Automatic proofs in combinatorial game theory

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http://www.discmath.ulg.ac.be/
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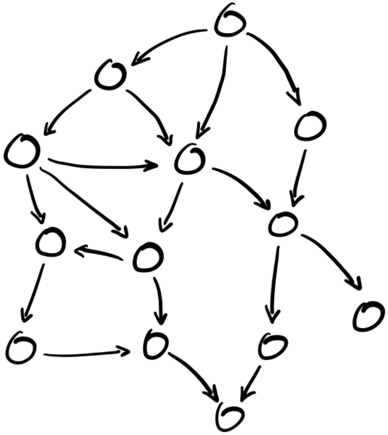
Université de Liège

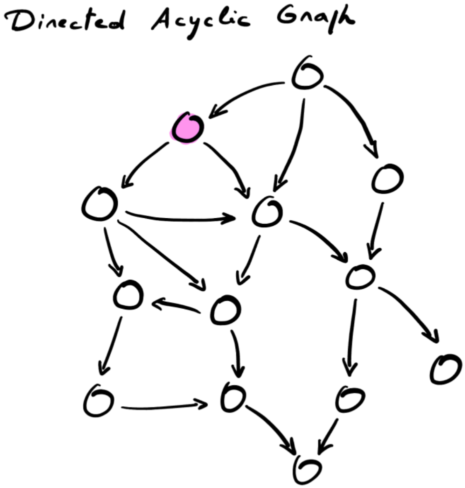
Directed Acyclic Graph



Game on a finite graph

- Player unable to move loses
- Pick a starting vertex
- Choose among options

Blue player has to choose



Game on a finite graph

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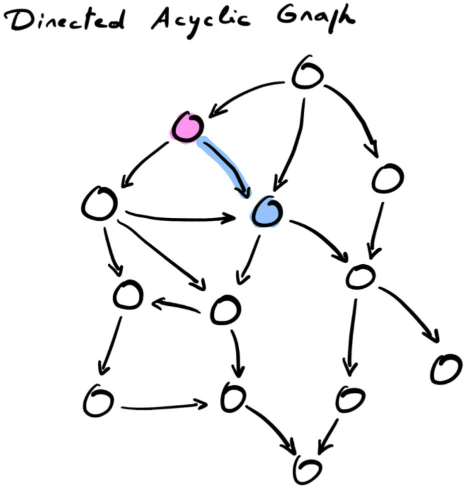
Pink player has to choose



Game on a finite graph

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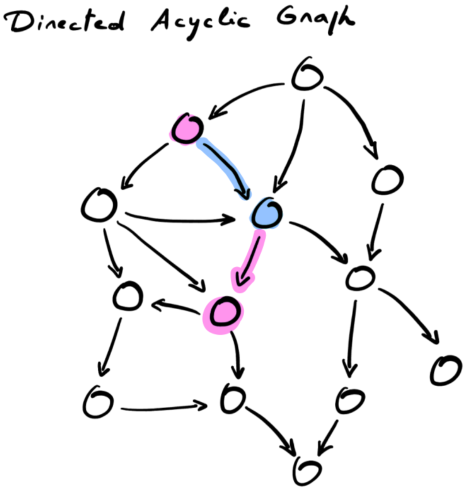
Blue player may choose but...

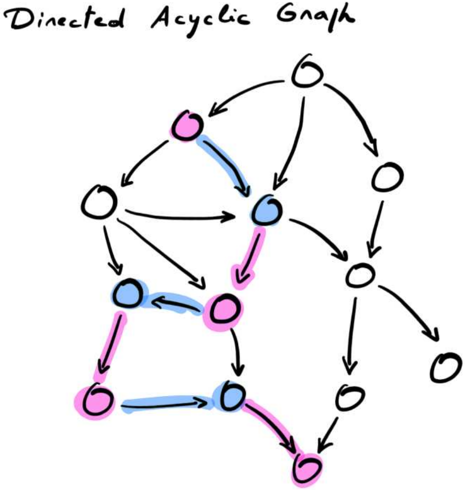


Game on a finite graph

- Player unable to move loses
- Pick a starting vertex
- Choose among options

The pink player was smart (choosing a *winning strategy*)

