u>: By Cassaigne–Richomme–Saari–Zamboni, $\mathbf{x}$ has bounded $\mathsf{b}_{\mathbf{x}}^{(1)}$.

$$\begin{array}{ccc} \mathbf{x} & \longrightarrow & f(\mathbf{x}) = \mathbf{x} \\ \text{bounded} & & \text{bounded} \\ \mathsf{b}_{\mathbf{x}}^{(1)} & & \mathsf{b}_{\mathbf{x}}^{(2)} \\ & \vdots & \end{array}$$
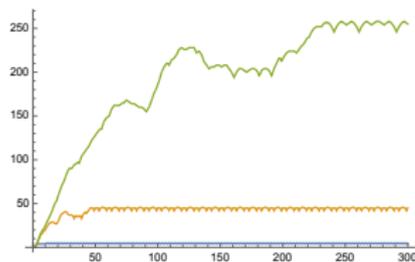
$f : 0 \mapsto 000111, 1 \mapsto 0110$

<u>Remark</u>: no stronger version

$f : 0 \mapsto 0^3 2^3, 1 \mapsto 0^3 1^3 2, 2 \mapsto 2^4 0^6 1^3$ adjacency matrix of rank 2

$\rightsquigarrow f^{\omega}(0)$ has unbounded $\mathsf{a} = \mathsf{b}^{(1)}$ (Adamczewski 2003)

$\rightsquigarrow f^{\omega}(0)$ has unbounded $\mathsf{b}^{(k)} \forall k$

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$ (f.p. of $\varphi\colon 0 \mapsto 01, 1 \mapsto 10$)

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1 \ \mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathsf{p}_{\mathbf{t}}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$
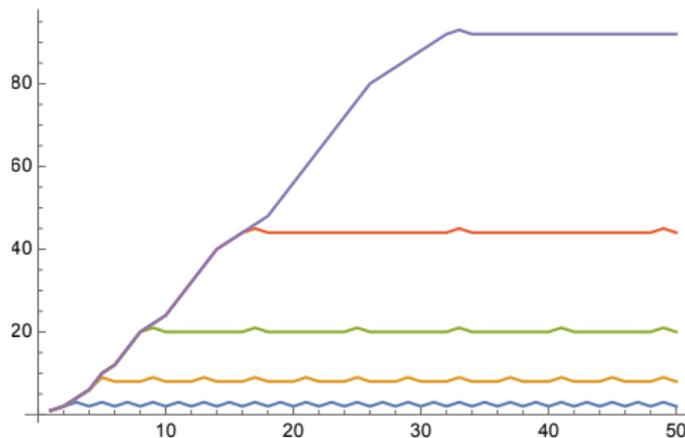
<u>Definition</u>: $\mathbf{x}$ has $\mathcal{P}_k$ if $\mathbf{b}_{\mathbf{x}}^{(j)} = \mathbf{b}_{\mathbf{t}}^{(j)} \ \forall j \in \{1, \dots, k\}$

> **Theorem** (Richomme–Saari–Zamboni 2011)
>
> $\mathbf{x}$ aperiodic binary has $\mathcal{P}_1$ iff $\exists \mathbf{y}$ s.t. $\mathbf{x} = a\varphi(\mathbf{y})$ with $a \in \{\varepsilon, 0, 1\}$

generalization of $\Longleftarrow$

> **Proposition**
>
> $\mathbf{x} = u\varphi^k(\mathbf{y})$ has $\mathcal{P}_k$ (with $\mathbf{y}$ aperiodic, $k \geq 1$, u suffix of $\varphi^k(0)/\varphi^k(1)$)

Proof with several cases using abelian Rauzy graphs

Example:

Thue–Morse word $\mathbf{t}$

Fibonacci word $\mathbf{f} = 010010100100\cdots$

$\mathbf{f} = \varphi^0(\mathbf{f})$ $\varphi^1(\mathbf{f})$

$\varphi^2(\mathbf{f})$ $\varphi^3(\mathbf{f})$

19
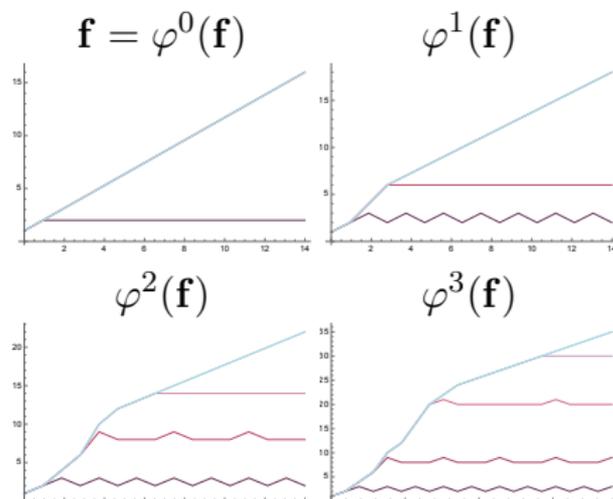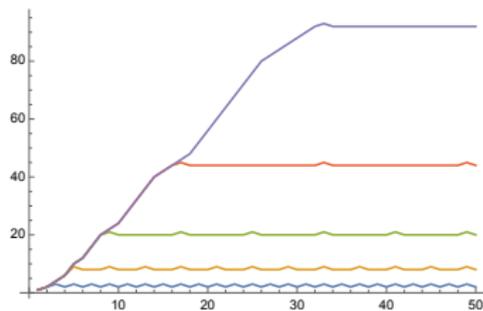
A partial converse

Definition: **x** is recurrent if each factor appears infinitely often

Example:
- recurrent: Thue–Morse, Fibonacci
- not recurrent: $101001000100001\cdots = 1010^2 10^3 10^4 1\cdots$
  e.g. 101 appears only once

Proposition

x recurrent binary with $\mathcal{P}_k$ $(k \geq 1)$
$\Longrightarrow \exists \mathbf{y}$ aperiodic s.t. $\mathbf{x} = u\varphi^k(\mathbf{y})$ with u suffix of $\varphi^k(0)/\varphi^k(1)$

Proof by induction
Base case: Richomme–Saari–Zamboni

Induction step: using a formula to compute the exact value of the $k{+}1$-binomial complexity

using abelian Rauzy graphs, using contradiction, and several cases to consider

Theorem (3rd characterization)

$\mathbf{x}$ aperiodic recurrent binary has $\mathcal{P}_k$ ($k \geq 1$) iff $\mathbf{x} = u\varphi^k(\mathbf{y})$ with $\mathbf{y}$ aperiodic and u suffix of $\varphi^k(0)/\varphi^k(1)$

Question: get rid of recurrence
⤳ close to a solution?

(Very) long proof with many cases and subcases using abelian Rauzy graphs

Part II

Observation

period-doubling word $\mathbf{pd} = 01000101010001\cdots$ (f.p. of $0 \mapsto 01, 1 \mapsto 00$)
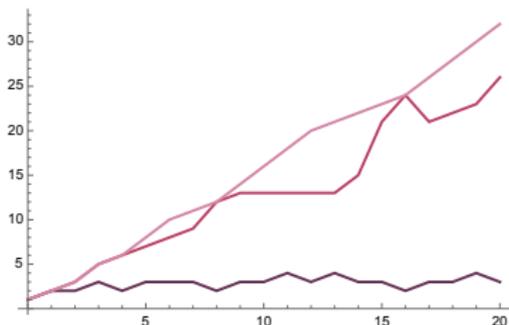
> **Proposition** (Lejeune-Rigo-S. in Lejeune 2021)
>
> $$\mathsf{b}_{\mathbf{pd}}^{(2)}(2^n) = \mathsf{p}_{\mathbf{pd}}(2^n) \qquad\qquad \mathsf{b}_{\mathbf{pd}}^{(2)}(m) < \mathsf{p}_{\mathbf{pd}}(m)$$
> $$\forall n \qquad\qquad\qquad\qquad \forall m \neq 2^n$$

