

Positionality of Dumont–Thomas numeration systems for integers

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March 11, 2025

Abstract

Introduced in 2001 by Lecomte and Rigo, abstract numeration systems provide a way of expressing natural numbers with words from a language L accepted by a finite automaton. As it turns out, these numeration systems are not necessarily positional, i.e., we cannot always find a sequence $U = (U_i)_{i \geq 0}$ of integers such that the value of every word in the language L is determined by the position of its letters and the first few values of U . Finding the conditions under which an abstract numeration system is positional seems difficult in general. In this paper, we thus consider this question for a particular sub-family of abstract numeration systems called Dumont–Thomas numeration systems. They are derived from substitutions and were introduced in 1989 by Dumont and Thomas. We exhibit conditions on the underlying substitution so that the corresponding Dumont–Thomas numeration is positional. We first work in the most general setting, then particularize our results to some practical cases. Finally, we link our numeration systems to existing literature, notably properties studied by Rényi in 1957, Parry in 1960, Bertrand-Mathis in 1989, and Fabre in 1995.

Keywords: morphism, substitution, periodic points, numeration system, positionality, Bertrand property, Fabre substitution.

1 Introduction

The need of representing natural numbers (or expressing them in writing) has occupied humans for centuries. In its most general form, this leads to so-called *abstract numeration systems*, introduced by Lecomte and Rigo in 2001 [12] (see also [2, Chapter 3] for a general presentation). Such a numeration system is defined by a triple $S = (L, A, \prec)$ where A is an alphabet ordered by the total order \prec and L is an infinite *regular* language over A (i.e., accepted by a deterministic finite automaton). We say that L is the *numeration language* of S . When we order the words of L

*Supported by the FNRS Research Fellow grant 1.A.789.23F

†Supported by the FNRS Research grant 1.C.104.24F

‡Supported by France’s Agence Nationale de la Recherche (ANR) project IZES (ANR-22-CE40-0011).

with the *genealogical* order (i.e., first by length, then using the dictionary order) induced by \prec , we obtain a one-to-one correspondence rep_S between \mathbb{N} and L . The (S) -*representation* of the non-negative integer n is then the $(n+1)$ st word of L , and the inverse map, called the (S) -*evaluation map*, is denoted by val_S . A simple example is given by the abstract numeration system S built on the language $L = 1^*2^*$ over the ordered alphabet $\{1, 2\}$. The first few words in the language are $\varepsilon, 1, 2, 11, 12, 22, 111$. For instance, $\text{rep}_S(5) = 22$ and $\text{val}_S(111) = 6$.

In general, a numeration system $S = (L, A, S)$ is *positional* if the underlying alphabet A is a set of consecutive integers $\{0, 1, \dots, c\}$ for some $c \in \mathbb{N}$ and if there exists a sequence $(U_i)_{i \geq 0}$ of non-negative integers such that the evaluation map is of the form $\text{val}_S: A^* \rightarrow \mathbb{N}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-1} w_i U_i$. We use the term *positional* because the *positions* of letters in representations are used to generate numbers. Observe that the numeration system built on $L = 1^*2^*$ cannot be positional: indeed, we have $\text{rep}_S(3) = 11$ and $\text{rep}_S(5) = 22$, so there is no hope to find an integer sequence $(U_i)_{i \geq 0}$ such that $3 = 1 \cdot U_1 + 1 \cdot U_0$ and $5 = 2 \cdot U_1 + 2 \cdot U_0$ (see also [2, Example 3.1.12]). We thus raise the following question:

Question A. What are the conditions for an abstract numeration system to be positional?

See also [2, Exercise 3.13]. As this question seems difficult to answer in its full generality, we consider a particular case with Question B below. Roughly, we consider numeration systems that are derived from *substitutions*, i.e., maps sending sequences to sequences and satisfying some mild properties. These numeration systems are due to Dumont and Thomas [6] in 1989. (Note that to properly state the weaker version of Question A, we introduce some notation and definitions in Section 2.) Since then, Dumont–Thomas numeration systems have been used to solve various problems as they provide a nice framework to work with. As the literature is quite vast, we focus on some recent papers from the 2020’s only. For instance, as Question B already testifies, Dumont–Thomas numerations fail to satisfy natural properties that are however somewhat desirable for numeration systems. In addition, they are typically neither greedy nor *addable*, i.e., addition is not computable by a finite automaton. The latter condition is needed in particular if one wants to make use of the theorem-prover **Walnut** [16, 20]. Recently, a characterization of such addable numeration systems was found [4]. Generalizations of Dumont–Thomas numeration systems as numeration systems *per se* can be found in [11, 21]. In [8, 9], classical Dumont–Thomas numeration systems are used to find string attractors for infinite words. In [15], the authors use extensions of Dumont–Thomas numeration systems to the setting of random substitutions. In [14], the specific case of the Thue–Morse substitution is used to establish uniform bounds for the twisted correlations for all elements in the Thue–Morse subshift. In this paper, we exhibit conditions on the underlying substitution for the corresponding Dumont–Thomas numeration for \mathbb{Z} to be positional.

The outline of the paper is as follows. In Section 2, we present the necessary background and preliminary results. In particular, we present the original generalization of Dumont–Thomas numerations to all integers in Section 2.1, then generalize it further in Section 2.2 by letting the lengths of representations vary. In Section 3, we study which Dumont–Thomas numeration systems are positional to answer Question B. We start with a sketch in Section 3.1, then we prove our main result in Section 3.2. We turn to particular cases in Section 3.3 and we finish by discussing the properties of our Dumont–Thomas numeration systems related to existing literature, e.g., the property of a numeration system to be Bertrand [3, 5].

2 Preliminaries

General combinatorics on words. An *alphabet* A is a finite set and its elements are called *letters*. A (*finite or infinite*) *word* over A is a (finite or infinite) sequence of letters belonging to A . For a finite word w over A , we let $|w|$ denote its *length*, i.e., the number of letters it is made of. The *empty word* is denoted by ε . For a word w over A , a *factor* of w is a word y such that there exist words x, z such that $w = xyz$. A *prefix* (resp. *suffix*) is a factor y such that x (resp. z) is empty. If $w = ps$ for some words p, s , then we write $p^{-1}w = s$ and $ws^{-1} = p$. For a word w we let w_i denote its i -th letter. We use the notation of intervals to indicate portions of words: if I

is an interval of integers, we let w_I denote the factor $(w_i)_{i \in I}$ of w . (Recall the difference between parentheses and square brackets to denote intervals.) We let A^* denote the set of all finite words over the alphabet A and A^+ the set of all non-empty words over A (note that $A^+ = A^* \setminus \{\varepsilon\}$). For $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Z}_{<0}\}$, we let $A^{\mathbb{D}}$ the set of words indexed over \mathbb{D} . We call $u_0 u_1 u_2 \cdots \in A^{\mathbb{N}}$ a *right-infinite* word and $\cdots u_{-3} u_{-2} u_{-1} \in A^{\mathbb{Z}_{<0}}$ a *left-infinite* word. A *two-sided* word is a word indexed over \mathbb{Z} . For convenience, we separate by a vertical bar its -1 -th and 0 -th elements to indicate the origin, i.e., $u = \cdots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \cdots$. If the alphabet A is totally ordered by \prec , we define the *lexicographic order* on A^* and $A^{\mathbb{N}}$ as follows. For two words $u, v \in A^*$, we write $u \prec_{\text{lex}} v$ if u is a proper prefix of v or if there exist $p, s, s' \in A^*$ and $a, a' \in A$ such that $u = pas$, $v = pa's'$, and $a \prec a'$. For two words $u, v \in A^{\mathbb{N}}$, we write $u \prec_{\text{lex}} v$ if there exist $p \in A^*$, $x, x' \in A^{\mathbb{N}}$, and $a, a' \in A$ such that $u = pax$, $v = pa'x'$, and $a \prec a'$. For finite or infinite words u, v , we write $u \preceq_{\text{lex}} v$ if $u \prec_{\text{lex}} v$ or $u = v$.

Morphisms and substitutions. Given alphabets A, B , a *morphism* is a map $\mu: A^* \rightarrow B^*$ such that $\mu(uv) = \mu(u)\mu(v)$ for all words $u, v \in A^*$. A morphism is entirely determined by the images of the letters of A . A *substitution* is a morphism $\mu: A^* \rightarrow A^*$ such that the image $\mu(a)$ is non-empty for every letter $a \in A$ and there exists a *growing* letter $a \in A$, i.e., $\lim_{n \rightarrow +\infty} |\mu^n(a)| = +\infty$. A morphism $\mu: A^* \rightarrow A^*$ is *primitive* if there exists an integer $k \in \mathbb{N}$ such that for all $a, b \in A$, the letter a appears in $\mu^k(b)$. Note that, if we consider the classical adjacency matrix M of $\mu: A^* \rightarrow A^*$, μ is primitive if and only if M is primitive. From the theory of primitive matrices, we then deduce that if a morphism is primitive, then there exists an integer $k \in \mathbb{N}$ such that for all $a, b \in A$ and for all $\ell \geq k$, the letter a appears in $\mu^\ell(b)$.

With a morphism $\mu: A^* \rightarrow A^*$, we can associate a directed graph in the following way: the nodes are the letters of A and for every $a \in A$, if we write $\mu(a) = c_0 \cdots c_\ell$ with $c_i \in A$ for every $i \in \{0, \dots, \ell\}$, then we draw an arrow labeled by i from a to c_i for every $i \in \{0, \dots, \ell\}$. This directed graph encodes the images of letters under μ . Similarly, we may unfold the directed graph into a directed tree starting with some fixed vertex that is then called the *root* of the tree. Given $a \in A$, we define the tree $\mathcal{T}_{\mu, a}$ as follows: its root is labeled by a , and if a node of the tree is labeled by x and $\mu(x) = y_0 \cdots y_\ell$ then that node has $\ell + 1$ children labeled from y_0 to y_ℓ , with the edge from x to y_i being labeled by i . Observe that the k -th level of $\mathcal{T}_{\mu, a}$ stores the k -th iteration of μ on the letter a . We say that a node of the tree is *in column* n if there are n nodes of the same level to its left.

Substitutions can naturally be applied to two-sided words by setting

$$\mu(\cdots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \cdots) = \cdots \mu(u_{-3}) \mu(u_{-2}) \mu(u_{-1}) | \mu(u_0) \mu(u_1) \mu(u_2) \cdots$$

Let $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Z}_{<0}\}$ and consider a substitution μ over A . A word $u \in A^{\mathbb{D}}$ is a *periodic point* of μ if there exists an integer $p \geq 1$ such that $\mu^p(u) = u$. In this case, p is called a *period* of the periodic point u . The smallest such integer is called the *period of* u . A periodic point of μ with period $p = 1$ is called a *fixed point* of μ . We let $\text{Per}_{\mathbb{D}}(\mu) = \{u \in A^{\mathbb{D}} \mid \mu^p(u) = u \text{ for some } p \geq 1\}$ denote the set of periodic points of μ . If $u \in \text{Per}_{\mathbb{Z}}(\mu)$, then the *seed* of u is the pair of letters $u_{-1} | u_0$; see [1, §4.1]. If both letters of the seed of a two-sided periodic point are growing, then the periodic point is defined entirely by its seed. More precisely, we have $u = \lim_{n \rightarrow +\infty} \mu^{np}(u_{-1}) | \lim_{n \rightarrow +\infty} \mu^{np}(u_0)$, where p is a period of u . By extension of the tree associated with a substitution and a letter as in the previous paragraph, we consider two-sided trees as follow. For a substitution μ over A and two letters $a, b \in A$, the tree $\mathcal{T}_{\mu, b|a}$ is obtained by setting a start root having two children: the left one is reached with an edge of label 1 and is the root of the tree $\mathcal{T}_{\mu, b}$, and the right one is reached with an edge of label 0 and is the root of the tree $\mathcal{T}_{\mu, a}$. We say that a node is *in column* n if either $n \geq 0$, the node is the right subtree and there are n nodes on the same level to its left in the right subtree, or $n < 0$, the node is in the left subtree and there are $-n - 1$ nodes on the same level to its right in the left subtree. Examples of such trees are given later in the paper.

