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Generalized Interpolation Methods and Pointwise Regularity through Continued Fractions and Diophantine Approximations

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Abstract

The study of function regularity has been extensively explored, with numerous tools developed to describe this property, such as continuity, differentiability, and Hölder conditions. One way to characterize regularity is by examining the functional spaces to which a function belongs, particularly those constructed through interpolation. A key objective of this work is to refine interpolation methods to define regularity spaces more precisely. A central idea is the use of Boyd functions, which replace the usual power functions in interpolation spaces. These functions, governed by specific continuity and growth conditions, provide a more precise means of capturing fine details in regularity, such as logarithmic effects, which appear, for instance, in Brownian motion. We divide this work into two main parts: the first one is about generalized interpolation spaces and the second one about pointwise regularity through continued fractions.

In the first Chapter, Boyd functions are studied to establish a solid foundation for defining new spaces. They are decomposed into two germs, leading to a representation Theorem that clarifies their structure. This also provides an opportunity to deepen the understanding of the connections between these functions and the concept of admissible sequences, which are likewise employed in the generalization of spaces. In particular, we demonstrate how to construct an adapted Boyd function from a given admissible sequence. It is noteworthy that Boyd functions are well suited for uniform spaces, whereas admissible sequences are more appropriate for pointwise spaces. Furthermore, an improved version of Merucci's Theorem is presented, facilitating work with generalized functional spaces involving parameter functions of higher regularity.

Next, in the second Chapter, generalized real interpolation methods are examined from a functorial perspective. Classical methods, such as the K-method, are extended by incorporating function parameters defined via Boyd functions. It is shown that the K-method and the J-method remain equivalent in this extended framework, ensuring consistency. A reiteration Theorem is also established, reinforcing the structural robustness of these interpolation spaces.

In the third Chapter, we investigate continuous interpolation spaces defined by function parameters, a construction central to trace theory and the analysis of boundary value problems for partial differential equations. Their functorial interpretation is explored, along with density results in specific cases. These spaces, characterized by asymptotic regularity properties, play a fundamental role in the analysis of operators in weighted functional spaces and in solving PDEs with precisely prescribed boundary behavior. We

apply these results to several examples, including Hölder, Lebesgue or Besov spaces. The extreme cases ($\theta = 0$ and $\theta = 1$) are specifically examined, providing an additional reason to use Boyd functions.

In the last Chapter of the first part, the scope is extended from interpolation between two spaces to the interpolation of multiple spaces. We attempt to generalize the results of the second Chapter to this context, emphasizing the usefulness of transitioning to multiple spaces.

The second part focuses on pointwise regularity, examining functions that are not necessarily locally bounded via Calderón-Zygmund spaces. A key example is the Brjuno function, which arises in dynamical systems and Diophantine approximation. Generalized versions of this function, linked to α-continued fractions, are studied. In the fifth Chapter, we explore the metric properties of these continued fractions, such as the notion of cells in this context, with the aim of providing a solid foundation for studying the pointwise regularity of generalized versions of the Brjuno function. We observe that certain values of α lead to cell structures that are easier to describe.

In the final Chapter, we study the pointwise regularity of the Brjuno-Yoccoz function, which corresponds to $\alpha = 1/2$, and we note that its regularity is identical to that of the standard Brjuno function: the behavior at each point is inversely proportional to the irrationality exponent at that point. Thomae's function is also investigated, an emblematic example of function that is discontinuous at rational points yet continuous elsewhere. This analysis offers a refined perspective on its irregularity, leveraging classical analytical tools to elucidate its fractal nature.

Résumé

L'étude de la régularité des fonctions a été largement considérée, avec de nombreux outils développés pour la décrire, comme la continuité, la dérivabilité, l'appartenance aux espaces de Hölder,... Une façon d'analyser cette régularité est d'examiner les espaces fonctionnels auxquels une fonction appartient, en particulier ceux construits par interpolation. L'un des objectifs de ce travail est d'améliorer les méthodes d'interpolation afin de définir des espaces de régularité plus précis. Une idée clé est l'utilisation des fonctions de Boyd, qui remplacent les fonctions puissances habituelles dans ces espaces. Ces fonctions, qui vérifient certaines propriétés de continuité et de croissance, permettent de mieux caractériser la régularité, comme la présence de logarithmes, qui apparaissent, par exemple, dans le mouvement Brownien. On divise ce travail en deux parties principales : la première porte sur les espaces d'interpolation généralisés, tandis que la seconde concerne la régularité ponctuelle de fonctions définies à partir des fractions continues.

