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Dissertation presented by

Thomas Lamby

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Notions of weight

Weights

A function $\phi : (0, \infty) \to (0, \infty)$ is a Boyd function if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0, \infty)$. The set of Boyd functions is denoted by $\mathcal{B}$. The lower and upper Boyd indices of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

Examples

Let ψ be a continuous slowly varying function on $(0, \infty)$:

$$\lim_{t \to 0} \frac{\psi(ts)}{\psi(t)} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{\psi(ts)}{\psi(t)} = 1$$

for any $s > 0$. For $\theta \in \mathbb{R}$, the function $t \mapsto t^{\theta}\psi(t)/\psi(1)$ is a Boyd function such that $\underline{b}(\phi) = \overline{b}(\phi) = \theta$. A standard choice for the slowly varying function is $\psi = (|\ln| + 1)^{\gamma}$, for $\gamma > 0$. One can deal with even more iterated logarithms : set

$$L_0(t) = t, \quad L_1(t) = 1 + |\log t| \quad \text{and} \quad L_m(t) = 1 + |\log(L_{m-1}(t))| \quad \text{for} \quad m > 1.$$

Then, if $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{R}^{n+1}$, $\phi_\alpha : (0, \infty) \to (0, \infty)$, defined by

$$\phi_\alpha(t) = \prod_{j=0}^{n} L_j(t)^{\alpha_j},$$

is a Boyd function such that $\underline{b}(\phi) = \overline{b}(\phi) = \alpha_0$.

Admissible Sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. One also associates Boyd indices to such a sequence. Let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\bar{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j}.$$

Relation between the two concepts

- Let $\phi \in \mathcal{B}$,

Proposition

Let $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$, then

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \text{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

- Let σ be an admissible sequence,

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}), j \in \mathbb{N}_0, \\ 1 & \text{if } t \in (0,1). \end{cases}$$

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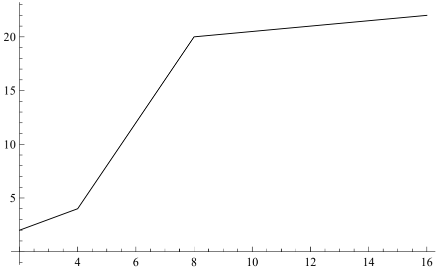
where $\underline{s}(\sigma) \leq s \leq \overline{s}(\sigma)$.

Construction of a smooth ϕ_σ

Suppose that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.



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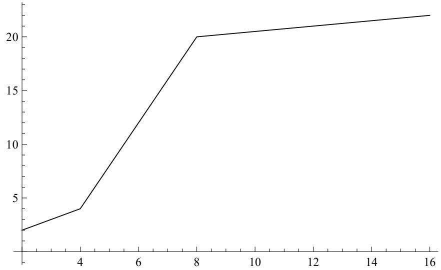

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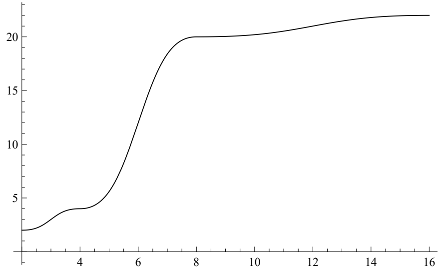

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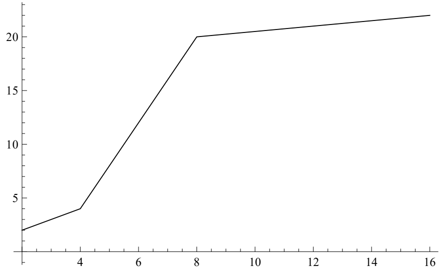

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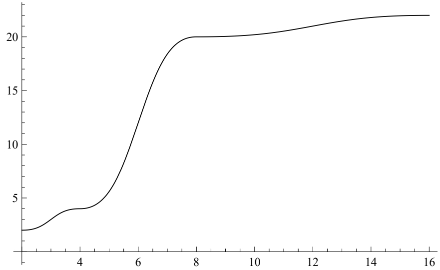


Theorem

If σ is an admissible sequence such that either $\underline{s}(\sigma) > 0$ or $\overline{s}(\sigma) < 0$, then there exists $\xi \in C^\infty(I)$ such that $(\xi(2^j))_j \asymp \sigma$ and $0 < \inf_{t>0} t\frac{|D\xi(t)|}{\xi(t)} \leq \sup_{t>0} t\frac{|D\xi(t)|}{\xi(t)} < \infty$.