Numeration systems. A *numeration system* over the domain $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}_{<0}, \mathbb{Z}\}$ is a pair of maps between \mathbb{D} and a set of words, i.e., the *representation map* $\text{rep}: \mathbb{D} \rightarrow A^*$ for some alphabet A , and the *evaluation map* $\text{val}: L \rightarrow \mathbb{D}$, where $\text{rep}(\mathbb{D}) \subset L \subset A^*$, such that $\text{val} \circ \text{rep} = \text{id}_{\mathbb{D}}$. The set $\text{rep}(\mathbb{D})$ is the *language* of the numeration system, while L contains some additional, non-canonical

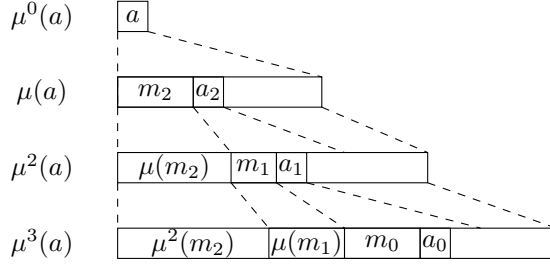


Figure 1: An illustration of what it means for a sequence $((m_i, a_i))_{0 \leq i \leq 2}$ to be a -admissible with respect to a substitution μ .

representations. In general, a numeration system over $\mathbb{D} = \mathbb{N}$ is *positional* if the underlying alphabet A is a set of consecutive integers $\{0, 1, \dots, c\}$ for some $c \in \mathbb{N}$ and the evaluation map is of the form $\text{val}: A^* \rightarrow \mathbb{N}, w_{k-1} \dots w_0 \mapsto \sum_{i=0}^{k-1} w_i U_i$ for some sequence $U = (U_i)_{i \geq 0} \in \mathbb{N}^{\mathbb{N}}$. Over $\mathbb{D} = \mathbb{Z}$, a numeration system is *positional* if the underlying alphabet A is a set of consecutive integers $\{0, 1, \dots, c\}$ for some $c \in \mathbb{N}$ and the evaluation map is of the form $\text{val}: A^* \rightarrow \mathbb{Z}, w_{k-1} \dots w_0 \mapsto \sum_{i=0}^{k-2} w_i U_i - w_{k-1} V_{k-1}$ for some sequences $U = (U_i)_{i \geq 0}, V = (V_i)_{i \geq 0} \in \mathbb{N}^{\mathbb{N}}$. The sequences U and V are the sequences of *weights* of the numeration system. Every position has a given weight, while the presence of an additional sequence V helps deal with the representation of negative numbers, in a fashion similar to the usual two's complement numeration system described in the next example.

Example 2.1. The two's complement numeration system allows representations of all integers in a binary system using powers of 2. Let $A = \{0, 1\}$. The evaluation map is defined as follows: for a word $w = w_{k-1} w_{k-2} \dots w_0$ over A , we set $\text{val}_{2c}(w) = -w_{k-1} 2^{k-1} + \sum_{i=0}^{k-2} w_i 2^i$. Observe now that, for any word w over A , we have $\text{val}_{2c}(00w) = \text{val}_{2c}(0w)$ and $\text{val}_{2c}(11w) = \text{val}_{2c}(1w)$. So for any integer $n \in \mathbb{Z}$, there exists a unique word $w \in A^* \setminus (00A^* \cup 11A^*)$ such that $n = \text{val}_{2c}(w)$. We let $\text{rep}_{2c}(n)$ denote this unique word and we call it the *two's complement representation* of n . The first few two's complement representations of integers are $(\text{rep}_{2c}(n))_{-4 \leq n \leq 4} = 100, 101, 10, 1, \varepsilon, 01, 010, 011, 0100$. Observe that the two's complement numeration system is positional: we may use the sequence of powers of 2 as weights.

2.1 Dumont–Thomas numeration systems

In this section, we expose the theory of Dumont and Thomas on numeration systems [6]. Roughly, a substitution μ defines a numeration system where the representation of n is the label of a path from the root to column n in a tree associated with μ .

We first recall the notion of admissible sequences for a substitution. Let $\mu: A^* \rightarrow A^*$ be a substitution. Fix a letter $a \in A$ and an integer k . For every $i \in \{0, \dots, k\}$, we consider a pair $(m_i, a_i) \in A^* \times A$. The sequence $((m_i, a_i))_{i=0, \dots, k}$ is *admissible with respect to μ* if for every $i \in \{1, \dots, k\}$, $m_{i-1} a_{i-1}$ is a prefix of $\mu(a_i)$. This sequence is *a-admissible with respect to μ* if it is admissible with respect to μ and $m_k a_k$ is a prefix of $\mu(a)$. When the context is clear, we simply say *admissible* or *a-admissible* without specifying the substitution. See Fig. 1 for an illustration.

Example 2.2. Consider the Tribonacci substitution $\tau: a \mapsto ab, b \mapsto ac, c \mapsto a$. The tree $\mathcal{T}_{\tau, a}$ in Fig. 2 is defined as in Section 2 and allows us to compute a -admissible sequences. For instance, the sequence $(m_0, a_0) = (\varepsilon, a), (m_1, a_1) = (\varepsilon, a), (m_2, a_2) = (a, b)$ is a -admissible and corresponds to the path in bold in $\mathcal{T}_{\tau, a}$. Note that the sequences $(a, c), (a, b)$ and $(a, c), (a, b), (\varepsilon, a)$ are also a -admissible and lead to nodes labeled by c and colored (in orange) in $\mathcal{T}_{\tau, a}$, both in column 3.

As illustrated in Example 2.2, given a right-infinite fixed point of a substitution, multiple paths in the corresponding tree can lead to nodes in the same column. However, there is a unique path starting from the root of the tree and labeled by a word not starting with 0. The following

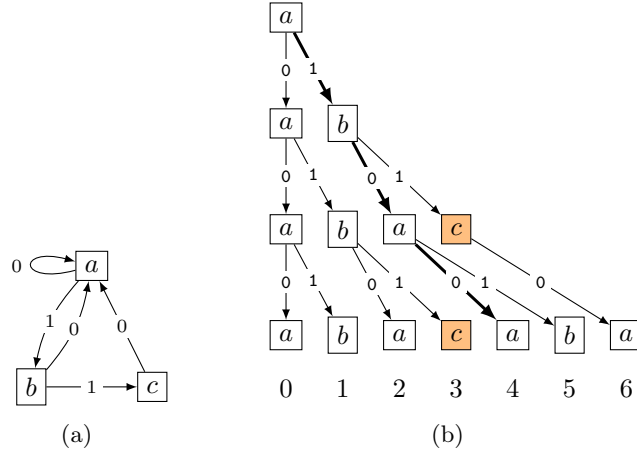


Figure 2: On the left, the directed graph associated with the Tribonacci substitution $\tau: a \mapsto ab, b \mapsto ac, c \mapsto a$. On the right, the tree $\mathcal{T}_{\tau,a}$ displays all a -admissible sequences of length at most 3 for the Tribonacci substitution τ and growing letter a .

Table 1: The representation of the first few non-negative integers in the Dumont–Thomas numeration system for τ and a , where $\tau: a \mapsto ab, b \mapsto ac, c \mapsto a$ is the Tribonacci substitution.

n	0	1	2	3	4	5	6
$\text{rep}_{\tau,a}(n)$	ε	1	10	11	100	101	110

result, shown by Dumont and Thomas, essentially formalizes this and states that there is a unique admissible sequence for every prefix of the fixed point.

Theorem 2.3. [6, Theorem 1.5] *Let $a \in A$ and let $\mu: A^* \rightarrow A^*$ be a substitution. Let u be a right-infinite fixed point of μ with growing seed $u_0 = a$. For every integer $n \geq 1$, there exist a unique integer $k = k(n)$ and a unique sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that the sequence is a -admissible, $m_{k-1} \neq \varepsilon$, and $u_0 u_1 \cdots u_{n-1} = \mu^{k-1}(m_{k-1}) \mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.*

Theorem 2.3 allows to define a numeration system for non-negative integers as defined below.

Definition 2.4. Let $\mu: A^* \rightarrow A^*$ be a substitution, a be a growing letter, and u be a right-infinite fixed point of μ with seed $u_0 = a$. Set $c = \max_{b \in A} |\mu(b)| - 1$ and define the set $D = \{0, 1, \dots, c\}$. We define the map $\text{rep}_{\mu,a}: \mathbb{N} \rightarrow D^*, n \mapsto \text{rep}_{\mu,a}(n)$ by

$$\text{rep}_{\mu,a}(n) = \begin{cases} \varepsilon, & \text{if } n = 0; \\ |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \geq 1; \end{cases}$$

where $k = k(n)$ is the unique integer and $((m_i, a_i))_{i=0, \dots, k-1}$ is the unique sequence from Theorem 2.3. This numeration system is called the *Dumont–Thomas numeration system associated with μ and u* .

Note that we use a special font for the digits representing numbers to distinguish them from integers. Note also that $\text{rep}_{\mu,a}(n)$ is the label of the shortest path from the root to a node in column n in the tree $\mathcal{T}_{\mu,a}$.

Example 2.5. We resume Example 2.2. The representation of the first few non-negative integers for the Tribonacci substitution τ and growing letter a are given in Table 1.

In [11], the authors proposed an extension of the numeration system given in Definition 2.4 to all integers. The next two results generalize Theorem 2.3 to right- and left-infinite periodic points of substitutions with a growing letter.

Theorem 2.6. [11, Theorem 4.1] Let $\mu : A^* \rightarrow A^*$ be a substitution with growing letter $a \in A$. Consider a right-infinite periodic point $u \in \text{Per}_{\mathbb{N}}(\mu)$ with $u_0 = a$ and period $p \geq 1$. For every integer $n \geq 1$, there exist a unique integer $k = k(n)$ such that p divides k and a unique sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that the sequence is a -admissible, $m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon$, and $u_0u_1 \cdots u_{n-1} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.

Theorem 2.7. [11, Theorem 4.2] Let $\mu : A^* \rightarrow A^*$ be a substitution with growing letter $b \in A$. Consider a left-infinite periodic point $u \in \text{Per}_{\mathbb{Z}_{<0}}(\mu)$ with $u_{-1} = b$ and period $p \geq 1$. For every integer $n \leq -2$, there exist a unique integer $k = k(n)$ such that p divides k and a unique sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that the sequence is b -admissible,

$$\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b),$$

$$\text{and } u_{-| \mu^k(b) |} \cdots u_{n-2}u_{n-1} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0).$$

Similarly to Definition 2.4, the combination of Theorems 2.6 and 2.7 allows to define a numeration system for all integers as defined below.

Definition 2.8. [11, Definition 4.3] Let $\mu : A^* \rightarrow A^*$ be a substitution and let $u \in \text{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $u_{-1}|u_0$ and period $p \geq 1$. Define $c = \max_{a \in A} |\mu(a)| - 1$ and the set $D = \{0, 1, \dots, c\}$. We define the map $\text{rep}_u : \mathbb{Z} \rightarrow \{0, 1\}D^*$, $n \mapsto \text{rep}_u(n)$ by

$$\text{rep}_u(n) = \begin{cases} 0 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0; \\ 1, & \text{if } n = -1; \\ 1 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \leq -2; \end{cases}$$

where if $n \geq 1$ (resp., $n \leq -2$), $k = k(n)$ is the unique integer and $((m_i, a_i))_{i=0, \dots, k-1}$ is the unique sequence obtained from Theorem 2.6 (resp., Theorem 2.7) applied on the right-infinite periodic point $u|_{\mathbb{N}} = u_0u_1 \cdots$ (resp., the left-infinite periodic point $u|_{\mathbb{Z}_{<0}} = \cdots u_{-2}u_{-1}$) with period p . This numeration system is called the *Dumont–Thomas complement numeration system associated with u* .

Observe that all representations of non-negative integers start with a letter 0 while those of negative integers start with a letter 1. Again, we use a special font for the digits in the representations of integers. Also note that, if we get rid of the initial letter, the lengths of all representations are multiples of p . Using the tree interpretation, this generalization corresponds to using the tree $\mathcal{T}_{\mu, u_{-1}|u_0}$ to accommodate negative numbers, and using only paths of length one plus a multiple of p to manage periodicity.

Example 2.9. Consider the substitution $\mu : a \mapsto abc, b \mapsto c, c \mapsto ac$ and the two-sided periodic point $u \in \text{Per}_{\mathbb{Z}}(\mu)$ with growing seed $c|a$. (Note that the period of u is 1, so it is actually a fixed point.) The corresponding tree $\mathcal{T}_{\mu, c|a}$ is depicted in Fig. 3a while the corresponding Dumont–Thomas numeration system is illustrated in Fig. 3b. For instance, the paths 02 and 002 lead to the nodes in orange, which both project onto $n = 2$ on the horizontal line below the tree. To represent this integer $n = 2$, we need to take the path leading to the earliest level having a node projecting onto $n = 2$, so $\text{rep}_u(2) = 02$.

In the next example, we consider two substitutions associated with the same real number. The key observation is that one gives rise to a numeration system that is positional, and the numeration system for the other is not. This example is already present in Lepšová’s PhD thesis [13].