Dans le premier chapitre, les fonctions de Boyd sont étudiées afin de poser une base solide pour la définition de nouveaux espaces. Elles sont décomposées en deux germes, ce qui mène à un théorème de représentation expliquant leur structure. Nous en profitons également pour mieux comprendre les liens entre ces fonctions et le concept de suites admissibles, utilisées également pour généraliser les espaces. En particulier, nous montrons comment construire une fonction de Boyd adaptée à partir d'une suite admissible donnée. Nous remarquons que les fonctions de Boyd sont bien adaptées aux espaces uniformes, tandis que les suites admissibles conviennent mieux aux espaces ponctuels. De plus, une amélioration du théorème de Merucci est présentée, permettant de travailler avec des espaces fonctionnels généralisés où les fonctions paramètres sont plus régulières.

Ensuite, dans le deuxième chapitre, nous étudions les méthodes d'interpolation réelle généralisées avec les fonctions de Boyd, en adoptant un point de vue fonctoriel. Nous démontrons que les méthodes K et J restent équivalentes dans ce nouveau contexte, garantissant ainsi la consistance de la théorie dévelopée. Un théorème de réitération est également prouvé, confirmant la structure robuste de ces espaces d'interpolation.

Dans le troisième chapitre, nous explorons les espaces d'interpolation "continus" définis par une fonction paramètre, un concept central dans la théorie des traces et dans l'étude des problèmes limites pour les équations aux dérivées partielles. Leur interprétation fonctorielle est examinée et des résultats de densité sont établis pour certains cas spécifiques. Ces espaces, caractérisés par des propriétés de régularité asymptotique, jouent un rôle clé dans l'analyse des opérateurs dans des espaces fonctionnels à poids et dans la résolution

d'équations aux dérivées partielles avec un comportement précis aux frontières. Nous appliquons ces résultats à plusieurs exemples, notamment les espaces de Hölder, de Lebesgue et de Besov. Les cas extrêmes ($\theta = 0$ et $\theta = 1$) sont examinés en détail, fournissant ainsi une justification supplémentaire à l'utilisation des fonctions de Boyd.

Dans le dernier chapitre de la première partie, le cadre est élargi de l'interpolation entre deux espaces à l'interpolation de plusieurs espaces. Nous tentons de généraliser les résultats du deuxième chapitre à ce contexte, en mettant en évidence l'intérêt de considérer plusieurs espaces.

La deuxième partie se focalise sur la régularité ponctuelle et étudie des fonctions qui ne sont pas nécessairement localement bornées à travers les espaces de Calderón-Zygmund. Un exemple clé est la fonction de Brjuno, qui apparaît en systèmes dynamiques et en approximation diophantienne. Nous étudions des versions généralisées de cette fonction en lien avec les fractions continues de type α. Dans le cinquième chapitre, on explore les propriétés métriques de ces fractions continues, comme par exemple les notions de cellule dans ce contexte, dans le but d'avoir des bases solides pour étudier la régularité ponctuelle des versions générales de la fonction de Brjuno. On s'aperçoit que certains α donnent des notions de cellules plus faciles à décrire.

Dans le dernier chapitre, on étudie la régularité ponctuelle de la fonction de Brjuno-Yoccoz, qui correspond à $\alpha = 1/2$ et nous remarquons qu'elle est identique à celle de la fonction de Brjuno usuelle : le comportement en chaque point est inversement proportionnel à l'exposant d'irrationalité au point. Nous nous intéressons également à la fonction de Thomae, un exemple classique de fonction qui est discontinue aux rationnels et continue ailleurs. Cette étude offre une perspective approfondie sur l'irrégularité de la fonction de Thomae, en utilisant des outils analytiques classiques pour déterminer sa nature fractale.

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Introduction

Given a function, one may naturally inquire about its regularity. Over time, various tools have been developed to characterize this property.

One approach involves analyzing the function from a uniform perspective by considering the functional spaces it belongs to and those it does not belong to. Some functional spaces can be constructed through the interpolation of other spaces. The primary objective of my thesis is to generalize interpolation methods to create more precise regularity spaces. A key tool for this generalization is the use of Boyd functions, which replace the standard functions $t \mapsto t^\theta$ commonly used in these spaces. For instance, in the case of Brownian motion, Boyd functions help capture the influence of the iterated logarithm, a feature not reflected in classical Hölder spaces.