Blue player eventually loses



Game on a finite graph

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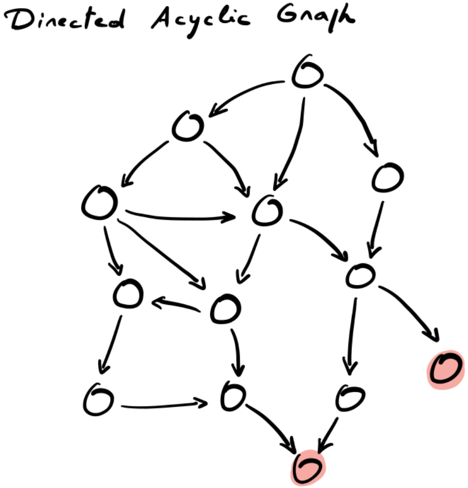
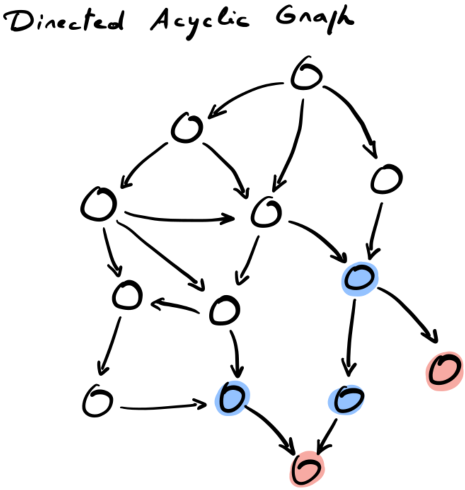
Should I start to play the game or not ?



What are the $\mathcal{P}$-positions?

- Bottom-up approach
- Sink states



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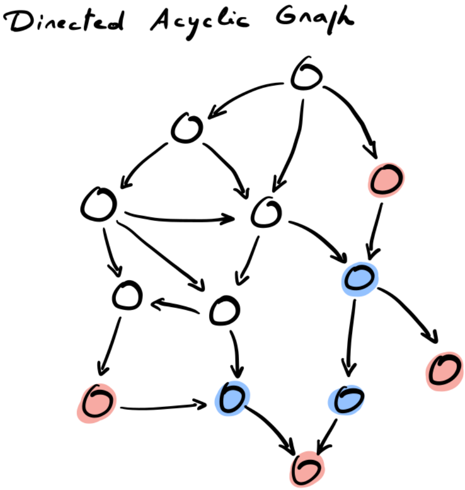
$$x \longrightarrow y \in \mathcal{P}$$

then $x \notin \mathcal{P}$



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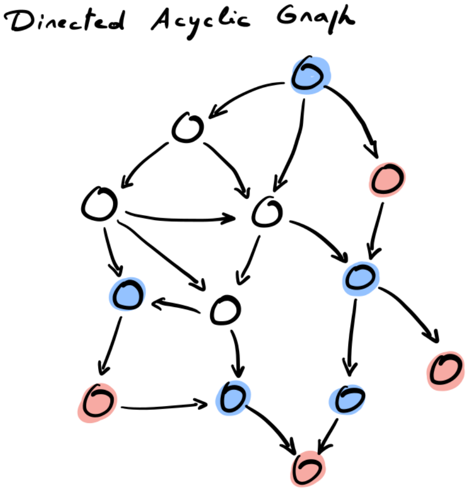