Generalized Interpolation Spaces

Setting for real interpolation

Let $\mathscr{N}$ denotes the category of all normed vector spaces. A_0 and A_1 in $\mathscr{N}$ are *compatible* if they are both subspaces of a Hausdorff topological vector space. We set

$$\|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

and

$$\|a\|_{A_0+A_1} := \inf_{a=a_0+a_1}\{\|a_0\|_{A_0}, \|a_1\|_{A_1}\}.$$

Let $\mathscr{C}$ be a sub-category of $\mathscr{N}$ and denote by $\mathscr{C}_c$ a category of compatible couples $\mathbf{A} = (A_0, A_1)$ (such that $A_0 \cap A_1$ and $A_0 + A_1$ are in $\mathscr{C}$). The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathscr{C}_c$ are bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that both $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are morphisms in $\mathscr{C}$. The two basic functors Δ and Σ from $\mathscr{C}_c$ to $\mathscr{C}$ are defined as follows: $\Delta(T) = \Sigma(T) = T$ and

$$\Delta(\mathbf{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\mathbf{A}) = A_0 + A_1.$$

Interpolation spaces and functors

Given a couple $\mathbf{A} = (A_0, A_1)$ in $\mathscr{C}_c$, a space $A \in \mathscr{C}$ is an *intermediate space* between A_0 and A_1 (or with respect to $\mathbf{A}$) if

$$\Delta(\mathbf{A}) \hookrightarrow A \hookrightarrow \Sigma(\mathbf{A}).$$

Such a space A is called an *interpolation space* between A_0 and A_1 (or with respect to $\mathbf{A}$) if in addition $T : \mathbf{A} \to \mathbf{A}$ implies $T : A \to A$.

If $\mathbf{B}$ is another couple in $\mathscr{C}_c$, two spaces A and B in $\mathscr{C}$ are *interpolation spaces with respect to* $\mathbf{A}$ *and* $\mathbf{B}$ if A and B are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$ respectively and if $T : \mathbf{A} \to \mathbf{B}$ implies $T : A \to B$.

An *interpolation functor* on $\mathscr{C}$ is a functor F from $\mathscr{C}_c$ into $\mathscr{C}$ such that if $\mathbf{A}$ and $\mathbf{B}$ are couples in $\mathscr{C}_c$, then $F(\mathbf{A})$ and $F(\mathbf{B})$ are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$ and

$$F(T) = T \text{ for all } T : \mathbf{A} \to \mathbf{B}.$$

General case

Given $\phi \in \mathcal{B}$, we will denote by ϕ_* the function explicitly defined by $\phi_*(t) = t/\phi(t)$ for $t > 0$. Let $\mathbf{A} = (A_0, A_1)$ and $\mathbf{B} = (B_0, B_1)$ be two couples in $\mathscr{C}_c$; two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$ respectively are *of exponent* $\phi \in \mathcal{B}$ if, for any $T : \mathbf{A} \to \mathbf{B}$,

$$\|T\|_{A,B} \leq C\bar{\phi}_*(\|T\|_{A_0,B_0})\bar{\phi}(\|T\|_{A_1,B_1}) \tag{1}$$

always holds for some constant $C > 0$.

F is an interpolation functor of *exponent* $\phi \in \mathcal{B}$ if $F(\mathbf{A})$ and $F(\mathbf{B})$ are of exponent ϕ for any couples $\mathbf{A}$, $\mathbf{B}$ in $\mathscr{C}_c$.

K-method

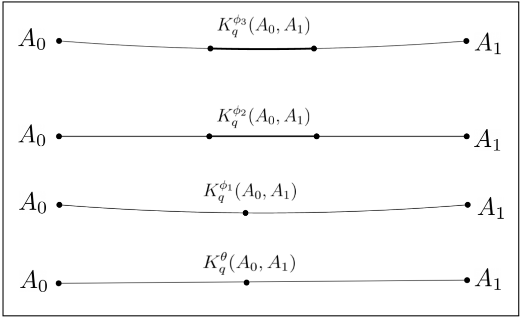
Given a couple $\mathbf{A}$, $t > 0$ and $a \in \Sigma(\mathbf{A})$,

$$K(t,a) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),$$

For $\phi \in \mathcal{B}$ and $q \in [1,\infty]$, let $K_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^\phi(\mathbf{A})} := \Big(\int_0^\infty \Big(\frac{1}{\phi(t)} K(t,a)\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.



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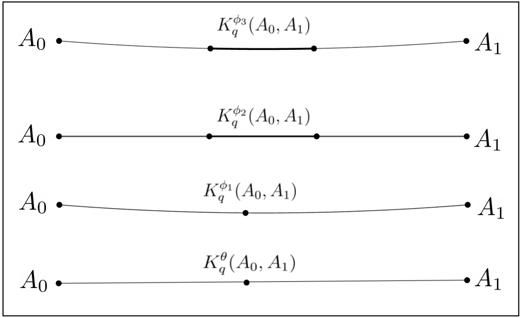

K_q^{ϕ} is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$.