Notation: $\mathsf{f}, \mathsf{g} \colon \mathbb{N} \to \mathbb{N}$ functions

$\quad\mathsf{f} \prec \mathsf{g} \equiv \quad \mathsf{f}(n) \leq \mathsf{g}(n) \ \forall n \quad \& \quad \mathsf{f}(n) < \mathsf{g}(n) \text{ for } \infty\text{ly many } n$

Example: $\mathsf{b}_{\mathbf{pd}}^{(2)} \prec \mathsf{p}_{\mathbf{pd}}$

Question B

binomial complexities are increasingly nested

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$

and $\mathsf{b}_{\mathbf{x}}^{(k)} = \mathsf{p}_{\mathbf{x}}$ ($\forall k \geq 2$) for $\mathbf{x}$ Sturmian, Tribonacci, billiard words

$\rightsquigarrow$ Can the factor complexity coincide with any binomial complexity?

> **Question B** (Stabilization)
>
> For $k \geq 1$, does there exist $\mathbf{w}_k$ s.t.
>
> $$\mathsf{b}_{\mathbf{w}_k}^{(1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(2)} \prec \cdots \prec \mathsf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(k)} = \mathsf{p}_{\mathbf{w}_k}?$$

inspired by Lejeune's PhD thesis (2021)

Examples:

$\mathbf{w} = 0\,0\,0\cdots$
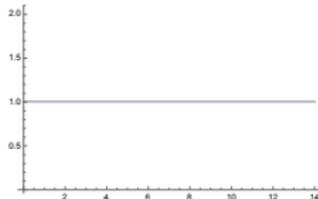
$\mathbf{b}_{\mathbf{w}}^{(1)} = \mathbf{p}_{\mathbf{w}}$

$\mathbf{w} = 01\,01\,01\cdots$
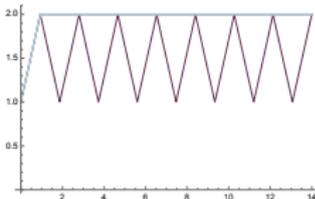
$\mathbf{b}_{\mathbf{w}}^{(1)} \prec \mathbf{b}_{\mathbf{w}}^{(2)} = \mathbf{p}_{\mathbf{w}}$

$\mathbf{w} = 011001\,011001\cdots$
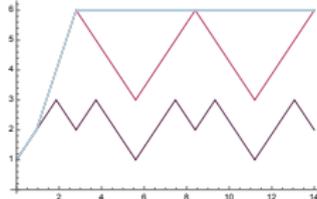
$\mathbf{b}_{\mathbf{w}}^{(1)} \prec \mathbf{b}_{\mathbf{w}}^{(2)} \prec \mathbf{b}_{\mathbf{w}}^{(3)} = \mathbf{p}_{\mathbf{w}}$

but these words have

- bounded complexities
- rather simple structure

more "interesting" words?

A second answer

Theorem

$k \geq 1$ **s** Sturmian $\implies \mathsf{b}^{(1)}_{\varphi^k(\mathbf{s})} \prec \mathsf{b}^{(2)}_{\varphi^k(\mathbf{s})} \prec \cdots \prec \mathsf{b}^{(k+1)}_{\varphi^k(\mathbf{s})} \prec \mathsf{b}^{(k+2)}_{\varphi^k(\mathbf{s})} = \mathsf{p}_{\varphi^k(\mathbf{s})}$

Proof using a characterization of Parikh-collinear morphisms via their bin. complexities, concepts and results from Lejeune–Leroy–Rigo 2020, and the theorem of Rigo–Salimov 2015 about Sturmian words