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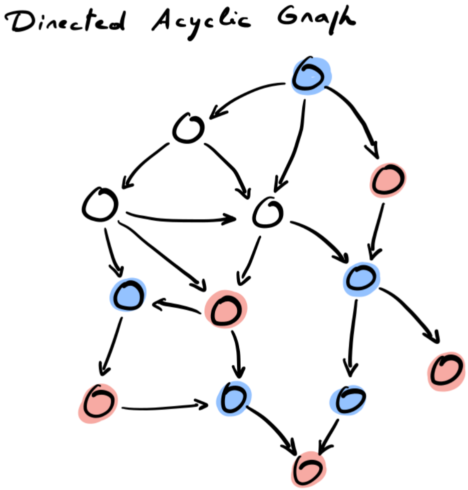

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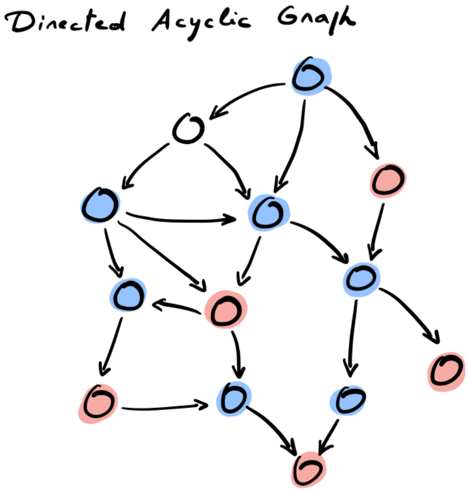

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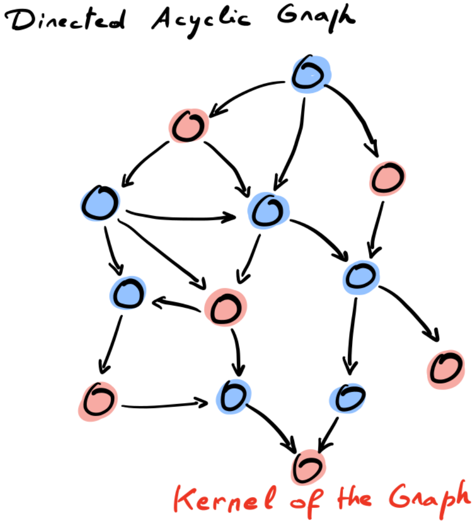

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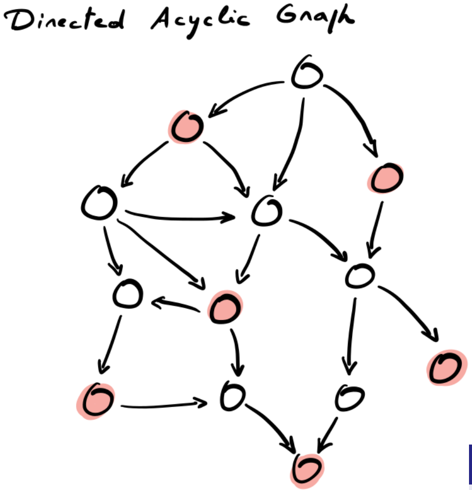


The *kernel* of the graph

- stable

$$\forall x, y \in \mathcal{P}, x \not\to y$$

- absorbing

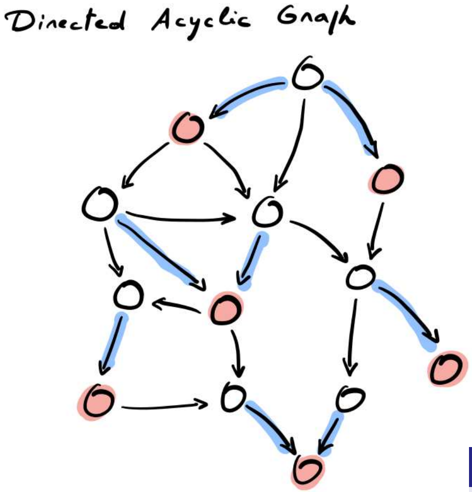
$$\forall x \notin \mathcal{P}, \exists y \in \mathcal{P} : x \to y$$

A DAG has a unique kernel

(C. Berge)

Remark

These expressions are "simple enough" for what comes next.