Example 2.10. [13, Example 6.5.6] Consider the *silver mean* $\beta = 1 + \sqrt{2}$. We define the substitutions $\mu : a \mapsto aab, b \mapsto a$ and $\rho : a \mapsto abb, b \mapsto ab$. The characteristic polynomials of the corresponding adjacency matrices of μ and ρ are equal and have dominant root β . We now consider the two-sided periodic points $u \in \text{Per}_{\mathbb{Z}}(\mu)$ with growing seed $b|a$ and period 2 and $v \in \text{Per}_{\mathbb{Z}}(\rho)$

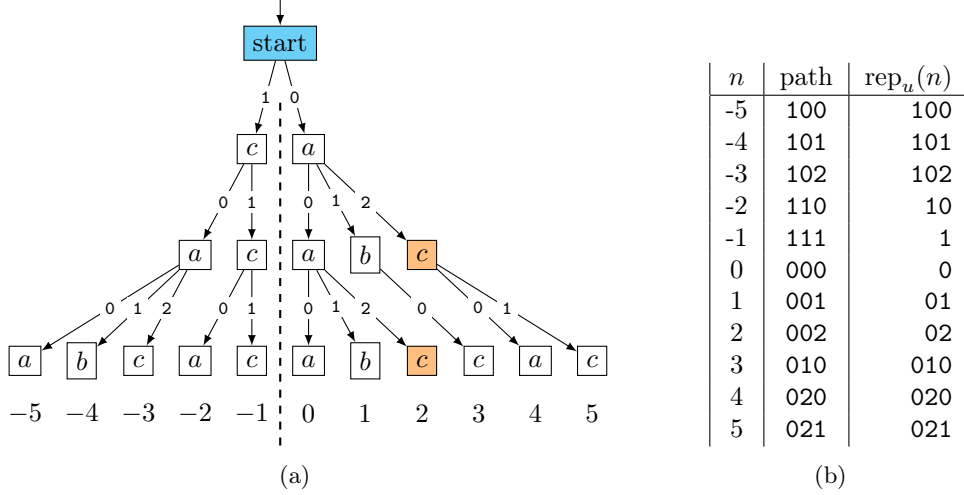


Figure 3: On the left, the tree associated $\mathcal{T}_{\mu, c|a}$ with the two-sided periodic point u of the substitution $\mu: a \mapsto abc, b \mapsto c, c \mapsto ac$ with growing seed $c|a$. On the right, the representations of the first few integers in the corresponding Dumont–Thomas complement numeration system.

Table 2: The representations of the first few integers in the Dumont–Thomas numeration system associated with $u \in \text{Per}_{\mathbb{Z}}(\mu)$ and $v \in \text{Per}_{\mathbb{Z}}(\rho)$ and $r = 0$ where $\mu: a \mapsto aab, b \mapsto a$ and $\rho: a \mapsto abb, b \mapsto ab$, both with seed $b|a$.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\text{rep}_{u,0}(n)$	10112	10120	100	101	1	0	001	002	010	011	012
$\text{rep}_{v,0}(n)$	100	101	102	10	1	0	01	02	010	011	020

with growing seed $b|a$ and period 1. The representations of the few integers in the corresponding Dumont–Thomas numeration systems with $r = 0$ are given in Table 2. We observe that the first numeration system is positional while the second is not. Indeed, if it were, there would be an evaluation map val such that $\text{val}(\text{rep}_{v,0}(n)) = n$ for all integers n . Since $\text{rep}_{v,0}(3) = 010$, we must have $U_1 = 3$, but then $\text{val}(\text{rep}_{v,0}(5)) = \text{val}(020) = 6 \neq 5$, a contradiction.

As suggested by the previous example, Dumont–Thomas numeration systems for \mathbb{Z} corresponding to periodic points of substitution are not always positional. Therefore, Lepšová raised the following natural question [13].

Question B. [13, Question 6.5.7] What are the conditions for a complement Dumont–Thomas numeration system to be positional?

2.2 Changing the lengths in the numeration language

As already observed, Theorems 2.6 and 2.7 both provide a numeration language whose words are of length multiple of the period p of the considered infinite word. In this section, we show that this is a superfluous restriction and we show that all lengths can be attained modulo p , a fact which is clear when considering the tree interpretation. We start with right-infinite periodic points.

Theorem 2.11. *Let $\mu: A^* \rightarrow A^*$ be a substitution with growing letter $a \in A$. Consider a right-infinite periodic point $u \in \text{Per}_{\mathbb{N}}(\mu)$ with $u_0 = a$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \dots, p-1\}$ modulo p and define $v_r = \mu^r(u)$. For every integer $n \geq 0$, there exist a unique integer $k = k(n)$*

with $k \equiv r \pmod p$ and a unique sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that the sequence is a -admissible,

$$m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon \text{ if } k \geq p, \quad (1)$$

and $(v_r)_{[0, n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.

The proof of Theorem 2.11 mimics that of Theorem 2.6 but we choose to present it for the sake of completeness. Some intermediate results are necessary, which we recall too.

Lemma 2.12. [6, Lemma 1.1] *Let $\mu: A^* \rightarrow A^*$ be a substitution and let $k \geq 0$ be an integer. If $((m_i, a_i))_{i=0, \dots, k}$ is an admissible sequence, then $\sum_{i=0}^k |\mu^i(m_i)| < |\mu^k(m_k a_k)|$.*

Lemma 2.13. [11, Lemma 3.9] *Let $\mu: A^* \rightarrow A^*$ be a substitution and let $k \geq 0$ be an integer. Let also $m \in A^*$ and $a \in A$ be such that m is a proper prefix of $\mu^k(a)$. Then there exists a unique a -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that $m = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.*

Proof of Theorem 2.11. Since a is a growing letter, the sequence $(|\mu^{\ell p+r}(a)|)_{\ell \geq 0}$ is (strictly) increasing. Note that, for every $r \in \{0, 1, \dots, p-1\}$, the word v_r is also a periodic point of μ with period p . Let $n \geq 0$ be an integer. Setting $|\mu^{-p+r}(a)| = 0$ by convention, there exists a unique integer $\ell = \ell(n) \geq 0$ such that $|\mu^{(\ell-1)p+r}(a)| \leq n < |\mu^{\ell p+r}(a)|$. Let $k = k(n) = \ell p + r$ be such that

$$|\mu^{k-p}(a)| \leq n < |\mu^k(a)|. \quad (2)$$

Now the word $m = v_{r,0}v_{r,1} \cdots v_{r,n-1}$ of length n is a proper prefix of $\mu^k(a)$ (for a visualization, recall that the k -th level of the tree $\mathcal{T}_{\mu,a}$ is a prefix of v_r since $k \equiv r \pmod p$). By Lemma 2.13, there exists a unique a -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that $m = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)$. It remains to show that $m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon$ if $k \geq p$. Towards a contradiction, we assume that $k \geq p$ and $m_{k-1}m_{k-2} \cdots m_{k-p} = \varepsilon$. By inspecting the definition of admissible sequences with periodic points, it means that $a_{k-p} = a$. Then Lemma 2.12 implies that

$$\begin{aligned} n = |m| &= \sum_{j=0}^{k-1} |\mu^j(m_j)| = \sum_{j=0}^{k-p-1} |\mu^j(m_j)| \\ &< |\mu^{k-p-1}(m_{k-p-1}a_{k-p-1})| \leq |\mu^{k-p-1}(\mu(a_{k-p}))| = |\mu^{k-p}(a)|, \end{aligned}$$

which contradicts (2). Consequently, $m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon$, which finishes the proof of existence.

For uniqueness, we already know that the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ constructed above is the only a -admissible sequence of length k that satisfies the conditions of the statement. It remains to show that other lengths do not lead to more such sequences. First, suppose that $((m'_i, a'_i))_{i=0, \dots, k'-1}$ is an a -admissible sequence of length k' with $k' < k$ that respects the conditions of the statement. Since $k' \equiv r \pmod p$, we have $k' \leq k - p$. From Lemma 2.12, we deduce that

$$|\mu^{k'-1}(m'_{k'-1}) \cdots \mu^0(m'_0)| < |\mu^{k'-1}(m'_{k'-1}a'_{k'-1})| \leq |\mu^{k'}(a)| \leq |\mu^{k-p}(a)| \leq n = |m|.$$

Thus it must be that $\mu^{k'-1}(m'_{k'-1}) \cdots \mu^0(m'_0) \neq m$, which violates the last condition of statement. Now for $k' > k$, we note that the sequence obtained by prefixing

$$(m_{k'-1}, a_{k'-1}) = (\varepsilon, \mu(a)_0), (m_{k'-2}, a_{k'-2}) = (\varepsilon, \mu^2(a)_0), \dots, (m_k, a_k) = (\varepsilon, \mu^{k'-k}(a)_0)$$

to the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ produces an a -admissible sequence of length k' which is such that $m = \mu^{k'-1}(m'_{k'-1}) \cdots \mu^0(m'_0)$. By Lemma 2.13, this is the only such sequence of length k' . But this sequence starts with $m_{k'-1} = \cdots = m_{k-p} = \varepsilon$, and thus it does not satisfy the statement. Thus the uniqueness is shown. \square

We now move to left-infinite periodic points.

Theorem 2.14. *Let $\mu : A^* \rightarrow A^*$ be a substitution with growing letter $b \in A$. Consider a left-infinite periodic point $u \in \text{Per}_{\mathbb{Z}_{<0}}(\mu)$ with $u_{-1} = b$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \dots, p-1\}$ modulo p and define $v_r = \mu^r(u)$. For every integer $n \leq -1$, there exist a unique integer $k = k(n)$ with $k \equiv r \pmod{p}$ and a unique sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that the sequence is b -admissible,*

$$\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b) \text{ if } k \geq p, \quad (3)$$

and $(v_r)_{[-|\mu^k(b)|, n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.

Proof. Note that, for every $r \in \{0, 1, \dots, p-1\}$, the word v_r is also a periodic point of μ with period p . Since b is a growing letter, the sequence $(-|\mu^{\ell p+r}(b)|)_{\ell \geq 0}$ is (strictly) decreasing. Let $n \leq -1$ be an integer. Setting $|\mu^{-p+r}(b)| = 0$ by convention, there exists a unique integer $\ell = \ell(n)$ such that $-|\mu^{\ell p+r}(b)| \leq n < -|\mu^{(\ell+1)p+r}(b)|$. Let $k = k(n) = \ell p + r$ be such that

$$-|\mu^k(b)| \leq n < -|\mu^{k-p}(b)|. \quad (4)$$

Now the word $m = v_{r, -|\mu^k(b)|} \cdots v_{r, n-2} v_{r, n-1}$ of length

$$|m| = |\mu^k(b)| + n < |\mu^k(b)| - |\mu^{k-p}(b)| \leq |\mu^k(b)| \quad (5)$$

is a proper prefix of $\mu^k(b)$. Using Lemma 2.13, there exists a unique b -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that $m = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$. We now show that (3) holds. Towards a contradiction, we assume that (3) is actually an equality. Then we have $a_{k-p} = b$. Writing

$$m = \mu^{k-p}(\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p}))\mu^{k-p-1}(m_{k-p-1}) \cdots \mu^0(m_0),$$

we obtain from (3) that

$$|m| = |\mu^{k-p}(\mu^p(b))| - |\mu^{k-p}(a_{k-p})| + \sum_{j=0}^{k-p-1} |\mu^j(m_j)| \geq |\mu^k(b)| - |\mu^{k-p}(b)|$$

which contradicts (5). Consequently, (3) is a non-equality, which finishes the proof of existence.

For uniqueness, we already know that the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ constructed above is the only b -admissible sequence of length k that satisfies the conditions of the statement. It remains to show that other lengths do not lead to more such sequences. First for $k' > k$, we note that when $k \equiv r \pmod{p}$ the sequence obtained by prefixing

$$\begin{aligned} (m_{k'-1}, a_{k'-1}) &= (\mu(b)_{[0, |\mu(b)|-1]}, \mu(b)_{|\mu(b)|-1}), \\ (m_{k'-2}, a_{k'-2}) &= (\mu(a_{k'-1})_{[0, |\mu(a_{k'-1})|-1]}, \mu(a_{k'-1})_{|\mu(a_{k'-1})|-1}), \\ &\vdots \\ (m_k, a_k) &= (\mu(a_{k+1})_{[0, |\mu(a_{k+1})|-1]}, \mu(a_{k+1})_{|\mu(a_{k+1})|-1}) \end{aligned}$$

to the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ produces a b -admissible sequence of length k' which is such that $v_{[-|\mu^{k'}(b)|, n-1]} = \mu^{k'-1}(m_{k'-1}) \cdots \mu^0(m_0)$. By Lemma 2.13, this is the only such sequence of length k' . But this sequence violates (3).