Example: Fibonacci word $\mathbf{f} = 010010100100\cdots$
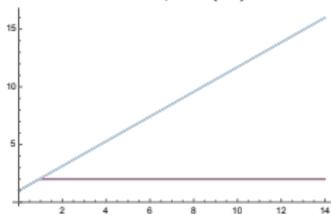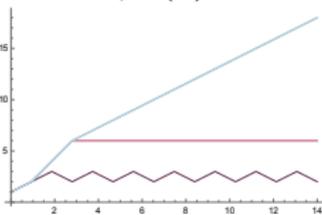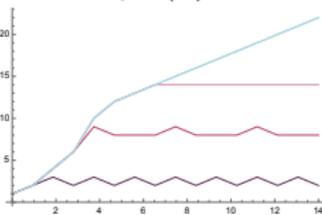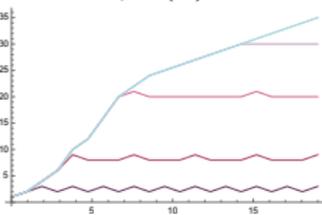
$\mathbf{f} = \varphi^0(\mathbf{f})$	$\varphi^1(\mathbf{f})$	$\varphi^2(\mathbf{f})$	$\varphi^3(\mathbf{f})$
$\mathsf{b}^{(2)} = \mathsf{p}$	$\mathsf{b}^{(3)} = \mathsf{p}$	$\mathsf{b}^{(4)} = \mathsf{p}$	$\mathsf{b}^{(5)} = \mathsf{p}$

True for Sturmian

(Rigo–Salimov 2015)

Exact values

Corollary

$k \geq 1$ **s** Sturmian

$$\forall 1 \leq j \leq k \qquad \mathsf{b}^{(j)}_{\varphi^k(\mathbf{s})}(n) = \mathsf{b}^{(j)}_{\mathbf{t}}(n) = \begin{cases} \mathsf{p_t}(n) & \text{if } n \leq 2^j - 1 \\ 3 \cdot 2^j - 3 & \text{if } n \equiv 0 \pmod{2^j} \text{ and } n \geq 2^j \\ 3 \cdot 2^j - 4 & \text{otherwise} \end{cases}$$

$$\mathsf{b}^{(k+1)}_{\varphi^k(\mathbf{s})}(2^k n + r) = \begin{cases} \mathsf{p_t}(r) & \text{if } n = 0 \text{ and } 0 \leq r < 2^k \\ 3 \cdot 2^k - 2 & \text{if } n = 1 \text{ and } r = 0 \\ 3 \cdot 2^k + r - 1 & \text{if } n = 1 \text{ and } r > 0 \\ 2^{k+2} - 2 & \text{otherwise} \end{cases}$$

$$\forall j \geq k + 2 \qquad \mathsf{b}^{(j)}_{\varphi^k(\mathbf{s})}(n) = \mathsf{p}_{\varphi^k(\mathbf{s})}(n) = \begin{cases} \mathsf{p_t}(n) & \text{if } n \leq 2^k \\ n + 2^{k+1} - 1 & \text{otherwise} \end{cases}$$

Proof for $j \leq k$, $\mathcal{P}_k$; for $k + 1$, abelian Rauzy graphs; for $k + 2$, results of Frid 1999
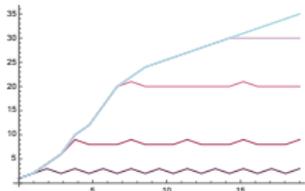
... but bounded $\mathsf{b}^{(1)}, \ldots, \mathsf{b}^{(k+1)}$

Question C

For $k \geq 1$, does there exist $\mathbf{w}_k$ s.t. $\mathsf{b}_{\mathbf{w}_k}^{(1)}$ is unbounded and

$$\mathsf{b}_{\mathbf{w}_k}^{(1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(2)} \prec \cdots \prec \mathsf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(k)} = \mathsf{p}_{\mathbf{w}_k}?$$

Answer for $k = 3$:
$\mathbf{h} = 0112122122212222\cdots$

(f.p. of $0 \mapsto 01, 1 \mapsto 12, 2 \mapsto 2$)