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Wythoff's game (1907)

A MODIFICATION OF THE GAME OF NIM,

BY

W. A. WYTHOFF.

(Amsterdam.)

1. The following arithmetical game is a modification of the game of „nim", described by C. L. Bouton in the *Annals of Mathematics*, 2nd series, vol. 3, p. 35 – 39.

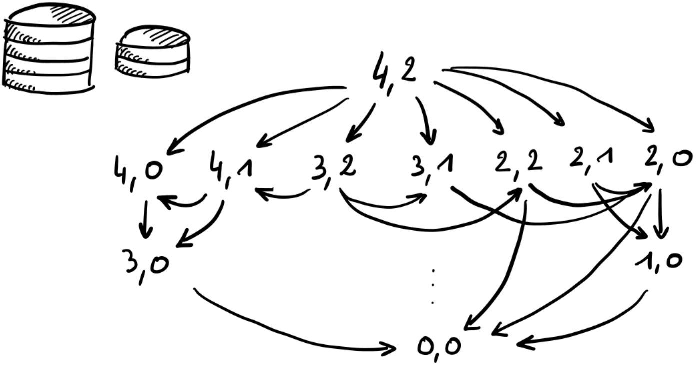
The game is played by two persons. Two piles of counters are placed on a table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an *equal* number. The player who takes up the last counter or counters, wins.

Nieuw Arch. voor Wiskunde

- 2 piles of token
- always remove a positive number of token
- remove any number of token from *one pile* (Nim game) or,
- remove the same number from *both piles*
- first player unable to move loses (normal convention).

Wythoff's game (1907)

It is still a game on a directed acyclic graph:



Hence, $\mathcal{P}$-positions are given by the kernel of the game graph.

Wythoff's game (1907)

Several characterizations of the $\mathcal{P}$-positions are known

- $(\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor)$
- $0100101001001010010 \cdots$
- some using MeX operation...

So every P-position (a, b) satisfies:

$$a = \lfloor k \cdot \phi \rfloor, \quad b = \lfloor k \cdot \phi^2 \rfloor \quad \text{or vice versa (since the order doesn't matter)}$$

These pairs are also called the **Wythoff pairs**.

Winning Strategy:

1. **Check if the current position** (a, b) is a Wythoff pair.
   - If yes: you're in a **P-position**, and if it's your turn, you're in trouble.
   - If no: then you can move to the **nearest lower Wythoff pair**, and that's your optimal move.
2. If your opponent makes a move and leaves you in a non-P-position, compute the nearest Wythoff pair and make the move that reaches it.

Wythoff's game (1907)

Using Fibonacci numeration system

Theorem (A. Fraenkel 1982)

A pair (a, b) such that $a \leq b$ is a $\mathcal{P}$-position IFF
1) $\mathrm{rep}_F(a)$ ends with an even number of zeroes and
2) $\mathrm{rep}_F(b) = \mathrm{rep}_F(a)0$.

Can `Walnut`[1] be of some use
with combinatorial games
like Wythoff's ?

R. Fokkink, G. F. Ortega, D. Rust (2022)

[1] Recall P. Popoli's talk from yesterday!

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Using walnut

`Walnut` handles Fibonacci system

- $1\{0,1\}^* \setminus \{0,1\}^*11\{0,1\}^*$
- $\mathrm{rep}_F(\{(x,y,z) \mid x+y=z\})$

Frougny's normalization (1992)

H. Mousavi, L. Schaeffer, J. Shallit (2016)

Büchi's thm. (1960) applies:

- $\mathbb{N}$ is U-recognizable
- addition is U-recognizable

i.e., to *addable* systems U

FO$(\langle \mathbb{N}, +, V_U \rangle)$ is decidable

V. Bruyère, G. Hansel, et al. (1994)

É. Charlier, N. Rampersad, J. Shallit (2012)