For the case $k' < k$, if we had $k \equiv r \pmod{p}$ and some sequence satisfying the statement existed, we could similarly extend that sequence to a sequence of length k that violates (3). But by Lemma 2.13 this would be the only sequence of length k with $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0) = m$. This is impossible as we have found another such sequence (that does not violate (3)) in the proof of existence above. \square

From these results, we generalize the definition of Dumont–Thomas numeration systems [6, 11].

Definition 2.15. Let $\mu: A^* \rightarrow A^*$ be a substitution and let $u \in \text{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $u_{-1}|u_0$ and period $p \geq 1$. Let $r \in \{0, 1, \dots, p-1\}$ be a residue modulo p . Define $c = \max_{a \in A} |\mu(a)| - 1$ and the set $D = \{0, 1, \dots, c\}$. We define the map $\text{rep}_{u,r}: \mathbb{Z} \rightarrow \{0, 1\}D^*$, $n \mapsto \text{rep}_{u,r}(n)$ by

$$\text{rep}_{u,r}(n) = \begin{cases} 0 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \geq 0; \\ 1 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \leq -1; \end{cases}$$

where $k = k(n)$ is the unique integer congruent to r modulo p and $((m_i, a_i))_{i=0, \dots, k-1}$ is the unique sequence obtained from Theorem 2.11 (resp., Theorem 2.14) applied on the right-infinite periodic point $u|_{\mathbb{N}} = u_0 u_1 \cdots$ (resp., the left-infinite periodic point $u|_{\mathbb{Z}_{<0}} = \cdots u_{-2} u_{-1}$) with period p .

This numeration system is called the *Dumont–Thomas complement numeration system associated with μ , u and r* . When the context is clear, we drop the dependence on μ , u and r .

Note that the lengths of the representations in the previous numeration system are congruent to $r+1$ modulo p . Observe that all representations of non-negative integers start with a letter 0 while those of negative integers start with a letter 1. Note that we again use a special font for the digits representing numbers to distinguish them from integers.

Example 2.16. Consider the substitution $\mu: a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$. Take the periodic point $u \in \text{Per}_{\mathbb{N}}(\mu)$ with growing seed $a|a$ and period $p = 2$. The tree $\mathcal{T}_{\mu,a|a}$ is depicted on Fig. 4. Now depending on whether we want representations of even or odd lengths, we obtain different numeration systems as illustrated on the table in Fig. 4.

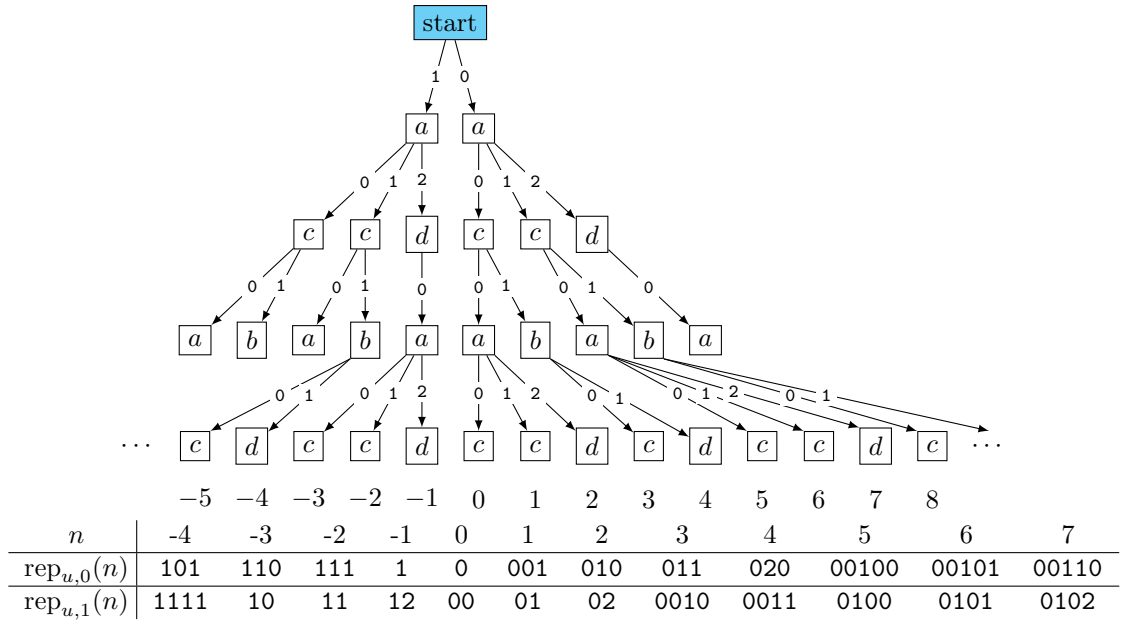


Figure 4: On the top, the tree $\mathcal{T}_{\mu,a|a}$ for the substitution $\mu: a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ and the periodic point u of period $p = 2$ and seed $a|a$. On the bottom, depending on the residue $r \in \{0, 1\}$, we obtain a Dumont–Thomas numeration system and we give $(\text{rep}_{u,r}(n))_{-4 \leq n \leq 7}$ whose lengths are congruent to $r+1 \pmod p$.

3 Positional Dumont–Thomas numeration systems

In this section, we study when a substitution generates a Dumont–Thomas numeration system that is positional to answer Question B. We state our theorem in the most general case, then

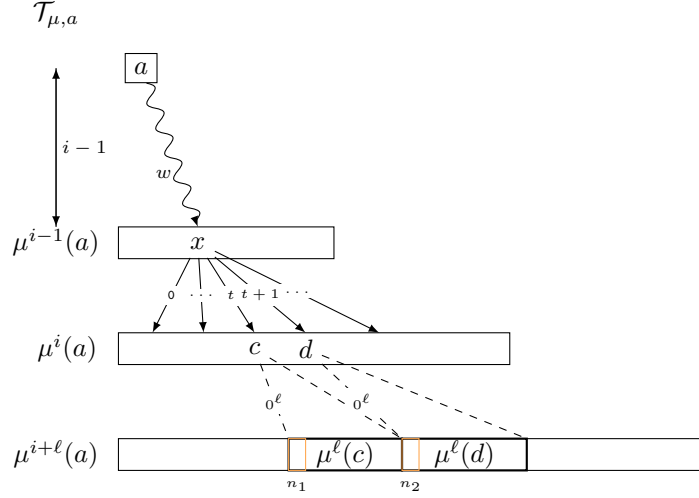


Figure 5: Comparing the values of $wt0^\ell$ and $w(t+1)0^\ell$ in the right part of $\mathcal{T}_{\mu,b|a}$.

present in Section 3.3 some particularizations as corollaries.

3.1 Sketch of the argument

The aim of this section is to informally sketch the argument that we will use to solve Question B. We will then present examples where this argument fails. This will allow us to motivate the technicalities that are introduced in Section 3.2 and to explain the reasoning without these technicalities getting in the way of the explanation.

Sketch 3.1. We let $\mu: A^* \rightarrow A^*$ be a substitution, $u \in \text{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point of μ with growing seed $b|a$ and period $p \geq 1$, and $r \in \{0, 1, \dots, p-1\}$ be a residue. We also consider the corresponding Dumont–Thomas complement numeration system associated with μ , u and r .

Now consider two words $wt0^\ell$ and $w(t+1)0^\ell$ that label two paths in the tree $\mathcal{T}_{\mu,b|a}$ (see Fig. 5, where we have arbitrarily chosen to represent the situation with $\mathcal{T}_{\mu,a}$ instead) and assume these words are the representations of the integers n_1 and n_2 respectively, i.e., the paths end on columns n_1 and n_2 , in addition to which we have some conditions for admissibility. We also let c and d be the letters attained after reading wt and $w(t+1)$ respectively.

If the numeration system is positional, then $n_2 - n_1$ must equal U_ℓ , the weight in position ℓ , as the representations of n_1 and n_2 differ only by one unit in position ℓ . However, $n_2 - n_1$ is the number of columns between those two nodes in the tree, which is $|\mu^\ell(c)|$, i.e., the number of level- ℓ descendants of c in the tree (see again Fig. 5). Since this reasoning could be applied to any letter c that has a sibling to its right in $\mathcal{T}_{\mu,b|a}$, this points towards the following implication: *If the Dumont–Thomas numeration system associated with μ , u , and r is positional, then $|\mu^\ell(c)| = U_\ell$ for every letter c that has a younger sibling in $\mathcal{T}_{\mu,b|a}$.*

The converse implication is also justified, almost by definition of a Dumont–Thomas numeration system. If the positive integer $n \geq 1$ is represented by the word $0 \cdot w_{k-1} \dots w_0$, this means that there is an a -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ such that

$$u_{[0, n-1]} = \mu^{k-1}(m_{k-1}) \dots \mu^0(m_0) \quad (6)$$

and $|m_i| = w_i$ for every $i \in \{0, \dots, k-1\}$. Because all letters in m_i have a_i as a younger sibling, their image by μ^ℓ has length U_ℓ by assumption. Taking the length in (6) yields $n = \sum_{i=0}^{k-1} U_i w_i$, which corresponds to the numeration system being positional with weights $(U_i)_{i \geq 0}$. The case of negative numbers is similar, with one correcting term corresponding to the value of V_{k-1} .

The above sketch leads us to formulate the next conjecture: *The Dumont–Thomas numeration system associated with μ , u , and r is positional if and only if $c \mapsto |\mu^\ell(c)|$ is constant on all letters c that have a younger sibling in $\mathcal{T}_{\mu,b|a}$, in which case this constant is the weight U_ℓ .*

However, trying to prove this conjecture reveals two issues with Sketch 3.1, which the following examples highlight.

Example 3.2. Recall the substitution $\mu: a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ from Example 2.16. Observe that μ is not primitive and is built by intertwining on distinct alphabets the Fibonacci substitution $x \mapsto xy, y \mapsto x$ and the substitution $x \mapsto xy, y \mapsto xy$ associated with the squared Golden Ratio. Furthermore, μ has periodic points with period $p = 2$.

The letters that have a younger sibling in $\mathcal{T}_{\mu,a}$ are a and c , but the sequences of the lengths of their consecutive images under μ are respectively $(|\mu^j(a)|)_{j \geq 0} = 1, 3, 5, 13, 21, 55, 89, 233, 377, \dots$ and $(|\mu^j(c)|)_{j \geq 0} = 1, 2, 5, 8, 21, 34, 89, 144, 377, \dots$. Our tentative condition for positionality is not satisfied. Despite this, the corresponding Dumont–Thomas numeration system is positional for both values of r , with the weight sequence $1, 2, 5, 8, 21, 34, \dots$ for $r = 0$ and $1, 3, 5, 13, 21, 55, \dots$ for $r = 1$, both of which are obtained by taking two out of every three terms in the Fibonacci sequence.

The way to see this is to examine our proof for the converse implication in the above sketch. With the same notation, for $r = 0$ we get that m_i is a power of a if i is even and a power of c if i is odd. Therefore, we obtain the sequence of weights by choosing $U_i = |\mu^i(a)|$ if i is even, and $U_i = |\mu^i(c)|$ if i is odd. Since the letters a and c are never present at the same level of the tree, the difference in image length is not a problem in our case.

Example 3.2 illustrates that, in the case where μ is not primitive, not all letters may appear at every level of the tree, and as such we may only control some of their image lengths. This will be the purpose of the sets E_j in the following section with Definition 3.5.

Example 3.3. Consider the substitution $\mu: a \mapsto bca, b \mapsto bb, c \mapsto b$ and the seed $a|b$. The fragment of the associated numeration system over $\mathbb{Z}^{<0}$ represented in Fig. 6. The letters b and c both appear in the tree with a younger sibling, and they have images of different lengths. However, the numeration system is still positional. One can show that the weights are $U_i = 2^i$ and $V_0 = 1, V_i = 3 \cdot 2^{i-1}$ for every $i \geq 1$ by showing that the language of this numeration system is $1\{0, 1\}^* \setminus 111\{0, 1\}^*$.

Our sketched argument fails, because if wt is a path to a node labeled by c in the tree, then $w(t+1)0^\ell$ is never the representation of any number, due to Condition (3) from Theorem 2.14. Thus we cannot constrain the lengths of the images of c .

The sort of argument of Example 3.3 can happen for letters that only appear in column -2 in $\mathcal{T}_{\mu,b|a}$ as $w(t+1)$ will lead to a node in column -1 and paths extending this one may not correspond to representations of numbers because of Condition (3). In fact, such letters may lead to even stranger behavior.