A function $\phi : (0, \infty) \to (0, \infty)$ is a Boyd function if it is continuous, satisfies $\phi(1) = 1$ and

$$\overline{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty, \quad \forall t \in (0, \infty).$$

In this context, the so-called Boyd indices have many applications (see, for example, [18, 88]). The lower and upper Boyd indices of a Boyd function ϕ are defined by

$$\underline{b}(\phi) = \lim_{t\to 0} \frac{\log \overline{\phi}(t)}{\log t}, \quad \text{and} \quad \overline{b}(\phi) = \lim_{t\to\infty} \frac{\log \overline{\phi}(t)}{\log t}.$$

In the first Chapter, we focus on this concept to establish a robust foundation for the generalization of spaces. We first present the basic properties of Boyd functions to show that they can be decomposed into two parts, leading to a representation Theorem:

A function $\phi : (0, \infty) \to (0, \infty)$ is a Boyd function if and only if $\phi(1) = 1$ and there exist four bounded continuous functions $\eta_0, \xi_0 : (0, 1] \to (0, \infty)$ and $\eta_\infty, \xi_\infty : [1, \infty) \to (0, \infty)$ such that

$$\phi(t) = e^{\eta_0(t)+\int_1^{1/t} \xi_0(s)\frac{ds}{s}}, \quad t \in (0, 1],$$

$$\phi(t) = e^{\eta_\infty(t)+\int_1^{t} \xi_\infty(s)\frac{ds}{s}}, \quad t \in [1, \infty).$$

We then present the essential properties of admissible sequences, which also play a role in space generalization, and investigate the relations between these two notions, with special attention to the Boyd indices. We show that a Boyd function can be identified with a couple of germs, while a germ can be associated with an admissible sequence. As a consequence, the admissible sequences are best suited for pointwise spaces since only asymptotic conditions are usually needed, while the Boyd functions are more adapted to uniform spaces on open sets.

Next, starting from an admissible sequence, we propose some constructions of Boyd functions. By doing so, we are led to improve a result of Merucci [88], where ξ is only differentiable:

If ϕ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in C^\infty(0, \infty)$, with

$$0 < \inf_{t>0} t\frac{|D\xi(t)|}{\xi(t)} \leq \sup_{t>0} t\frac{|D\xi(t)|}{\xi(t)} < \infty, \tag{1}$$

such that ξ is equivalent to ϕ, i.e., there exists a constant $C > 0$ such that $C^{-1}\xi \leq \phi \leq C\xi$. Condition (1) is commonly employed for variable changes of the form $s = f(t)$ in integrals involving the measure dt/t.

The origins of interpolation theory can be traced back to Marcinkiewicz [84] and the Riesz-Thorin Theorem [100], which states that if a linear operator is continuous on L^p and L^q, then it remains continuous on L^r for any r between p and q. Later, with the realization that Sobolev spaces consist of functions exhibiting fractional orders of differentiability [98], various techniques were devised to construct analogous spaces. A common approach in real interpolation methods, extended via a function parameter, is based on the K-method.

Let A_0 and A_1 be two Banach spaces continuously embedded in a Hausdorff topological vector space, such that $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined Banach spaces. The K-functional is defined as

$$K(t, a) = \inf_{a=a_0+a_1} \|a_0\|_{A_0} + t\|a_1\|_{A_1}$$

for $t > 0$ and $a \in A_0 + A_1$. For $0 < \theta < 1$ and $q \in [1, \infty]$, an element a belongs to the interpolation space $K_q^\theta(A_0, A_1)$ if $a \in A_0 + A_1$ and

$$(2^{-j\theta}K(2^j, a))_j \in \ell^q.$$

The generalized interpolation method is obtained by replacing the sequence $(2^{-j\theta})_j$ with a Boyd function.

The second Chapter investigates generalized real interpolation methods from a functorial perspective, aiming to extend their applicability and theoretical foundations. In 1965, Aronszajn and Gagliardo established that any interpolation space of a given Banach couple can be realized as the value of a minimal or maximal interpolation functor on the category of all Banach couples [5]. We adapt this Theorem to our framework. The J-method can be generalized in a similar fashion, and we show that both methods yield the same interpolation spaces. Furthermore, we establish a reiteration Theorem, compactness Theorems and results about retracts, reinforcing the robustness of these methods within this analytical framework. We also observe that generalizing the complex interpolation method C_θ using Boyd's functions appears to be challenging, as only the exponent θ is involved, rather than the function $t \mapsto t^\theta$. We still obtain connections with the generalized real interpolation method.