J-method

Given a couple $\mathbf{A}$, $t > 0$ and $a \in \Delta(\mathbf{A})$,

$$J(t,a) := \max\{(\|a\|_{A_0}, t\|a\|_{A_1}),$$

For $\phi \in \mathcal{B}$ and $q \in [1,\infty]$, let $J_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ which can be represented by $a = \int_0^\infty b(t)\,dt/t$, with convergence in $\Sigma(\mathbf{A})$, where b is measurable, takes its values in $\Delta(\mathbf{A})$ for $t > 0$ and

$$t \mapsto \frac{J(t,b(t))}{\phi(t)} \in L_*^q.$$

This space is equipped with the norm

$$\|a\|_{J_q^\phi(\mathbf{A})} := \inf_b \|\frac{J(t,b(t))}{\phi(t)}\|_{L_*^q}.$$

the infimum being taken on all $b : (0,\infty) \to \Delta(\mathbf{A})$ measurable such that $a = \int_0^\infty b(t)\,dt/t$.

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Equivalence Theorem

For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi), \overline{b}(\phi) < 1$ and $q \in [1, \infty]$, we have $J_q^\phi(\mathbf{A}) = K_q^\phi(\mathbf{A})$.

Reiteration

X is of class $\mathcal{C}(\phi; \mathbf{A})$ if it is an intermediate spaces with respect to $\mathbf{A}$ such that, for all $a \in X$, $K(t, a) \leq C\phi(t)\|a\|_X$ and, for all $a \in \Delta(\mathbf{A})$, $\phi(t)\|a\|_X \leq CJ(t, a)$.

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Theorem

If for $j \in \{0,1\}$, X_j is of class $\mathcal{C}(\phi_j; \mathbf{A})$ with $\underline{b}(\phi_j) \geq 0$ and $\overline{b}(\phi_j) \leq 1$, let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$ and set $\theta = \phi_1/\phi_0$, $\psi = (\phi \circ \theta)\phi_0$; if $\underline{b}(\theta) > 0$ or $\overline{b}(\theta) < 0$ then

$$K_q^{\phi}(\mathbf{X}) = K_q^{\psi}(\mathbf{A}).$$

In particular, for $\underline{b}(\phi_j) > 0$ and $\overline{b}(\phi_j) < 1$, the spaces $K_{q_j}^{\phi_j}(\mathbf{A})$ are complete ($j \in \{0,1\}$), then

$$K_q^{\phi}(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) = K_q^{\psi}(\mathbf{A}).$$

Examples : Hölder spaces

For $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi), \overline{b}(\phi) < 1$, let $C_b^\phi(\mathbb{R}^d)$ is the space of the so-called bounded and uniformly ϕ-Hölder continuous functions, equipped with the norm

$$\|f\|_{C_b^\phi(\mathbb{R}^d)} := \|f\|_\infty + |f|_{C^\phi} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\phi(|x-y|)}.$$

Theorem

- Let $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi), \overline{b}(\phi) < 1$,

$$K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) = C_b^\phi(\mathbb{R}^d).$$

- Let $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ to set $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma \circ f)$. If $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ then

$$K_\infty^\gamma(C_b^{\phi_0}(\mathbb{R}^d), C_b^{\phi_1}(\mathbb{R}^d)) = C_b^\psi(\mathbb{R}^d).$$

Examples : Lebesgue spaces

Let $q \in [1, \infty]$ and $\phi \in \mathcal{B}$; if X is a Banach space, the space $\ell^q_\phi(X)$ consists of all sequences $(a_j)_j$ of X such that

$$(\phi(2^j)\|a_j\|_X)_j \in \ell^q.$$

Theorem

Let $q_0, q_1, q \in [1, \infty]$ and $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ to set $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma \circ f)$. If $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then

$$K^\gamma_q(\ell^{q_0}_{\phi_0}(X), \ell^{q_1}_{\phi_1}(X)) = \ell^q_\psi(X).$$

Examples : Besov spaces

Let $\phi \in \mathcal{B}$ and $(\varphi_j)_j$ be a Paley-Littlewood system of test functions. For $p, q \in [1, \infty]$, we define the generalized Besov Space by

$$B^{\phi}_{p,q} := \{f \in \mathcal{S}' : (\|\varphi_j * f\|_{L^p})_j \in \ell^q_\phi\}.$$