Example 3.4. Consider the eight-letter substitution μ defined by $a_1 \mapsto bca_2, f \mapsto b^2, a_2 \mapsto a_3, b \mapsto d^2, c \mapsto d^2e, a_3 \mapsto a_1, d \mapsto f^2, e \mapsto f^4$, and its periodic point u with seed $a_1|$ with period $p = 3$ (specifying the right part is not of importance for the sequel). When considering the residue class $r = 2$, this numeration system is not positional as we have $\text{rep}_{u,r}(-6) = 100$, $\text{rep}_{u,r}(-4) = 110$ and $\text{rep}_{u,r}(-1) = 120$, leading to both $U_1 = 2$ and $U_1 = 3$, a contradiction. However, if we change $\mu(c)$ to de instead of d^2e , the numeration system becomes positional for $r = 2$ despite the fact that the lengths of most images under μ of b and c are still different.

These examples motivate the detours and technical details present in the next section, notably Definition 3.6. Keeping these hurdles in mind, we now move on to proving the main result (Theorem 3.8), using the sketch above but proceeding in a more careful manner.

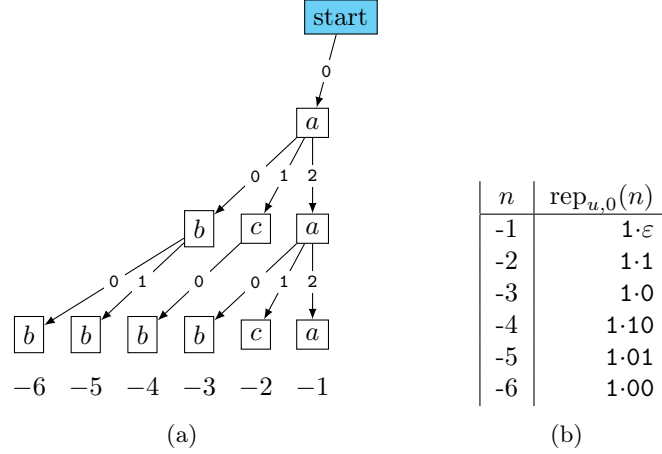


Figure 6: The tree $\mathcal{T}_{\mu, a|}$ and the first few representations of negative integers in the numeration system associated with the substitution $\mu: a \mapsto bca, b \mapsto bb, c \mapsto b$, the left-infinite periodic point $u = \cdots bbca$ and residue $r = 0$.

3.2 Main result

To state the main result, we need to define some particular sets of letters. The setting of this section is as follows: we consider a substitution $\mu: A^* \rightarrow A^*$ and a two-sided periodic point $u \in \text{Per}_{\mathbb{Z}}(\mu)$ of μ with growing seed $b|a$ and period $p \geq 1$. We draw the tree $\mathcal{T}_{\mu, b|a}$. For a fixed residue $r \in \{0, \dots, p-1\}$, we also consider the corresponding Dumont–Thomas numeration system $\text{rep}_{u,r}: \mathbb{Z} \rightarrow \{0, 1\}D^*$ from Definition 2.15.

Definition 3.5. Let $j \in \{0, \dots, p-1\}$. We let E_j be the set of letters $c \in A$ such that there exist some integer $k \geq 1$ and some sequence $((m_i, a_i))_{i=0, \dots, k-1} \in (A^* \times A)^k$ that verifies the following:

- the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ is a - or b -admissible;
- we have $k \equiv j \pmod{p}$;
- the letter c appears at the end of the word m_0 ;
- if $((m_i, a_i))_{i=0, \dots, k-1}$ is b -admissible, then $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 \neq \mu^k(b)$.

Thinking in terms of the tree $\mathcal{T}_{\mu, b|a}$, the first three conditions simply describe letters that appear in the tree at some level congruent to $j \pmod{p}$ and have a younger sibling on that level. In view of Fig. 1, the last condition excludes letters c that only appear in column -2 in the tree $\mathcal{T}_{\mu, b|a}$. To deal with these letters, a dedicated condition is required, as follows. Recall that the j th level of the tree $\mathcal{T}_{\mu, b|a}$ corresponds to the j th iteration of μ on $b|a$ (by convention, the root of $\mathcal{T}_{\mu, b|a}$ is on level -1).

Definition 3.6. Let $j \in \{0, \dots, p-1\}$. On level j in the tree $\mathcal{T}_{\mu, b|a}$, we let c be the letter in column -2 and d be the letter in column -1 (i.e., d is immediately to the right of c). If c and d share the same parent, we consider two cases to modify E_j . If $|\mu^{p-j}(d)| > 1$, then we add c to E_j if it was not already present. If $|\mu^{p-j}(d)| = 1$ and $j \leq r < p$, then we add the following condition:

$$|\mu^{r-j}(c)| \text{ must be equal to } |\mu^{r-j}(e)| \text{ for every letter } e \in E_j. \quad (7)$$

As we will see below, all these values are equal if the numeration system is positional.

Example 3.7. For the substitution μ from Example 3.2, consider its two-sided periodic point with seed $a|a$, so $p = 2$. In this case, we obtain $E_0 = \{a\}$ and $E_1 = \{c\}$. Note that the letters

b, d do not have younger siblings in $\mathcal{T}_{\mu, a|a}$. For $j = 1$, we observe that c is on level j and column -2 . It also has d as younger sibling and we have $|\mu^{p-j}(d)| = |\mu(d)| = 1$. Condition (7) is trivially satisfied for $r = 1$.

Consider the substitution $\mu: a \mapsto bcd, d \mapsto ba, b \mapsto b^2, c \mapsto b$, its two-sided periodic point of period 2 with seed $a|b$ and the residue $r = 0$. If we go only by Definition 3.5, we will find $E_0 = E_1 = \{b\}$, but if we add Definition 3.6, c is added to E_1 , and we now correctly find that the system is not positional (which we can also see from the representations of $-5, -3$ and -2).

We can now state the main result.

Theorem 3.8. *Let $\mu: A^* \rightarrow A^*$ be a substitution and let $u \in \text{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $b|a$ and period $p \geq 1$. The Dumont–Thomas complement numeration system associated with μ, u , and r is positional if and only if for every $j \in \{0, \dots, p-1\}$, the map $c \mapsto |\mu^\ell(c)|$ is constant over E_j for every ℓ such that $\ell + j \equiv r \pmod{p}$, and condition (7) is satisfied for the letters where it was added.*

In this case, the sequences U, V of weights of the numeration system are given as follows: for every $\ell \geq 0$, we define $U_\ell = |\mu^\ell(c)|$ for a letter $c \in E_j$ where $j \in \{0, \dots, p-1\}$ and $\ell + j \equiv r \pmod{p}$; and $V_\ell = |\mu^\ell(b)|$ for every $\ell \in \mathbb{N}$.

Remark 3.9. It may be, although rarely, that E_j is empty. (For instance, consider the case of $\mu: a \mapsto b, b \mapsto aa$ and the set E_1 .) In this case, if condition (7) applies to some letter c , we may give the weight $|\mu^{r-j}(c)|$ to position $r - j$. Otherwise, this means that only the digit 0 appears at positions congruent to $j \pmod{p}$ in this numeration system, and as such the weight given to these positions is arbitrary.

The proof of our main result relies on the following technical lemma. Note that the conditions in the statement are those of Theorems 2.11 and 2.14.

Lemma 3.10. *Let $j \in \{0, \dots, p-1\}$. Fix a letter c that is in E_j according to Definition 3.5.*

If there exists a b -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ that verifies the four conditions in Definition 3.5, then there exists a b -admissible sequence that verifies those conditions, plus the extra condition that $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$ or $k < p$.

If there exists an a -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ that verifies the three conditions in Definition 3.5, then there exists an a -admissible sequence that verifies those conditions, plus the extra condition that $m_{k-1} \cdots m_{k-p} \neq \varepsilon$ or $k < p$.

Proof. For the first part, if we assume that $k \geq p$ and $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} = \mu^p(b)$, then the fact that u has period p implies that $a_{k-p} = b$. The cropped sequence $((m_i, a_i))_{i=0, \dots, k-p-1}$ is thus again b -admissible. We also have $k - p \equiv j \pmod{p}$ and c is the last letter of m_0 . We also have $\mu^{k-p-1}(m_{k-p-1}) \cdots \mu^0(m_0)a_0 \neq \mu^{k-p}(b)$ for otherwise

$$\begin{aligned} \mu^k(b) &= \mu^{k-p}(\mu^p(b)) = \mu^{k-p}(\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})b) \\ &= \mu^{k-1}(m_{k-1}) \cdots \mu^{k-p}(m_{k-p})\mu^{k-p}(b) \\ &= \mu^{k-1}(m_{k-1}) \cdots \mu^{k-p}(m_{k-p})\mu^{k-p-1}(m_{k-p-1}) \cdots \mu^0(m_0)a_0, \end{aligned}$$

which is forbidden by the condition of Definition 3.5. We have showed that the shorter sequence $((m_i, a_i))_{i=0, \dots, k-p-1}$ also satisfies the condition of the previous definition. Iterating this process, we get the existence of the required sequence.

Similarly, if the sequence is a -admissible with $k \geq p$ and $m_{k-1}m_{k-2} \cdots m_{k-p} = \varepsilon$, then the fact that u has period p implies that $a_{k-p} = a$. We conclude in the same way: the cropped sequence $((m_i, a_i))_{i=0, \dots, k-p-1}$ is thus again a -admissible, we also have $k - p \equiv j \pmod{p}$, and c is the last letter of m_0 . \square

We now prove our main result.

Proof of Theorem 3.8. We start by assuming that the Dumont–Thomas complement numeration system associated with μ , u and r is positional and we let U, V be its sequences of weights as in Section 2. Fix some $j \in \{0, \dots, p-1\}$. We prove that $c \mapsto |\mu^\ell(c)|$ is constant over E_j for all suitable values of ℓ . Indeed, consider some letter $c \in E_j$. From Definition 3.5 and Definition 3.6, we know that there is some a - or b -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ with $k \equiv j \pmod p$ and where c is the last letter of m_0 . Additionally, if the sequence is b -admissible, then we may assume that one of the following holds:

- we have $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 \neq \mu^k(b)$ and either $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$ or $k < p$ (by Definition 3.5 and Lemma 3.10);
- we have $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 = \mu^k(b)$ (by Definition 3.6).

On the other hand, if the sequence is a -admissible, then from Lemma 3.10 we may assume that $m_{k-1} \cdots m_{k-p} \neq \varepsilon$ or $k < p$.

We define two new sequences $((m'_n, a'_n))_{n=0, \dots, k-1+\ell}$ and $((m''_n, a''_n))_{n=0, \dots, k-1+\ell}$ of elements of $A^* \times A$ as follows. The first sequence is defined by

$$(m'_n, a'_n) = \begin{cases} (m_{n-\ell}, a_{n-\ell}), & \text{if } n \in \{\ell, \dots, k-1+\ell\}; \\ (\varepsilon, \mu(a'_{n+1})_0), & \text{if } n \in \{0, \dots, \ell-1\}. \end{cases}$$

For the second sequence, the definitions of m'' and a'' are the same, except at index $n = \ell$ for which we set $m'_\ell c = m'_\ell$ and $a'_\ell = c$. (Note that $m'_\ell = m_0$ ends with c , so m'_ℓ is the prefix of m_0 without its last letter, which we may write $m_0 c^{-1}$.) Thinking in terms of the tree, these sequences correspond to the paths labeled by $w(t+1)0^\ell$ and $wt0^\ell$ from our Sketch 3.1.

Now, notice that the sequence $((m'_n, a'_n))_{n=0, \dots, k-1+\ell}$ satisfies the conditions of Theorem 2.14 in the b -admissible case and the conditions of Theorem 2.11 in the a -admissible case. Indeed, the only case where this is not trivial is the b -admissible case with $k < p$, in which case the result is obtained by considering

$$|\mu^{p-1}(m_{k-1}) \cdots \mu^{p-k}(m_0)| \leq |\mu^{p-1}(m_{k-1}) \cdots \mu^{p-k}(m_0)\mu^{p-k}(a_0)| - 1 \leq |\mu^{p-k}(\mu^k(b))| - 1,$$

where one of the two inequalities must be strict. The same result is true for the sequence $((m''_n, a''_n))_{n=0, \dots, k-1+\ell}$, with the sole exception of sequences of this form with $k < p$, $m_{k-1} = \cdots = m_1 = \varepsilon$, and $m_0 = c$.