We can express Fraenkel's characterization

```
reg end_even_zeros msd_fib "0*(00|0*1)*":
reg left_shift {0,1} {0,1} "([0,0]|([0,1][1,1]*[1,0]))*":
def ppos_asym "?msd_fib $end_even_zeros(a) & $left_shift(a,b)":
def ppos "?msd_fib $ppos_asym(a,b) | $ppos_asym(b,a)":
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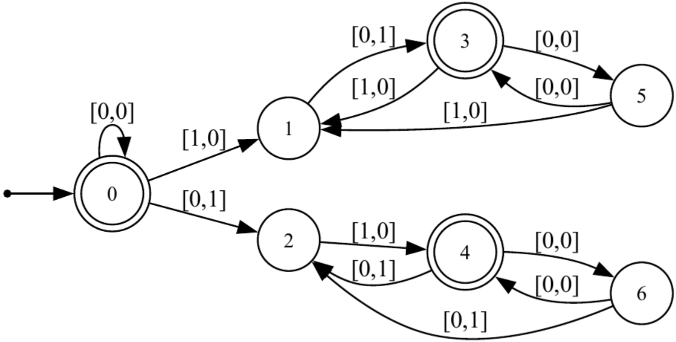
USING WALNUT

`$ppos(9,15) True`

```
0 1 0 0 0 1
1 0 0 0 1 0
```



DFA accepting $\mathcal{P}$-positions game written in the Fibonacci system.

Using walnut

We can also express stability and absorption

```
eval w_stable "?msd_fib Ap,q,r,s (($ppos(p,q) & $ppos(r,s)
=> ((p=r & q=s) | (p>r & q>s & p+s!=q+r)) )":

eval w_absorbing "?msd_fib Ap,q (~$ppos(p,q) => Ex,y
( x<=p & y<=q & $ppos(x,y) & (p+y=q+x | p=x | q=y) )) ":
```

True

$\Longrightarrow$ More than a century after Wythoff's proof, we get an *automatic* proof of the characterization of the set of $\mathcal{P}$-positions!

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What are the ingredients ?

Let us recap (if I had to stop my talk now)

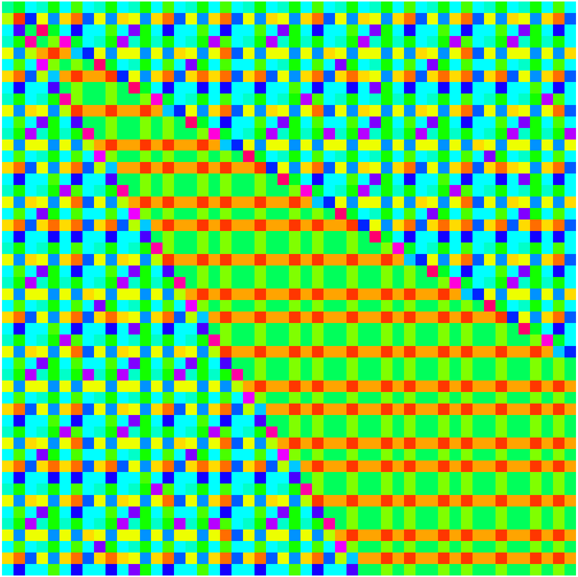
- The rules of the game can be expressed in FO$(\langle\mathbb{N}, +, V_U\rangle)$.
- We have an addable numeration system U
  "decidability of the theory comes from automata".
- The set of $\mathcal{P}$-positions, when expressed within this system, is a regular language.

Note that we had a candidate to test for the set $\mathcal{P}$.

I skip some of our results around Wythoff's game:
https://orbi.uliege.be/handle/2268/323845

Can you add/remove rules such that $\mathcal{P}$ is not affected?

- Solving a "long-standing" conjecture on extensions preserving the set of $\mathcal{P}$-positions E. Duchêne, A. Fraenkel, R. Nowakowski, M.R. (2010)
- Exploring redundant moves
- Nhan Bao Ho's variant restrictions or extensions JCTA (2012)



Fraenkel's variations

- One may remove $k > 0$ tokens from one pile and $\ell > 0$ from the other one, provided that $|k - \ell| < m$

  $m = 1$ is Wythoff's game

Consider the quadratic irrational $\alpha = \frac{2-m+\sqrt{4+m^2}}{2} = [1, \overline{m}]$ and the Ostrowski p-system based on the convergents of the CF.

Theorem (A. Fraenkel 1982)

A pair (a, b) such that $a \leq b$ is a $\mathcal{P}$-position IFF
1) $\mathrm{rep}_\alpha(a)$ ends with an even number of zeroes and
2) $\mathrm{rep}_\alpha(b) = \mathrm{rep}_\alpha(a)0$.

$\alpha = \sqrt{2}$ A. Baranwal, L. Schaeffer, J. Shallit (2021)