- $\mathsf{b}_{\mathbf{h}}^{(1)}$ unbounded
- $\mathsf{b}_{\mathbf{h}}^{(1)} \prec \mathsf{b}_{\mathbf{h}}^{(2)} \prec \mathsf{b}_{\mathbf{h}}^{(3)} = \mathsf{p}_{\mathbf{h}}$

Proof for $\mathsf{b}_{\mathbf{h}}^{(1)} \prec \mathsf{b}_{\mathbf{h}}^{(2)} \neq \mathsf{p}_{\mathbf{h}}$: exhibiting specific factors

Proof for $\mathsf{b}_{\mathbf{h}}^{(3)} = \mathsf{p}_{\mathbf{h}}$: by contradiction + special form of the f.p.
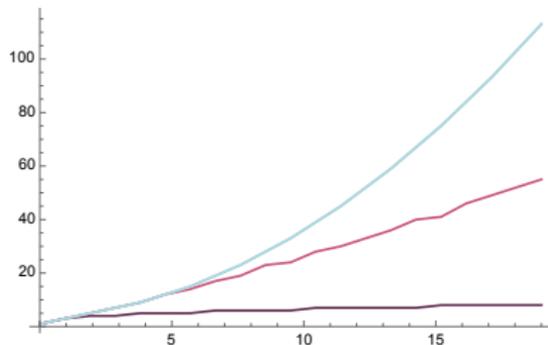
Conjecture for $k = 4$: period-doubling word $\mathbf{pd} = 01000101010001\cdots$
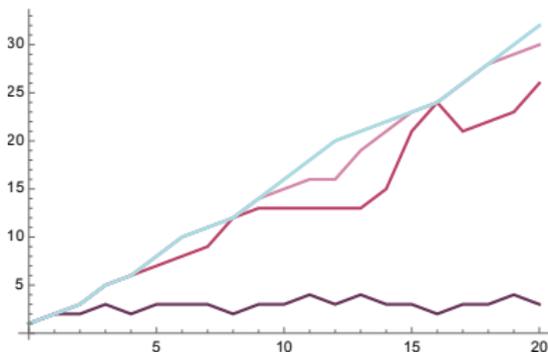
- $\mathsf{b}_{\mathbf{pd}}^{(1)}$ unbounded (Karhumäki–Saarela–Zamboni 2017)
- $\mathsf{b}_{\mathbf{pd}}^{(1)} \prec \mathsf{b}_{\mathbf{pd}}^{(2)} \prec \mathsf{p}_{\mathbf{pd}}$ (Lejeune-Rigo-S. in Lejeune 2021)
- computer experiments: $\mathsf{b}_{\mathbf{pd}}^{(3)} \prec \mathsf{p}_{\mathbf{pd}}$ and $\mathsf{b}_{\mathbf{pd}}^{(4)} = \mathsf{p}_{\mathbf{pd}}$

Open question: what about larger values of k?

binomial complexities are increasingly nested

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$

Can the non-equality happen ∞ly many times?

Question A

Does there exist $\mathbf{w}$ s.t. $\mathsf{b}_{\mathbf{w}}^{(k)} \prec \mathsf{b}_{\mathbf{w}}^{(k+1)} \ \forall k \geq 1$?

inspired by Lejeune's PhD thesis

A festival of answers: opening act

binary Champernowne word
$$\mathbf{c} = 0\,1\,10\,11\,100\,101\,110\,111\cdots$$

(concatenate all binary representations)

- unbounded $\mathsf{b}_{\mathbf{c}}^{(1)}$
- not morphic
- not uniformly recurrent

(recurrent with bounded gaps)