From Definition 2.15, we obtain the following. On the one hand, if the sequences are b -admissible, we get that $1 \cdot |m_{k-1}| \cdots |m_0| \cdot 0^\ell$ is the representation of the integer

$$|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0)\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)| - |\mu^{k+\ell}(b)|, \quad (8)$$

while $1 \cdot |m_{k-1}| \cdots |m_0 c^{-1}| \cdot 0^\ell$ is the representation of the integer

$$|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0 c^{-1})\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)| - |\mu^{k+\ell}(b)|. \quad (9)$$

Subtracting (9) from (8), we get that $U_\ell = |\mu^\ell(c)|$ (recall that U_ℓ is the weight in position ℓ in the numeration system). On the other hand, if the sequences are a -admissible, we get that $0 \cdot |m_{k-1}| \cdots |m_0| \cdot 0^\ell$ is the representation of

$$|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0)\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)|, \quad (10)$$

while $0 \cdot |m_{k-1}| \cdots |m_0 c^{-1}| \cdot 0^\ell$ is the representation of

$$|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0 c^{-1})\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)|. \quad (11)$$

Subtracting again (11) from (10) leads to the same conclusion. In the exceptional case that was singled out above, (11) does not hold anymore but (10) becomes that $0^{k-1}10^\ell$ is a representation of $|\mu^\ell(c)|$, which leads to the same conclusion once more. All in all, for every $c \in E_j$ and every ℓ

such that $j + \ell \equiv r \pmod{p}$, we have shown that $|\mu^\ell(c)| = U_\ell$, thus the map $c \mapsto |\mu^\ell(c)|$ is constant over E_j as desired.

To end the proof of this implication, we now show that condition (7) is satisfied for the required letters, still in the case of a positional numeration system. We let c be a letter for which the hypotheses of condition (7) are fulfilled. As c is on the j th level of the tree $\mathcal{T}_{\mu,b|a}$, there exists some b -admissible sequence $((m_n, a_n))_{n=0, \dots, j-1}$ with c being the last letter of m_0 and $\mu^{j-1}(m_{j-1}) \cdots \mu^0(m_0)a_0 = \mu^j(b)$. Since $|\mu^{p-j}(a_0)| = 1$ and $j \leq r < p$, it follows that the sequence

$$(m_{j-1}, a_{j-1}), \dots, (m_0, a_0), (\varepsilon, \mu(a_0)), \dots, (\varepsilon, \mu^{r-j}(a_0))$$

is b -admissible and

$$\mu^{r-1}(m_{j-1}) \cdots \mu^0(m_0)\mu^{r-j}(a_0) = \mu^r(b). \quad (12)$$

Therefore, the representation of -1 must be $1 \cdot |m_{j-1}| \cdots |m_0| \cdot 0^{r-j}$. On the other hand, we have as above that $1 \cdot |m_{j-1}| \cdots |m_0 c^{-1}| \cdot 0^{r-j}$ is the representation of

$$|\mu^{r-1}(m_{j-1}) \cdots \mu^{r-j}(m_0 c^{-1})\mu^{r-j-1}(\varepsilon) \cdots \mu^0(\varepsilon)| - |\mu^r(b)|. \quad (13)$$

Utilizing Equality (12) to substitute $\mu^r(b)$, the integer in (13) is $-1 - |\mu^{r-j}(c)|$. Comparing the two representations we discussed, we obtain that $|\mu^{r-j}(c)| = U_{r-j}$. This value is also the value of $|\mu^{r-j}(e)|$ for every letter $e \in E_j$ as proven above, so condition (7) is satisfied as expected.

We turn to the converse implication. We assume that the map $c \mapsto |\mu^\ell(c)|$ is constant over E_j for all suitable values of j and ℓ and that condition (7) is satisfied, and we prove the positionality of the numeration system, i.e., we find suitable sequences of weights. We let U_ℓ be the constant value imposed on μ^ℓ by the discussed conditions and we also define $V_\ell = |\mu^\ell(b)|$ for every $\ell \in \mathbb{N}$.

We now consider an integer $n \in \mathbb{Z}$ and we divide the argument into three different cases.

Case 1. Assume that n is non-negative. Then Definition 2.15 implies that

$$\text{rep}_{u,r}(n) = 0 \cdot |m_{k-1}| \cdots |m_0| \quad (14)$$

for some a -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ satisfying

$$\mu^k(a)_{[0, n-1]} = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0).$$

In particular, we have that

$$n = \sum_{i=0}^{k-1} |\mu^i(m_i)|. \quad (15)$$

If c is a letter in m_i , we may write $m_i a_i = p_i c s_i$ for some words p_i, s_i and we note that the existence of the sequence $(m_{k-1}, a_{k-1}), \dots, (p_i c, (s_i)_0)$ means that c is in $E_{k-i \pmod{p}}$. Since $i + k - i \equiv k \equiv r \pmod{p}$, we obtain that $|\mu^i(c)| = U_i$ for any letter c of m_i . As a result, (15) yields that $n = \sum_{i=0}^{k-1} |m_i| U_i$. Due to (14), we obtain that the numeration system is positional with weights $(U_i)_{i \in \mathbb{N}}$ as expected.

Case 2. Assume that n is negative and different from -1 . Again Definition 2.15 implies that

$$\text{rep}_{u,r}(n) = 1 \cdot |m_{k-1}| \cdots |m_0| \quad (16)$$

for some b -admissible sequence $((m_i, a_i))_{i=0, \dots, k-1}$ satisfying

$$\mu^k(b)_{[-|\mu^k(b)|, n-1]} = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0).$$

A bit of care is required to prove that all letters in m_i are in $E_{k-i \pmod{p}}$. Indeed, if c is such a letter with $m_i a_i = p_i c s_i$ for some words p_i, s_i , then we must verify that either we have $\mu^{k-i-1}(m_{k-1}) \cdots \mu^0(p_i c)(s_i)_0 \neq \mu^{k-i}(b)$ or we have an equality but $k-i \leq p$ and $|\mu^{p-k+i}(s_i)| > 1$.

The only case where the equality can hold is if $p_i c = m_i$ and $s_i = a_i$. Next, remember that the sequence $((m_i, a_i))_{i=0, \dots, k-1}$ must satisfy $\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$. Thus, the equality can only occur if $k-i < p$. But in this case, it must be that $|\mu^{p-k+i}(a_i)| > 1$, otherwise we would have

$$\begin{aligned} & \mu^{k-i-1}(m_{k-1}) \cdots \mu^0(m_i)a_i = \mu^{k-i}(b) \\ \Rightarrow & \mu^{p-k+i}(\mu^{k-i-1}(m_{k-1}) \cdots \mu^0(m_i)a_i) = \mu^p(b) \\ \Rightarrow & \mu^{p-1}(m_{k-1}) \cdots \mu^{p-k+i}(m_i)\mu^{p-k+i}(a_i) = \mu^p(b) \\ \Rightarrow & \mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} = \mu^p(b) \end{aligned}$$

as $\mu^{p-k+i-1}(m_{i-1}) \cdots a_{k-p}$ is a prefix of length at least 1 of $\mu^{p-k+i}(a_i)$, which has length 1. The last equality we obtain is a contradiction, so $k-i < p$ and $|\mu^{p-k+i}(a_i)| > 1$. Thus, we have proven that c belongs to $E_{k-i \bmod p}$ in all cases. From there, we get that

$$n = -|\mu^k(b)| + \sum_{i=0}^{k-1} |\mu^i(m_i)| = -|\mu^k(b)| + \sum_{i=0}^{k-1} |m_i|U_i,$$

where the second equality follows because all the letters of m_i are in $E_{k-i \bmod p}$. Due to (16), we get that the numeration system is positional with sequences U, V of weights as defined above.

Case 3. Only the case where $n = -1$ remains. Here, we have a sequence $((m_i, a_i))_{i=0, \dots, r-1}$ with $\mu^{r-1}(m_{r-1}) \cdots \mu^0(m_0)a_0 = \mu^r(b)$. All the letters of m_i except the last one are guaranteed to be in $E_{k-i \bmod p}$ as in the previous case. For the last letter of m_i , either it is in $E_{k-i \bmod p}$ or condition (7) applies. In any case, we have $|\mu^i(c)| = U_i$, and the rest of the argument is as for Case 2.

In all three cases, we have shown the numeration system to be positional: for every $i \geq 0$, the weights U_i are defined by the constant value of $c \mapsto |\mu^i(c)|$ over $E_{r-i \bmod p}$, while the weights V_i are equal to $|\mu^i(b)|$. This concludes the proof of the main result. \square

Remark 3.11. Note that replacing the substitution by one of its powers may lead to lose the positionality of the corresponding numeration system. For instance, consider the substitution $\mu: a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ from Example 2.16 and its square renamed $\nu: a \mapsto ababa, b \mapsto aba, c \mapsto ccdcd, d \mapsto ccd$. From before, we already know that the Dumont–Thomas numeration system associated with μ and the periodic point of μ starting with a of period 2 is positional. However, that associated with ν and its fixed point starting with a is not. Indeed, drawing the first two levels of the tree $\mathcal{T}_{\nu, a}$ we obtain that the representation of 5 is 10 and that of 8 is 20, making it impossible to find a suitable evaluation map for the numeration system to be positional. This is as expected from our Theorem 3.8 as a and b are both in E_0 but have images of different lengths.

3.3 Particular cases

In this section, we highlight some general cases where the technicalities of Section 3.2 do not occur, leading to results that are more concise and legible. We also discuss possible simplifications of the substitution at play, as well as a parallel to Bertrand numeration systems.

From now on, we assume that the alphabet A of the substitution μ is minimal, i.e., all the letters in A are present in $\mu^n(b|a)$ for some n , where $b|a$ designates the seed of the periodic point u of μ . Given a substitution μ , we say that a letter c is *non-final* if there exist $d \in A$, $x \in A^*$, and $y \in A^+$ such that $\mu(d) = xcy$. We let E_μ denote the set of non-final letters of μ . When there is no ambiguity, we drop the subscript.

We start with the case where the domain \mathbb{D} is equal to \mathbb{N} and the substitution μ has a fixed point. In this case, the period p is equal to 1 and there is only one sequence of weights, with no particular care given to the most significant digit in a representation.

Corollary 3.12. *Let $\mu: A^* \rightarrow A^*$ be a substitution and $u = u_0 u_1 \dots$ be a right-infinite fixed point of μ with growing seed a . Then the Dumont–Thomas numeration system (for \mathbb{N}) associated with μ , u and $r = 0$ is positional if and only if the map $c \mapsto |\mu^\ell(c)|$ is constant over E_μ for every ℓ , in which case the sequence of weights is equal to $U_\ell = |\mu^\ell(a)|$ for every $\ell \geq 0$.*

Proof. Note that since a is the growing seed of a fixed point of μ , we must have $\mu(a) = ay$ for some non-empty word y , so $a \in E_\mu$. Note also that since the domain is \mathbb{N} , Definition 3.6 does not apply. It then suffices to apply the proof of Theorem 3.8 restricted to the domain \mathbb{N} , taking into account that, since $p = 1$ and $r = 0$ in our case, the set E_μ is equal to the set E_0 of Theorem 3.8, which is the only set for which we have conditions to check. \square

Another special case is that of primitive substitutions, even on the complete domain $\mathbb{D} = \mathbb{Z}$. Recall that the definition of primitive substitutions was given in Section 2, where we mentioned that if μ is primitive then there exists k such that $a \in \mu^\ell(b)$ for all $a, b \in A$ and $\ell \geq k$.

Corollary 3.13. *Let $\mu: A^* \rightarrow A^*$ be a primitive substitution and let $u \in \text{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $b|a$ and period $p \geq 1$. The Dumont–Thomas complement numeration system associated with μ , u , and r is positional if and only if the map $c \mapsto |\mu^\ell(c)|$ is constant over E_μ for every $\ell \geq 0$. In this case, for every $\ell \geq 0$, U_ℓ is the constant value of $|\mu^\ell(\cdot)|$ over E_μ and $V_\ell = |\mu^\ell(b)|$.*

Proof. Let k be an integer such that, for all letters $c, d \in A$, for all $\ell > k$, c appears in $\mu^\ell(d)$. We then know that every letter appears at every sufficiently large level in the tree $\mathcal{T}_{\mu,a}$. In particular, b appears in $\mu^k(a)$, so $\mathcal{T}_{\mu,b}$ appears as a subtree of $\mathcal{T}_{\mu,a}$. Therefore, any letter that appears with a younger sibling on column -2 also appears with a younger sibling on some column with non-negative index. As a result, Definition 3.6 can be skipped when determining the sets E_j and condition (7) is never relevantly added to any letter.