Theorem

Given $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, define $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma \circ f)$. If $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then

$$K^{\gamma}_{q}(B^{\phi_0}_{p,q_0}, B^{\phi_1}_{p,q_1}) = B^{\psi}_{p,q} \quad \text{for} \quad p, q, q_0, q_1 \in [1, \infty]$$

and

$$K^{\gamma}_{q}(F^{\phi_0}_{p,q_0}, F^{\phi_1}_{p,q_1}) = B^{\psi}_{p,q}, \quad \text{for} \quad p, q_0, q_1 \in (1, \infty), q \in [1, \infty].$$

Examples : Sobolev and Besov spaces

We denote by $\mathcal{B}''$ the set of functions $\phi \in \mathcal{B}$ which are C^∞ on $[1,\infty)$ and such that for all $m \in \mathbb{N}$, $t^m|\phi^{(m)}(t)| \leq C_m\,\phi(t)$ is satisfied for all $t \in [1,\infty)$. Given $\phi \in \mathcal{B}''$, the generalized Bessel operator $\mathcal{J}^\phi$ is defined on $\mathcal{S}'$ by

$$\mathcal{J}^\phi f = \mathcal{F}^{-1}(\phi(\sqrt{1+|\cdot|^2})\mathcal{F}f).$$

It is clear that $\mathcal{J}^\phi$ is a linear bijective operator from $\mathcal{S}'$ to $\mathcal{S}'$ such that $(\mathcal{J}^\phi)^{-1} = \mathcal{J}^{1/\phi}$ and $\mathcal{J}^\phi(\mathcal{S}) = \mathcal{S}$. From there, the generalized (fractional) Sobolev space H_p^ϕ is defined by

$$H_p^\phi = \{f \in \mathcal{S}' : \|\mathcal{J}^\phi f\|_{L^p} < \infty\}.$$

Theorem

Let $\gamma \in \mathcal{B}, \phi_0, \phi_1 \in \mathcal{B}''$, $f = \phi_0/\phi_1, \psi = \phi_0/(\gamma \circ f)$ and $p, q \in [1,\infty]$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$, then

$$K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1}) = B_{p,q}^\psi.$$

Continuous Interpolation Spaces and Limiting cases

Continuous Interpolation Spaces

Proposition

For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi), \overline{b}(\phi) < 1$ and $q \in [1, \infty)$, $\Delta(\mathbf{A})$ is dense in $K_q^\phi(\mathbf{A})$.

When $q = \infty$, $\Delta(\mathbf{A})$ is not dense in $K_\infty^\phi(\mathbf{A})$.

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Let $K_\infty^{0,\phi}(\mathbf{A})$ denote the space comprising all $a \in \Sigma(\mathbf{A})$ such that

$$\lim_{t \to 0} \frac{1}{\phi(t)} K(t, a) = \lim_{t \to \infty} \frac{1}{\phi(t)} K(t, a) = 0.$$

We naturally equip $K_\infty^{0,\phi}(\mathbf{A})$ with the norm induced by $K_\infty^{\phi}(\mathbf{A})$. It is evident that $K_\infty^{0,\phi}(\mathbf{A})$ constitutes a closed subspace of $K_\infty^{\phi}(\mathbf{A})$.

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For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$ is $K_\infty^{0,\phi}(\mathbf{A})$.

Examples : little Hölder Spaces

Theorem

For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, we have

$$K_{\infty}^{0,\phi}(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) = h^{\phi}(\mathbb{R}^d),$$

where $h^{\phi}(\mathbb{R}^d)$ is the space consisting of bounded functions f such that

$$\lim_{h\to 0} \sup_{x\in\mathbb{R}^d} \frac{|f(x+h) - f(x)|}{\phi(|h|)} = 0.$$

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Corollary

For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $C_b^1(\mathbb{R}^d)$ in $K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))$ is $h^\phi(\mathbb{R}^d)$.

Examples : Lebesgue and Besov Spaces

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Theorem

Let $q_0, q_1 \in [1, \infty]$, $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ and define $f = \phi_0/\phi_1$. If $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then

- $$K_\infty^{0,\gamma}(\ell_{\phi_0}^{q_0}(X), \ell_{\phi_1}^{q_1}(X)) = c_{0,\psi}(X),$$
  with $\psi = \phi_0/(\gamma \circ f)$, where $c_{0,\psi}(X)$ is the subspace of $\ell_\psi^\infty(X)$ such that $\lim_j \psi(2^j)\|a_j\|_X = 0$.
- $$K_\infty^{0,\gamma}(H_p^{\phi_0}, H_p^{\phi_1}) = b_{p,\infty}^\psi,$$
  for $\psi = \phi_0/(\gamma \circ f)$, where $b_{p,\infty}^\psi$ is the subspace of the elements a of $B_{p,\infty}^\psi$ (equipped with the induced norm) such that $\lim_j \psi(2^j)\|\varphi_j * a\|_{L^p} = 0$.