The trace method is a widely employed technique in the study of boundary value problems for partial differential equations [74]. It is particularly effective for handling problems involving boundary values or non-homogeneous boundary conditions. Within this setting, trace Theorems offer compelling interpolation methods [81, 107]. Let $K^{\infty}_{0,\theta}(A_0, A_1)$ denote the space of all $a \in K^{\theta}_{\infty}(A_0, A_1)$ such that

$$\lim_{t\to 0} \frac{K(t,a)}{t^\theta} = \lim_{t\to\infty} \frac{K(t,a)}{t^\theta} = 0.$$

In the third Chapter, we examine the properties of these newly defined spaces using function parameter, which we refer to as generalized continuous interpolation spaces, and establish their interpretation within a functorial framework. Specifically, we address the density problem in the context of L^∞ spaces. Additionally, we highlight the versatility of these spaces through various examples. Notably, the generalized continuous interpolation between the space of bounded continuous functions and the space of continuously differentiable functions with bounded derivatives yields a weighted variant of the so-called little Hölder space [81]. In the classical case, if $\theta = 0$ or $\theta = 1$, the spaces $K^{\theta}_{q}(A_0, A_1)$ are degenerate. It may not be the case when $0 = \underline{b}(\phi)$ and $\overline{b}(\phi) = 1$ (same question with the continuous setting). One may also question the validity of the equivalence Theorem between the K- and J-methods when $b(\phi) = 0$ or $b(\phi) = 1$. We derive conditions on ϕ ensuring the validity of the Theorem in these cases.

In the next Chapter, we extend interpolation theory beyond the conventional framework of two Banach spaces with a function parameter by incorporating methods involving more than two spaces. Additionally, we provide a functorial interpretation of this extended approach. This framework lays the foundation for generalizing the Stein-Weiss interpolation Theorem, inspired by the approach in [6], which necessitates a reiteration formula for triples. Unlike the interpolation of Banach couples, families of smooth functions are not necessarily stable under real interpolation. However, the situation changes significantly when interpolating between triples of smooth spaces.

We explore the equivalence between the K- and J-methods in this extended setting, shedding light on essential properties of the resulting spaces. We establish that, in a fairly general context, these methods involve only 1-functors. Finally, we apply our developed theory to demonstrate that the interpolation of multiple generalized Sobolev spaces still results in a generalized Besov space.

Another approach to characterizing regularity consists of studying the pointwise regularity of a function at each of its points. For functions that are not locally bounded, the notion of Calderón-Zygmund spaces becomes relevant [26, 66]. Given a point x, $p \in [1, \infty]$, and $u \geq -1/p$, the space $T^p_u(x)$ consists of functions $f \in L^p$ for which there exists a polynomial P of degree strictly less than u such that

$$r^{-d/p} \|f - P\|_{L^p(B(x,r))} \leq C r^u$$

for all $r > 0$. The p-exponent of a function f at x is then defined as the supremum of u such that $f \in T^p_u(x)$.

A well-known example of a function that is not locally bounded is the Brjuno function, whose pointwise regularity has been established [48]. Consider the Gauss map A, which associates to a real number $x \in (0,1)$ the fractional part of its inverse. The Brjuno function is defined by the series

$$B(x) = \sum_{n\geq 0} xA(x)\cdots A^{n-1}(x)\log\frac{1}{A^n(x)}.$$

The Brjuno function plays a fundamental role in the study of dynamical systems generated by iterations of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. In [85], generalized versions of A are also considered: given $\alpha \in [1/2, 1]$, define

$$[x]_\alpha = \min\{p \in \mathbb{Z} : x < p + \alpha\}$$

for a real x. The (generalized) Gauss map is then given by

$$A_\alpha(x) = \left|\frac{1}{x} - [\frac{1}{x}]_\alpha\right|.$$

Considering iterations of A_α leads to new continued fraction expansions: set $x_0 = |x - [x]_\alpha|$ and $a_0 = a_0(x) = [x]_\alpha$. Consequently, $x_0 = a_0 + \varepsilon_0 x_0$, where

$$\varepsilon_0 = \begin{cases} 1 & \text{if } x \geq a_0, \\ -1 & \text{otherwise.} \end{cases}$$

This initialization defines $x_{n+1} = A_\alpha(x_n)$ and

$$a_{n+1} = a_{n+1}(x) = [\frac{1}{x_n}]_\alpha \geq 1,$$

for $n \in \mathbb{N}_0$ if it is meaningful. Subsequently, $x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1}$, where

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

If $x_n = 0$ for some $n \in \mathbb{N}_0$, the process concludes. Otherwise, this algorithm establishes three sequences $(x_n)_{n\in\mathbb{N}_0}$, $(a_n)_{n\in\mathbb{N}_0}$ and $(\varepsilon_n)_{n\in\mathbb{N}_0}$.