Next, we show that, for every $j \in \{0, \dots, p-1\}$, $E_j = E_\mu$. Fix some $j \in \{0, \dots, p-1\}$. As the inclusion $E_j \subseteq E_\mu$ is clear by Definition 3.5 we show the other one. If $c \in E_\mu$, there exist $d \in A$, $x \in A^*$, and $y \in A^+$ such that $\mu(d) = xcy$. Since μ is primitive, d appears in $\mu^{j-1+\ell p}(a)$ for some sufficiently large ℓ , so c appears with a younger sibling at some level $j + \ell p$ in the tree $\mathcal{T}_{\mu,b|a}$. This implies that $c \in E_j$, as desired.

Finally, notice that since a is a growing letter of μ , at least one letter must have an image by μ that is of length at least 2, thus E_μ is nonempty.

Now that we have shown that $E_j = E_\mu \neq \emptyset$ for every $j \in \{0, \dots, p-1\}$ and that condition (7) is not relevant to any letter, applying Theorem 3.8 leads to the statement. \square

The substitutions that generate the positional Dumont–Thomas numeration systems discussed in the two corollaries above turn out to have a special form as we will see. This form lends itself well to simplifications and will allow us to link these positional numeration systems to some known families of numeration systems. The following lemma shows that these substitutions are equivalent (in the sense that they generate the same Dumont–Thomas numeration system) to a substitution that contains only one non-final letter.

Lemma 3.14. *Let $\mu: A^* \rightarrow A^*$ be a substitution such that the map $c \mapsto |\mu^\ell(c)|$ is constant over E_μ for every integer $\ell \geq 0$. Then there exist an alphabet $B \subseteq A$, a substitution $\nu: B^* \rightarrow B^*$ such that $|E_\nu| = 1$ and some letters $a', b' \in B$ such that the trees $\mathcal{T}_{\mu,b|a}$ and $\mathcal{T}_{\nu,b'|a'}$ differ only by their labeling.*

Proof. Before showing that the statement holds, we start with some notation. For an integer $k \geq 0$ and a letter $a \in A$, we let $\mathcal{T}_{\mu,a}^{\leq k}$ denote the first k levels of the tree $\mathcal{T}_{\mu,a}$ (agreeing that the root is on level 0). Additionally, we extend our definition of trees associated with substitutions to forests associated with substitutions: if $w = w_1 \dots w_n$ is a word over A , then $\mathcal{T}_{\mu,w}$ is the ordered forest where the first tree is \mathcal{T}_{μ,w_1} , the second tree is \mathcal{T}_{μ,w_2} , and so on. We define similarly $\mathcal{T}_{\mu,w}^{\leq k}$ for any integer $k \geq 0$.

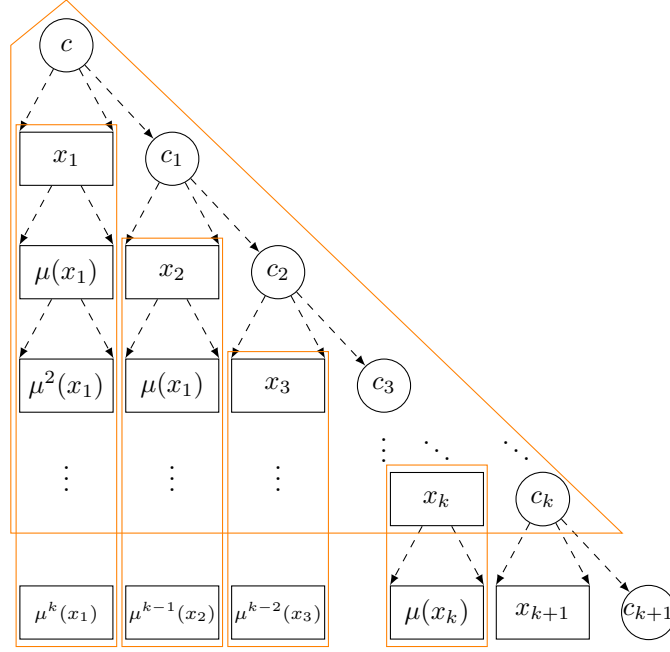


Figure 7: In the proof of Lemma 3.14, given a letter c , we inductively define $\mu(c) = x_1 c_1$ and $\mu(c_i) = x_{i+1} c_{i+1}$ for every $i \geq 1$, where x_i is a word and c_i is a letter for every $i \geq 1$. We draw the tree $\mathcal{T}_{\mu,c}^{\leq k+1}$ as defined in the proof of Lemma 3.14. Black rectangles correspond to words (note that we do not give information on the length here) and black circles to letters. The induction hypothesis tells us that, up to labeling, every color-framed subtree is equal to its corresponding subtree in $\mathcal{T}_{\mu,d}^{\leq k+1}$.

Now we prove the following claim: for every integer $k \geq 0$ and for every pair of letters $c, d \in E_\mu$, the trees $\mathcal{T}_{\mu,c}^{\leq k}$ and $\mathcal{T}_{\mu,d}^{\leq k}$ differ only by their labeling. Proving this for every k will also show it for the entire trees. We proceed by induction on $k \geq 0$. The case $k = 0$ is trivial as we start with letters. The case $k = 1$ is obtained directly from the hypothesis for $\ell = 1$. Indeed, c and d must have the same number of children as they are both in E_μ by assumption. Now, assume that the claim holds for k and let us prove it for $k + 1$. We inductively define $\mu(c) = x_1 c_1$ and $\mu(c_i) = x_{i+1} c_{i+1}$ for every $i \geq 1$, where $x_i \in A^*$ and $c_i \in A$ for every $i \geq 1$ (see Fig. 7). Similarly, we define $\mu(d) = y_1 d_1$ and $\mu(d_i) = y_{i+1} d_{i+1}$ for every $i \geq 1$ with the same constraints on y_i and d_i . The induction hypothesis ensures that $\mathcal{T}_{\mu,c}^{\leq k}$ and $\mathcal{T}_{\mu,d}^{\leq k}$ only differ by their labeling. Still from the induction hypothesis, the same is true for $\mathcal{T}_{\mu,x_1}^{\leq k}$ and $\mathcal{T}_{\mu,y_1}^{\leq k}$ and, more generally, for $\mathcal{T}_{\mu,x_i}^{\leq k+1-i}$ and $\mathcal{T}_{\mu,y_i}^{\leq k+1-i}$ with $i \in \{1, \dots, k\}$ (letters of the words x_i, y_i are non-final by definition). See Fig. 7 for a more visual explanation.

As a result, to conclude the proof of the claim, we only need to check that c_k and d_k have the same number of children, i.e., $|\mu(c_k)| = |\mu(d_k)|$. Since

$$|\mu^{k+1}(c)| = \sum_{i=1}^k |\mu^{k+1-i}(x_i)| + |\mu(c_k)| \quad \text{and} \quad |\mu^{k+1}(d)| = \sum_{i=1}^k |\mu^{k+1-i}(y_i)| + |\mu(d_k)|,$$

we have

$$|\mu(c_k)| = |\mu^{k+1}(c)| - \sum_{i=1}^k |\mu^{k+1-i}(x_i)| = |\mu^{k+1}(d)| - \sum_{i=1}^k |\mu^{k+1-i}(y_i)| = |\mu(d_k)|,$$

where the second equality results from our hypothesis (all the letters in x_i and y_i belong to E_μ , as do c and d). Thus the claim is proved.

Now that we have proven that $\mathcal{T}_{\mu,c}$ and $\mathcal{T}_{\mu,d}$ only differ by their labeling, we may replace every instance of the letter d in an image of μ by the letter c and this will only change the labeling of the tree $\mathcal{T}_{\mu,b|a}$, not its shape. If we proceed like this for every letter d in $E_\mu \setminus \{c\}$ and then remove any such superfluous letter from the alphabet A , we will have constructed the desired substitution ν . \square

Example 3.15. Consider the primitive substitution $\mu: a \mapsto ab, b \mapsto ba$ (often referred to as the Thue–Morse substitution). We note that both a, b are non-final letters and $|\mu^\ell(c)| = 2^\ell$ for $c \in \{a, b\}$ and for every $\ell \geq 0$. Applying our result gives the substitution $\nu: a \mapsto a^2$. The corresponding Dumont–Thomas numeration system (over \mathbb{N}) is the usual binary system.

As a consequence, when considering positional Dumont–Thomas numeration systems associated with either primitive substitutions or fixed points on the domain \mathbb{N} (namely, the assumptions of Corollaries 3.12 and 3.13) we may look only at substitutions having one single non-final letter e by Lemma 3.14. In such a substitution, the image of a letter is determined by the number of repetitions of the non-final letter e and the choice of final letter.

If we let a be the seed of the periodic point u , it must be that $\mu^\ell(a)$ starts with e for some ℓ . In order for u to be a periodic point, we must have either $e = a$ (Case 1) or a chain of letters that all have images of length 1, say $e \mapsto b_1 \mapsto b_2 \mapsto \dots \mapsto b_k \mapsto a$ (Case 2). In all other cases, no iterated image of e starts with a , and u cannot be a periodic point as a result.

In what follows, we will try to link these substitutions to established literature on numeration systems. However, note that in most common systems, representations may take any length, which corresponds in our case to the substitution having a fixed point, and therefore to Case 1 above. As a result, we will solely focus on this case where $e = a$ is both the seed of the periodic point and the only non-final letter. In this case, with the extra condition to operate on the minimal alphabet, the substitution is equivalent to one of the form

$$\mu: a_1 \mapsto a_1^{d_1} a_2, a_2 \mapsto a_1^{d_2} a_3, \dots, a_n \mapsto a_1^{d_n} a_k, \quad (17)$$

where n is a non-negative integer, $\{a_1, \dots, a_n\}$ is the alphabet of the substitution, $k \in \{1, \dots, n\}$, $d_1 > 0$ and d_i is a non-negative integer for every $i \in \{1, \dots, n\}$. An example of such a substitution is the Fibonacci (resp. , Tribonacci) substitution given by $\varphi: a \mapsto ab, b \mapsto a$ (resp. , $\tau: a \mapsto ab, b \mapsto ac, c \mapsto a$), for which $n = 2$, $a_1 = a$, $a_2 = b$, $d_1 = 1$, $d_2 = 0$, and $k = 1$ (resp. , $n = 3$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $d_1 = d_2 = 1$, $d_3 = 0$, and $k = 1$). See also their generalization to *generic n -bonacci* substitutions in [18, Ex 2.11].

The similarity with the substitutions studied by Fabre in [7] is striking, so we will call *Fabre-like* the substitutions of the form (17). We will now study this particular class of substitutions in additional detail. First, we note the following property.

Proposition 3.16. *Let μ and μ' be Fabre-like substitutions and consider their respective fixed points u and u' with respective seeds a and a' . If the Dumont–Thomas numeration systems over \mathbb{N} corresponding to u and u' respectively and $r = 0$ have the same sequence of weights, then every natural number has the same representation in both numeration systems.*

In a sense, the weight sequence completely determines the numeration system provided that it is based on a Fabre-like substitution. In what follows, we will call two numeration systems over \mathbb{N} *equal* if every natural number is represented by the same word in both systems. Note that this property does not hold in general as shown in the next example.

Example 3.17. Consider the substitutions $\mu: a \mapsto bcd, b \mapsto ef, c \mapsto e^2g, d \mapsto d, e \mapsto ad, f \mapsto a^2d, g \mapsto a$ and μ' defined in the same way but with $\mu'(a) = cbd$ instead of bcd . Then, if u and u' are the periodic points of period $p = 3$ starting with a of μ and μ' respectively and $r = 0$, the corresponding Dumont–Thomas numeration systems associated with u and r and with u' and r have the same sequence of weights, but the representations of some numbers differ: for example, we have $\text{rep}_{u,0}(4) = 012$, $\text{rep}_{u',0}(4) = 020$, $\text{rep}_{u,0}(9) = 120$, and $\text{rep}_{u',0}(9) = 112$.

Sketch of the proof of Proposition 3.16. From the equality of the weight sequences, we deduce that $|\mu^\ell(a)| = |\mu'^\ell(a')|$ for every integer ℓ as $a \in E_\mu$ and $a' \in E_{\mu'}$. Then, we prove that, for every $k \geq 0$, the pair of trees $\mathcal{T}_{\mu,a}^{\leq k}$ and $\mathcal{T}_{\mu',a'}^{\leq k}$ differ only by their labeling, using the same method as for the claim in Lemma 3.14. Since the two trees differ only by their labeling, $\text{rep}_{u,0}(n) = \text{rep}_{u',0}(n)$ for every natural number n . \square

Let us quickly recall some other milestones of numeration systems. For the reader not familiar with the classical theory and definitions, see, for instance, [2, Chapter 2] and [18, Chapter 2]. In 1957, Rényi [19] introduced β -numeration systems, a way to represent positive real numbers with the help of a real base $\beta > 1$. Among all non-negative real numbers, 1 is the most distinguishable, and the notation $d_\beta(1)$ is used for its representation in base β . Additionally, we define its *quasi-greedy representation* $d_\beta^*(1)$ to be equal to $(d_1 \cdots d_{\ell-1}(d_\ell - 1))^\omega$ if $d_\beta(1) = d_1 \cdots d_\ell 0^\omega$, and $d_\beta(1)$ otherwise. Of particular interest are the so-called *Parry numbers*, which are real numbers β such that $d_\beta(1)$ is either finite (β is then a *simple Parry number*) or ultimately periodic [17].