Limiting cases

Proposition

Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi), \overline{b}(\phi) \leq 1$ and $q \in [1, \infty]$.

(i) If

$$\int_0^1 \left(\frac{t}{\phi(t)}\right)^q \frac{dt}{t} = \infty \quad \text{or} \quad \int_1^\infty \left(\frac{1}{\phi(t)}\right)^q \frac{dt}{t} = \infty \tag{2}$$

with the usual modification if $q = \infty$, then $K_q^\phi(\mathbf{A}) = \{0\}$.

(ii) If

$$\int_0^\infty \left(\frac{1}{\phi(t)} \min\{1, t\}\right)^q \frac{dt}{t} < \infty, \tag{3}$$

with the usual modification if $q = \infty$, then K_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$.

Interpolation of several Spaces with function parameters

A word on the Interpolation with several spaces

Let $q \in [1,\infty]$, $\phi_1,\ldots,\phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$ and f be a function from $(0,\infty)^n$ to $(0,\infty)$. Set

$$\Phi_q^{\phi_1,\ldots,\phi_n}(f) = \Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(t_1)} \cdots \frac{1}{\phi_n(t_n)} f(t_1,\ldots,t_n)\big)^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big)^{1/q},$$

with the usual modification in the case $q=\infty$. Given $a \in \mathbf{A}$ and $t \in (0,\infty)^n$, let

$$K(t,a) = \inf_a \|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + \cdots + t_n\|a_n\|_{A_n},$$

where the infimum is taken over all the decompositions $a = a_0 + \cdots + a_n$, $a_j \in A_j$.

A word on the Interpolation with several spaces

We define $K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})$ as the set of $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})} = \Phi_q^{\phi_1,\dots,\phi_n}\big(K(t,a)\big) < \infty.$$

Let f be a function from $(0,\infty)^{n+1}$ to $(0,\infty)$; an interpolation functor F is of type f if there exists a constant $C \geq 1$ such that

$$\|T\|_{F(\mathbf{A}),F(\mathbf{B})} \leq C f\left(\|T\|_{A_0,B_0},\dots,\|T\|_{A_n,B_n}\right),$$

for any morphism $T : \mathbf{A} \to \mathbf{B}$.

Proposition

The functor $K_q^{\phi_1,\dots,\phi_n}$ is an exact interpolation functor of type f where $f(t_0,\dots,t_n) = t_0\overline{\phi_1}(t_1/t_0)\cdots\overline{\phi_n}(t_n/t_0)$.

A word on the Interpolation with several spaces

Proposition

Let $q \in [1,\infty]$, $\phi_1,\dots,\phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; then $J_q^{\phi_1,\dots,\phi_n}(\mathbf{A}) \hookrightarrow K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})$.

Let $\sigma(\mathbf{A})$ the subspace of all $a \in \Sigma(\mathbf{A})$ for which $\int \frac{K(t,a)}{\max t} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} < \infty$. The condition $\mathcal{F}(\mathbf{A})$ is satisfied if, for every $a \in \sigma(\mathbf{A})$, there exists a function $u : (0,\infty)^n \to \Delta(\mathbf{A})$ such that

$$a = \int u(t) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \quad \text{in } \Sigma(\mathbf{A}) \quad \text{and} \quad J\big(t, u(t)\big) \leq C(\mathbf{A}) K(t,a).$$

Theorem

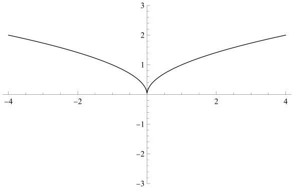
Let $q \in [1,\infty]$, $\phi_1,\dots,\phi_n \in \mathcal{B}$ such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; if $\mathcal{F}(\mathbf{A})$ is satisfied, then

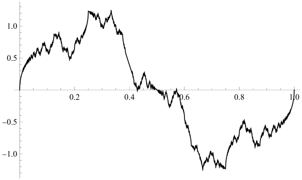
$$J_q^{\phi_1,\dots,\phi_n}(\mathbf{A}) = K_q^{\phi_1,\dots,\phi_n}(\mathbf{A}).$$

Pointwise Regularity

Hölder

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha$$





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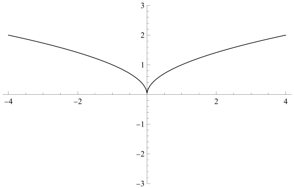
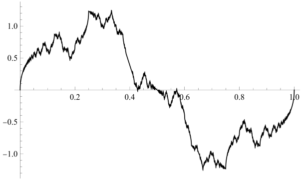
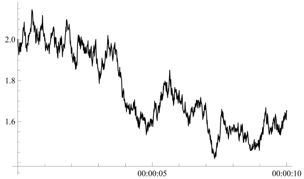
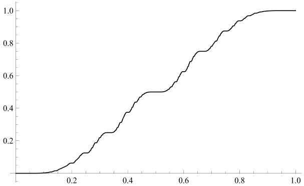