Thue–Morse word $\mathbf{t}$
- bounded $\mathsf{b}_{\mathbf{t}}^{(k)}$

(Lejeune–Leroy–Rigo 2020)
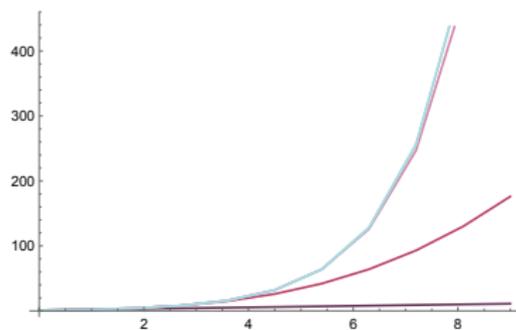
Theorem (Ochsenschläger, 1981)

$\varphi \colon 0 \mapsto 01, 1 \mapsto 10$
$\varphi^k(0) \sim_k \varphi^k(1)$ and $\varphi^k(0) \not\sim_{k+1} \varphi^k(1) \quad \forall k$

A festival of answers: more "structured" words

$\mathbf{v} = \tau(f^\omega(a))$

$f \colon a \mapsto a0\alpha, 0 \mapsto 01, 1 \mapsto 10, \alpha \mapsto \alpha^2$

$f^\omega(a) := \lim_{n \to \infty} f^n(a)$

$\tau \colon a \mapsto \varepsilon, 0 \mapsto 0, 1 \mapsto 1, \alpha \mapsto 1$

- unbounded $\mathsf{b}_{\mathbf{v}}^{(1)}$
- binary
- morphic
- not uniformly recurrent
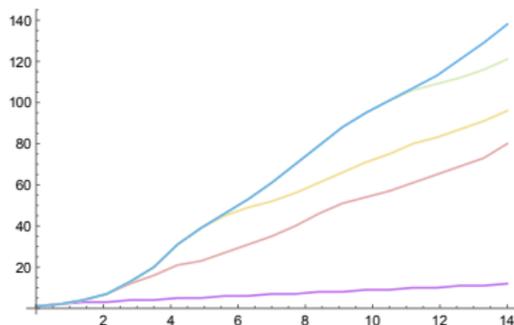
Grillenberger's word
$\mathbf{w} = 0100010101100111\cdots$

- unbounded $\mathsf{b}_{\mathbf{w}}^{(1)}$
- binary
- uniformly recurrent

Construction (Grillenberger 1973)

Start: $D_0 = \{0,1\}$ with $0 < 1$

Induction:
$w_n = $ concatenate words in D_n in lex. order
$D_{n+1} = w_n D_n^2$

End: uniformly recurrent $\mathbf{w} = \lim_{n \to \infty} w_n$

n	D_n	w_n
0	$\{0,1\}$	0 1
1	$01\{0,1\}^2$	0100 0101 0110 0111

topological entropy: $\lim_n \frac{\log \mathsf{p}(n)}{n}$

- Characterization 1

  $\mathbf{x}$ Sturmian iff $\exists k \geq 2$ s.t $\mathsf{b}_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

- Characterization 2

  f Parikh-collinear iff (bounded $\mathsf{b}_{\mathbf{x}}^{(k)} \implies$ bounded $\mathsf{b}_{\mathbf{x}}^{(k+1)}$)

- Characterization 3

  $\mathbf{x}$ aperiodic, recurrent, binary has $\mathcal{P}_k$ $(k \geq 1)$ iff $\mathbf{x} = u\varphi^k(\mathbf{y})$ with $\mathbf{y}$ aperiodic

- Question A

  $\exists \mathbf{w}$ s.t. $\mathsf{b}_{\mathbf{w}}^{(k)} \prec \mathsf{b}_{\mathbf{w}}^{(k+1)}$ $\forall k \geq 1$?

- Questions B-C

  For $k \geq 1$, $\exists \mathbf{w}_k$ s.t. ($\mathsf{b}_{\mathbf{w}_k}^{(1)}$ unbounded and) $\mathsf{b}_{\mathbf{w}_k}^{(1)} \prec \cdots \prec \mathsf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(k)} = \mathsf{p}_{\mathbf{w}_k}$?

Next CANT school-conference at CIRM   29/09 – 03/10 2025

Happy birthday Émilie!