In an article of Bertrand-Mathis [3], later corrected by Charlier, Cisternino and Stipulanti [5], a link is made between Rényi numeration systems and some greedy positional numeration systems for natural numbers (numeration systems over \mathbb{N} that are positional in our sense, with the extra condition that the representation can be computed by a greedy algorithm). If we write $d_\beta^*(1) = d_1 d_2 \cdots$, we define the sequence of weights $(U_i)_{i \geq 0}$ by $U_0 = 1$ and the relation $U_i = d_1 U_{i-1} + d_2 U_{i-2} + \cdots + d_i U_0 + 1$ for each $i \geq 1$. In the case where β is a Parry number, this relation can be simplified to a linear recurrence relation for $(U_i)_{i \geq 0}$. This weight sequence, when combined with a greedy algorithm, defines a positional numeration system called a *canonical Bertrand numeration system*, which enjoys properties such as the numeration language having the same set of factors as the language of the Rényi numeration system with base β . Two other notable properties of such numeration systems are the following:

Property 1. A word w is the representation of some natural number if and only if $w0$ is.

Property 2. The lexicographically greatest representations of every length are all prefixes of one another.

However, these properties characterize Bertrand numeration systems only if we consider *all* Bertrand numeration systems: in addition to canonical ones, we consider the *non-canonical* Bertrand numeration systems, obtained just like the canonical ones but with the word $d_\beta^*(1)$ replaced by $d_\beta(1)$, and the *trivial* Bertrand numeration system based on the weights $U_i = i + 1$, with $i \geq 0$. This is the nature of the correction brought by Charlier et al. in [5].

A few of years after Bertrand-Mathis' paper, Fabre [7] introduced another way to view canonical Bertrand numeration systems. If β is a Parry number with $d_\beta^*(1) = d_1 \cdots d_n(d_{n+1} \cdots d_{n+m})^\omega$ for some m, n , we introduce the substitution

$$\mu_\beta: 1 \mapsto 1^{d_1} 2, 2 \mapsto 1^{d_2} 3, \dots, (n+m-1) \mapsto 1^{d_{n+m-1}}(n+m), (n+m) \mapsto 1^{d_{n+m}}(n+1). \quad (18)$$

Note that Fabre defines his substitutions not by using the word $d_\beta^*(1)$ like we do but using the word $d_\beta(1)$. Still, the substitution in (18) is the same as the one defined by Fabre. The case where $d_\beta(1)$ is finite corresponds to the case where $n = 0$ with our notation.

Fabre then shows that $|\mu_\beta^\ell(1)|$ is the integer U_ℓ defined for the canonical Bertrand numeration system, and his [7, Theorem 2] establishes the equality between the Dumont–Thomas numeration system associated with μ_β and the canonical Bertrand numeration system associated with β .

As an aside, we note that the term “conjugate substitutions” used by Fabre in [7, Section 3.1] can be thought of in terms of the trees associated with those substitutions. Following Fabre's notation, for two substitutions $\mu: A^* \rightarrow A^*$ and $\nu: B^* \rightarrow B^*$ with seeds $a \in A$ and $b \in B$ respectively, we write $\mu \rightarrow \nu$ if there exists a morphism $h: A^* \rightarrow B^*$ such that $h(a) = b$ and $h(\mu(c)) = \nu(h(c))$ for every $c \in A$. We then say that μ and ν are *conjugates* if there exists a third substitution τ such that $\tau \rightarrow \mu$ and $\tau \rightarrow \nu$.

Proposition 3.18. *Two substitutions $\mu: A^* \rightarrow A^*$ and $\nu: B^* \rightarrow B^*$ with respective seeds a and b are conjugate exactly when the trees $\mathcal{T}_{\mu,a}$ and $\mathcal{T}_{\nu,b}$ are equal up to relabeling.*

Proof. On the one hand, if the trees $\mathcal{T}_{\mu,a}$ and $\mathcal{T}_{\nu,b}$ are equal up to relabeling, we may construct a tree with the same graph structure but with labels in $A \times B$ obtained by simply joining the labels of the corresponding nodes in the two trees. Then, this new tree gives a substitution $\mu \times \nu$ on $A \times B$ that is such that $\mu \times \nu \rightarrow \mu$ and $\mu \times \nu \rightarrow \nu$, which implies that μ and ν are conjugate.

On the other hand, if $\tau \rightarrow \mu$ for some substitution τ , applying the morphism h on the label of every node in the tree associated with τ gives the tree associated with μ (which must have the same shape since we only have changed the labels). By applying this result twice, we conclude that the trees associated with two conjugate substitutions have the same shape and differ only by their labeling. \square

With these milestones recalled, let us go back to the study of our Dumont–Thomas numeration systems based on Fabre-like substitutions. We first note that all Bertrand numeration systems associated with Parry numbers are a particular case of Dumont–Thomas numeration systems (and not just the canonical ones as proven by Fabre).

Proposition 3.19. *Every Bertrand numeration system associated with a Parry number is equal to some Dumont–Thomas numeration system associated with a Fabre-like substitution.*

Proof. It is clear that the substitution μ_β in (18) introduced by Fabre is Fabre-like when β is a Parry number (note that the first digit of $d_\beta^*(1)$ is nonzero, so $d_1 > 1$ as expected). Therefore, the canonical Bertrand numeration systems can all be seen as special cases of our Dumont–Thomas numeration systems.

The non-canonical Bertrand numeration systems introduced in [5] also fall in this framework. If $d_\beta^*(1) = (d_1 \cdots d_m)^\omega$, we create μ'_β from μ_β by adding the letter $m+1$ to the alphabet, setting $\mu'_\beta(m) = 1^{d_m+1}(m+1)$ instead of $\mu_\beta(m) = 1^{d_m+1}$, and setting $\mu'_\beta(m+1) = (m+1)$. In a sense, this corresponds to improperly setting $d_\beta^*(1) = d_1 \cdots d_{m-1}(d_m+1)0^\omega$, then defining a Fabre substitution from this ultimately periodic expansion.

Since the trivial Bertrand numeration system based on the weight sequence $(i+1)_{i \geq 0}$ is the Dumont–Thomas numeration system associated with the substitution $\mu: a \mapsto ab, b \mapsto b$, which is Fabre-like, we conclude the proof of the statement. \square

Although our Dumont–Thomas numeration systems clearly verify the two Properties 1 and 2 that characterize Bertrand numeration systems, the converse of Proposition 3.19 is not true, as we see in the following example.

Example 3.20. Consider the Fabre-like substitution $\mu: a \mapsto aab, b \mapsto aaaa$ with fixed point $u = aabaabaaaa \cdots$. The corresponding Dumont–Thomas numeration system is positional, with the sequence of weights starting by 1, 3, 10, 32, \dots . We note that $\text{rep}_{u,0}(9) = 23$, but this cannot happen in a Bertrand numeration system as the representation of 9 would be 30 with the given sequence of weights.

The catch, of course, is that Dumont–Thomas numeration systems, while positional, do not represent numbers by applying a greedy algorithm on the sequence of weights, but rather by applying a greedy algorithm on the factorization of their fixed point, which can lead to different results. This can be understood with an adaptation of the *Parry condition*. To recall, this condition (first seen in [17], Corollary 1) states that a word $d_1 d_2 \cdots$ is equal to $d_\beta(1)$ for some $\beta > 1$ if, and only if, $d_1 d_2 \cdots \succ 10^\omega$ and $d_1 d_2 \cdots \succ d_i d_{i+1} \cdots$ for every $i \geq 1$. In the case of Example 3.20, if the substitution μ were the Fabre substitution associated with some Parry number β , we would have $d_\beta^*(1) = (23)^\omega$ and $d_\beta(1) = 240^\omega$, which contradicts the Parry condition. Thus the system cannot be equal to a Bertrand numeration system.

In fact, the Parry condition (adapted for use with $d_\beta^*(1)$ instead of $d_\beta(1)$) is all that is necessary to guarantee that the system is greedy and therefore equal to a Bertrand numeration system, as we will see in the following result.

Proposition 3.21. *Let μ be a Fabre-like substitution as in (17). Construct the word $d_1 d_2 \cdots = d_1 \cdots d_{k-1}(d_k \cdots d_n)^\omega$. The Dumont–Thomas numeration system associated with μ and the seed*

a_1 is equal to a Bertrand numeration system if and only if we have $d_i d_{i+1} \cdots \preceq_{\text{lex}} d_1 d_2 \cdots$ for each $i \geq 1$.

Proof. Assume first that $d_i d_{i+1} \cdots \succ_{\text{lex}} d_1 d_2 \cdots$ for some $i \geq 1$. In this case, we must have $d_i \cdots d_{j-1} \succ_{\text{lex}} d_1 \cdots d_{j-i}$ for some i, j . However, it is not hard to see that the lexicographically greatest representations of length j and $j - i$ in our numeration system must be $d_1 \cdots d_j$ and $d_1 \cdots d_{j-i}$ respectively. Since the suffix of length $j - i$ of the former word is lexicographically greater than the latter, our system cannot be equal to a greedy positional numeration system (it is not suffix-closed), and therefore cannot be equal to a Bertrand numeration system.

For the other direction, assume that

$$d_i d_{i+1} \cdots \preceq_{\text{lex}} d_1 d_2 \cdots, \quad \forall i \geq 1. \quad (19)$$

holds. We consider two cases depending on the periodicity of $d_1 d_2 \cdots$.

As a first case, if $d_1 d_2 \cdots$ is not purely periodic, the inequality sign can be replaced by a strict inequality in (19) (which corresponds to the Parry condition). If $d_1 d_2 \cdots = 10^\omega$, then $\mu: a \mapsto ab, b \mapsto b$ and our Dumont–Thomas system is equal to the trivial Bertrand numeration system associated with weights $(i+1)_{i \geq 0}$. If $d_1 d_2 \cdots$ ends with 0^ω and is not 10^ω , then $d_1 d_2 \cdots$ is equal to $d_\beta(1)$ for some simple Parry number β . Then, μ is equal to the Fabre substitution μ'_β modified to fit the non-canonical Bertrand system (as in the proof of Proposition 3.19) and our Dumont–Thomas numeration system is equal to the non-canonical Bertrand numeration system associated with β . Finally, if $d_1 d_2 \cdots$ does not end in 0^ω , then it is equal to $d_\beta(1) = d_\beta^*(1)$ for some non-simple Parry number β . In this case, $\mu = \mu_\beta$ and our Dumont–Thomas numeration system is equal to the canonical Bertrand numeration system associated with β .

As a second case, if $d_1 d_2 \cdots$ is purely periodic with minimal period ℓ and is equal to $(d_1 \cdots d_\ell)^\omega$, then $d_1 \cdots d_{\ell-1} (d_\ell + 1) 0^\omega$ also verifies (19), this time with a strict inequality (see [5, Lemma 4]). This new word is equal to $d_\beta(1)$ for some simple Parry number β , thus $d_1 d_2 \cdots$ is equal to $d_\beta^*(1)$ for the same β , our μ is the associated Fabre substitution, and our numeration system is the canonical Bertrand numeration system associated with β .

In all cases, we have shown that the Dumont–Thomas numeration system associated with μ is equal to some Bertrand numeration system, as desired. \square

4 Open problems

We end this article by presenting three open problems related to our topic.

First, we ask about the practical aspects of the results presented here. Can the results in this paper be turned into an algorithmic process and, if so, what would its algorithmic complexity be?

Second, we note that abstract numeration systems may be defined using trees (by representing a number using the label of a path in a tree) even when those trees are not defined using an underlying substitution. This process corresponds to abstract numeration systems whose language is prefix-closed and right-extendable. Can our methods be adapted to this case, which would lead us one step closer to a characterization of positionality for general abstract numeration systems?

Finally, we have used the term Dumont–Thomas *complement* numeration systems by analogy with the two’s complement numeration system. What properties are shared between our systems and two’s complement? In particular, can the addition of integers be performed similarly to the addition of natural numbers? For the analogue of the two’s complement numeration system in the Fibonacci context, this question was answered positively in [10].

Acknowledgments

We thank Émilie Charlier for useful discussions.

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