Weighted Hölder

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq C\phi(r)$$





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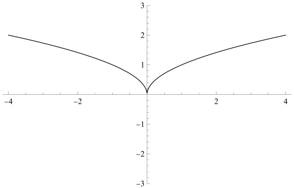

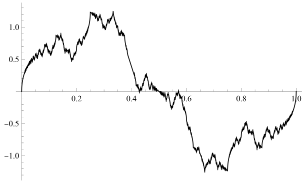

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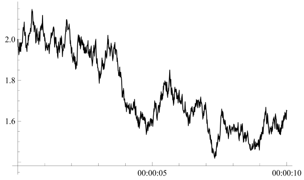
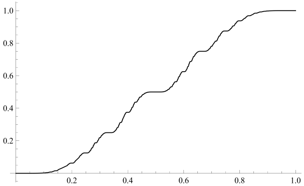
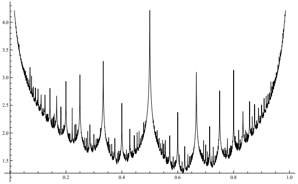
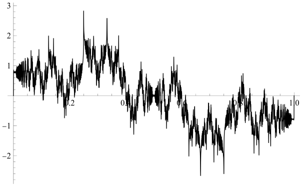




Calderon-Zygmund

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq Cr^\alpha$$





Pointwise Regularity

Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$, $\alpha > -d/p$, a function $f \in L^p_{\text{loc}}$ is in $T^p_\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq C r^\alpha$$

for sufficiently small r.

Pointwise Regularity

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p-exponent

$$h_p(x_0) := \sup\{\alpha > -d/p : f \in T^p_\alpha(x_0)\}.$$

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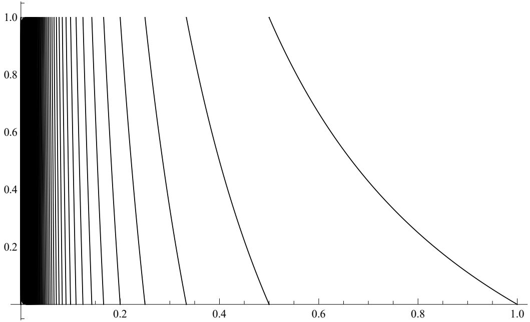
p-spectrum

$$d_p(h) = \dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : h_p(x) = h\}).$$

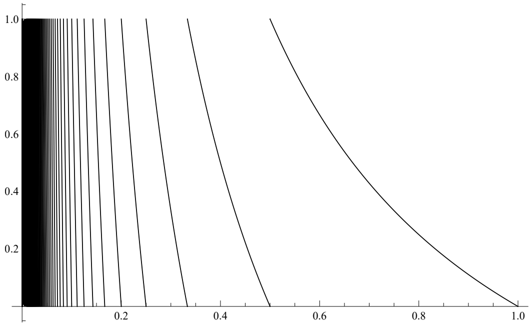
Regularity of functions defined through continued fractions

Brjuno function



$$A : (0,1) \to [0,1] \quad x \mapsto |\frac{1}{x} - \lfloor \frac{1}{x} \rfloor|.$$

Brjuno function



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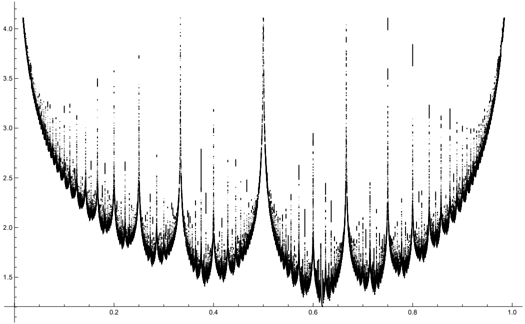

$$B : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} x_0 x_1 \ldots x_{n-1} \log x_n,$$

where $x_0 = |x - \lfloor x \rfloor|$ and $x_{n+1} = A(x_n)$.