```
ost ost2 [1] [2]:
def ost2_move "?msd_ost2 (a+b>0) & (a=0 | b=0 |
                    (a>=b & a<b+2) | (a<b & b<a+2))";
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Fraenkel's variations

Corollary

For any fixed m, we may apply the same approach as before.

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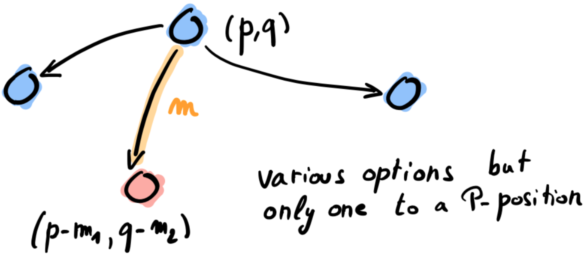
Something new: A move is redundant, if the set of $\mathcal{P}$-positions is unchanged when the move is deleted from the rule-set.

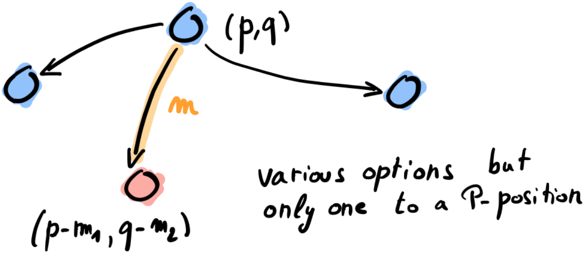
A move $m = (m_1, m_2)$ is not redundant, if there exists a $\mathcal{N}$-position (p, q) such that m is the unique winning move from (p, q) to some $\mathcal{P}$-position.



Fraenkel's variations

If moves and $\mathcal{P}$-positions are expressed in FO($\langle \mathbb{N}, +, V_U \rangle$), then non-redundancy can also be expressed:

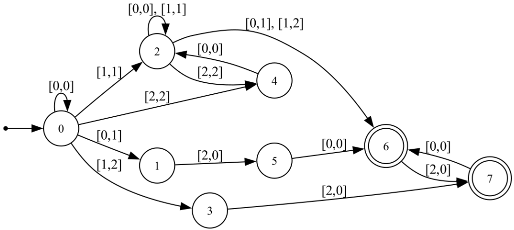
```
def ost2_non_redundant "?msd_ost2 $ost2_move(a,b)
  & Ep,q (~$ost2_ppos(p,q) & $ost2_ppos(p-a,q-b)
  & (Ac,d((a!=c|b!=d) & $ost2_move(c,d) & c<=p & d<=q)
              => ~$ost2_ppos(p-c,q-d)))":

def ost2_redundancies "?msd_ost2 $ost2_move(a,b)
                              & ~$ost2_non_redundant(a,b)":
```



Fraenkel's variations

Proposition ($m = 2$)

The variation of Wythoff's game where $|k - \ell| < 2$, has infinitely many redundant moves: $(n, n+1)$ and $(n+1, n)$ for all $n \geq 3$.