Regularity of B

Theorem (S. Jaffard, B. Martin)

Let $p \in [1, \infty)$; the p-exponents of B are given by

$$h_p^{(B)}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise,} \end{cases}$$

where

$$\tau(x) = \sup \left\{ u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N} : \left| x - \frac{p}{q} \right| < \frac{1}{q^u} \right\}.$$

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Moreover, the p-spectrum is given by

$$d_p(h) = \begin{cases} 2h & \text{si } h \in [0, 1/2], \\ -\infty & \text{sinon.} \end{cases}$$

α-continued fractions

Given $\alpha \in [1/2, 1]$ and $x \in \mathbb{R}$, define

$$[x]_\alpha = \min\{p \in \mathbb{Z} : x < p + \alpha\}.$$

We introduce the (generalized) Gauss map:

$$A_\alpha : (0, \alpha) \to [0, \alpha] : x \mapsto \left|\frac{1}{x} - \left[\frac{1}{x}\right]_\alpha\right|.$$

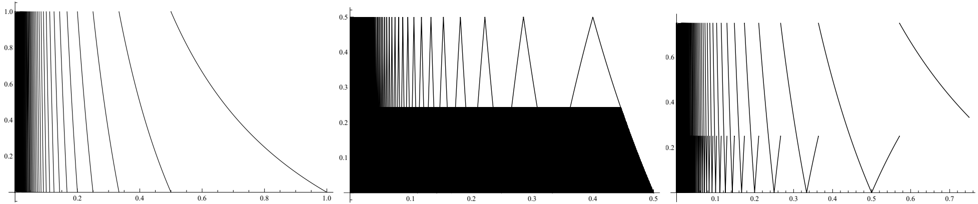


Figure 1: $A_\alpha : (0, \alpha) \to [0, \alpha]$ with resp. $\alpha = 1$, $\alpha = 1/2$ and $\alpha = 3/4$.

α-continued fractions

Set $x_0 = |x - [x]_\alpha|$ and $a_0 = [x]_\alpha$. Consequently, $x_0 = a_0 + \varepsilon_0 x_0$, where

$$\varepsilon_0 = \begin{cases} 1 & \text{if } x \geq a_0, \\ -1 & \text{otherwise.} \end{cases}$$

This initialization defines $x_{n+1} = A_\alpha(x_n)$ and

$$a_{n+1} = [\frac{1}{x_n}]_\alpha \geq 1,$$

for $n \in \mathbb{N}_0$ if it is meaningful. Subsequently, $x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1}$, where

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

α-continued fractions

The n-th α-convergent of x is given by

$$\frac{p_n}{q_n} = [(a_0, \varepsilon_0), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n] = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{\ddots + a_{n-1} + \cfrac{\varepsilon_{n-1}}{a_n}}}$$

Let $x \in \mathbb{R} \setminus \mathbb{Q}$, we introduce the α-irrationality exponent of x as $\tau^{(\alpha)}(x) = \limsup_{n\to\infty} \frac{\log |x - \frac{p_n}{q_n}|}{\log \frac{1}{q_n}}$.

Theorem

For all $\alpha \in [1/2, 1]$, $x \in \mathbb{R} \setminus \mathbb{Q}$,

$$\tau^{(\alpha)}(x) = \tau(x).$$

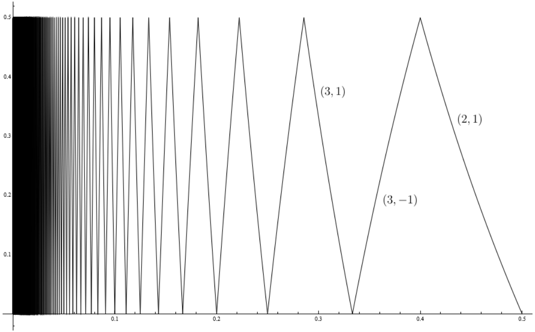
α-cells

We set

$$\mathfrak{c}(a,\varepsilon) = \begin{cases} \left(\frac{1}{a+\alpha}, \frac{1}{a}\right) \cap (0,\alpha) \text{ if } \varepsilon = 1, \\ \left(\frac{1}{a}, \frac{1}{a+\alpha-1}\right) \cap (0,\alpha) \text{ if } \varepsilon = -1. \end{cases}$$

and

$$\mathfrak{c}[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)] = \bigcap_{j=1}^{n} A_\alpha^{-(j-1)}(\mathfrak{c}(a_j,\varepsilon_j)).$$



If $\mathfrak{c}[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)]$ is non empty, we say that $\mathfrak{c}[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)]$ is an *α-cell of depth n* and $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)$ is called *admissible*.

Möbius Transformations

The image of $\mathfrak{c}(a, \varepsilon)$ under A_α,

$$J(a, \varepsilon) = A_\alpha\big(\mathfrak{c}(a, \varepsilon)\big),$$

is an open interval, and the inverse of A_α on $\mathfrak{c}(a, \varepsilon)$ is given by

$$\begin{aligned} \psi_{(a,\varepsilon)} \colon J(a, \varepsilon) &\to \mathfrak{c}(a, \varepsilon) \\ t &\mapsto \frac{1}{a + \varepsilon t}. \end{aligned}$$

We set

$$\psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)} = \psi_{(a_1,\varepsilon_1)} \circ \psi_{(a_2,\varepsilon_2)} \circ \cdots \circ \psi_{(a_n,\varepsilon_n)},$$

so that

$$\psi_{(a_1,\varepsilon_1),\dots,(a_n,\varepsilon_n)}(t) = \frac{p_n + t\varepsilon_n p_{n-1}}{q_n + t\varepsilon_n q_{n-1}}.$$

Advantageous numbers

Let $n \geq 1$, $(a_1, \varepsilon_1)\ldots(a_n, \varepsilon_n) \in \mathcal{A}^*$. We have

$$\mathfrak{c}\big[(a_1, \varepsilon_1)\ldots(a_n, \varepsilon_n)\big] = \bigcap_{j=1}^{n} \psi_{(a_1,\varepsilon_1)\ldots(a_j,\varepsilon_j)}\big(J(a_j, \varepsilon_j)\big) \tag{4}$$

A number $\alpha \in [1/2, 1]$ is called *advantageous* if for all $n \geq 1$ and for all $(a_1, \varepsilon_1)\ldots(a_n, \varepsilon_n) \in \mathcal{L}_n(\alpha)$,

$$\mathfrak{c}\big[(a_1, \varepsilon_1)\ldots(a_n, \varepsilon_n)\big] = \psi_{(a_1,\varepsilon_1)\ldots(a_n,\varepsilon_n)}\big(J(a_n, \varepsilon_n)\big).$$

Proposition

A number $\alpha \in [1/2, 1]$ is advantageous if and only if

$$\alpha \in \{1/2, g, 1\} \cup \left\{1 - \frac{1}{k}, k \geq 3\right\} \cup \left\{\frac{-k + \sqrt{k^2 + 4k}}{2}, k \geq 2\right\}.$$

Brjuno functions

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$B_\alpha : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} x_0 x_1 \ldots x_{n-1} \log x_n.$

Theorem

Let $p \in [1, \infty)$; the p-exponents of $B_{1/2}$ are given by $h_p(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$

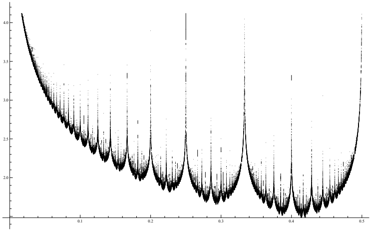

Figure 2: Brjuno function $B_{1/2}$.

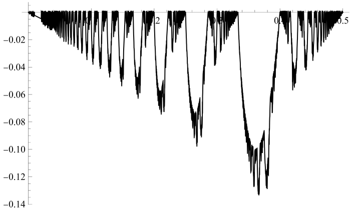

Figure 3: Difference $B - B_{1/2}$.

Thomae's function

$$T_\theta(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

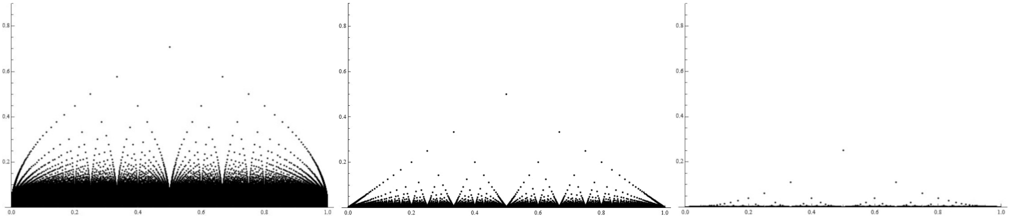

Figure 4: Representation of the function T_θ on $(0, 1)$ for $\theta = 1/2, 1$ and 2.

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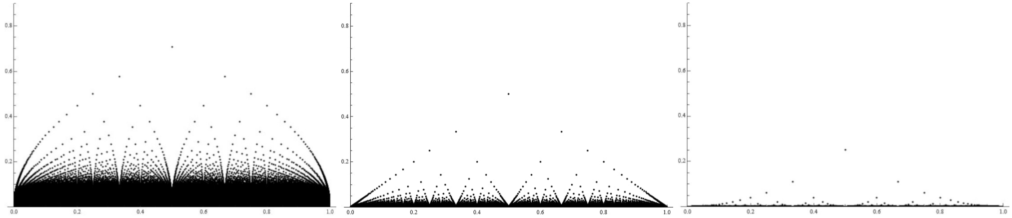
Proposition

For $\theta \in (0, 2]$,

$$h_{T_\theta}^{(\infty)}(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ \theta/\tau(x) & \text{if } x \text{ is irrational.} \end{cases}$$



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Thank you for your attention !