Intermediate computations : $\simeq 2500$ states, 7Gb of memory
We can do the same for $m = 3, 4$ up to 45Gb (21 minutes)

Beyond Ostrowski systems

Fraenkel (1998) $s, m > 0$ are integer parameters

- Remove a positive number of tokens from one pile,
- remove k tokens from one pile and ℓ from the other one, provided that $0 < k \leq \ell < sk + m$.

  For $s = 1$, this is the previous game with parameter m.

  For $s = m = 1$, this is Wythoff's game.

Consider the numeration system U defined by

$$U_{n+2} = (s + m - 1)\, U_{n+1} + U_n \quad \text{and} \quad U_0 = 1, U_1 = m + s$$

Theorem (A. Fraenkel 1998)

A pair (a, b) such that $a \leq b$ is a $\mathcal{P}$-position IFF
1) $\mathrm{rep}_U(a)$ ends with an even number of zeroes and
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Beyond Ostrowski systems

We are "lucky" to be in the Pisot case,

- We have an *addable* system

  Frougny's normalization (1992)

- We have a *regular* candidate for the set of $\mathcal{P}$-positions.
  We have the "same" Fraenkel's result for the third time.
- The rules can be expressed in FO$(\langle \mathbb{N}, +, V_U \rangle)$.

Hence, in principle, we may use `Walnut`.

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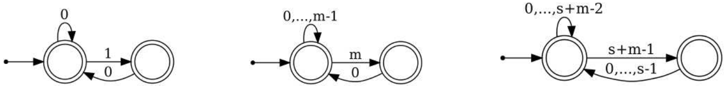

We still have to build an *adder*.

Beyond Ostrowski systems

Let $A = \{0, \dots, m+s-1\}$ and $d = 2(m+s-1)$

Follow the procedure given by C. Frougny, J. Sakarovitch (CANT 2010) and build the *zero automaton* over $\{-d, \dots, d\}$, states in $\mathbb{Z}[\beta]$. Since β is a Pisot number, the automaton is finite.

Now replace label ℓ with $(a, b, c) \in A^3$ s.t. $a + b - c = \ell$.

We provide `Walnut` with two automata:

- one for the U-representations
- one for addition

A similar result

A procedure and a tool to get an adder for Dumont–Thomas numeration in O. Carton, J.-M. Couvreur, M. Delacourt, and N. Ollinger (2024)

Beyond Ostrowski systems

As observed by Carton et al. validity of the adder can be effectively checked:

```
eval test1 "?msd_a3ba2 Ax,y Ez x+y=z":
eval test2 "?msd_a3ba2 Ax,y,z,t (x+y=z & x+y=t) => z=t":
eval test3 "?msd_a3ba2 Ax,z (x+0=z) <=> x=z":
eval test4 "?msd_a3ba2 Ax,t (x+1=t) <=>
                              (x<t & (Ay x<y => t <= y))":
eval test5 "?msd_a3ba2 Ax,y,z,u,t (u=y+1 & t=z+1) =>
                                       (x+y=z <=> x+u=t)":
```

Conclusions

Fraenkel's combinatorial games and `Walnut`:
a marriage made in heaven!

- The rules of the game can be expressed in $\mathsf{FO}(\langle \mathbb{N}, +, V_U \rangle)$.
- We have an addable numeration system U
- The set of $\mathcal{P}$-positions, when expressed within this system, is a regular language.

automatic proofs of old and new results ! Build new games, etc.

However,

- automatic proofs are obtained for fixed parameters
- state complexity could be problematic, Presburger arithmetic is beyond NP: triple exponential thight bound F. Klaedtke (2005)
- difficult to cope with Tribonacci adder E. Duchêne, M.R. (2008)