The goal of the fifth Chapter is to explore the metric properties of these continued fractions, such as the notion of cells in this context, with the aim of providing a solid foundation for studying the pointwise regularity of generalized versions of the Brjuno function using those A_α, thus addressing certain questions raised at the end of [48]. Since the regularity of B at x is linked to the irrationality measure of x, one can naturally define an irrationality exponent linked to the convergents (p_n/q_n) obtained by iterations of A_α :

$$\tau^{(\alpha)}(x) = \limsup_n \frac{\log|x - \frac{p_n}{q_n}|}{\log\frac{1}{q_n}}.$$

We proved that those exponents are, in fact, the same. We also consider α-cells: fixing $a_0, ..., a_n$, $\varepsilon_0, ..., \varepsilon_n$, it is the sets of points x such that

$$\forall j \in \{1, \dots, n\}, a_j(x) = a_j, \varepsilon_j(x) = \varepsilon_j.$$

When $\alpha = 1$, the structure of the cells plays a crucial role in [9] and [48]. In particular, it seems crucial to know the endpoints of a given cell. It is the case for certain values of α, which we call advantageous : α is advantageous if and only if

$$\alpha \in \{1/2, g, 1\} \cup \Big\{1 - \frac{1}{k}, k \geq 3\Big\} \cup \Big\{\frac{-k + \sqrt{k^2 + 4k}}{2}, k \geq 2\Big\}.$$

The multifractal analysis of $B_{1/2}$, corresponding to the nearest integer continued fraction expansion, is examined in the last Chapter: the fractal properties of $B_{1/2}$ mirror those of B, with the p-exponent at x equal to the inverse of the irrationality exponent of x. This result for $\alpha = 1/2$ follows from the fact that $B - B_{1/2}$ is 1/2-Hölder continuous [85] and results from [9]. Here, we prove it by adapting each outcome from [48], utilizing the results of the preceding Chapter for $\alpha = 1/2$.

We also investigate the pointwise regularity of another function arising from Diophantine approximation: Thomae's function [71], a paradigmatic example of a function that is continuous on the irrationals and discontinuous elsewhere. Defined for a parameter $\theta \in (0, 2]$, it exhibits a rich self-similar structure and intriguing regularity properties. After revisiting its fundamental characteristics, we analyze its Hölder continuity, emphasizing the interplay between its discrete spikes and its behavior on dense subsets of the real line. This study provides a refined perspective on the irregularity of Thomae's function, using classical analytical tools to elucidate its fractal nature.

For all the aforementioned functions, the regularity exponent at a given point is inversely proportional to the irrationality measure of that point. This implies that the better a point is approximable by rationals, the less regular these functions are at that point. This raises the question of whether all multifractal functions associated with diophantine approximation follow a similar pattern. To explore this, we introduce the Takagi function, which is monofractal with regularity 1, and the Minkowski function, whose pointwise regularity remains unknown but which is multifractal with a different regularity behavior than the inverse of the irrationality measure.

Notations

We use the letter C for a generic positive constant, while d is the dimension of the space if it makes sense.

Sets

$\mathbb{N}$	Naturals without zero
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
I	$(0, \infty)$
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{\pm\infty\}$
$\mathcal{B}$	Set of Boyd functions
SV	Set of Slowly varying functions
OR	Set of O-regular functions
$\mathcal{B}'$	Subset of $\mathcal{B}$ with regularity assumptions on the derivative
$\mathcal{B}''$	Subset of $\mathcal{B}$ with C^∞ regularity assumptions
$\mathcal{B}^\infty$	Set of 1-germ Boyd function
$\mathbb{P}^n_+$	$(0,\infty)^{n+1}/(0,\infty)$ id
$\mathcal{I}(a,b)$	(a,b) if $a < b$ and (b,a) otherwise
$\mathfrak{c}_n$	Cell $\mathfrak{c}[(a_0,\varepsilon_0)\ldots(a_n,\varepsilon_n)]$
$J(a,\varepsilon)$	$A_\alpha\big(\mathfrak{c}(a,\varepsilon)\big)$
BN	Set of Brjuno numbers
$\mathscr{D}_1(\alpha)$	$\left\{\frac{1}{k} \,:\, k \in \mathbb{N}\right\} \cap (0,\alpha)$
$\mathscr{D}_2(\alpha)$	$\left\{\frac{1}{k+\alpha} \,:\, k \in \mathbb{N}\right\} \cap (0,\alpha)$
$\mathscr{D}(\alpha)$	$\mathscr{D}_1(\alpha) \cup \mathscr{D}_2(\alpha)$
$Z(\alpha)$	$\bigcup_{j\geq 0} A_\alpha^{-j}\big(\mathscr{D}(\alpha)\big)\big)$
X	$X(\alpha) = (0,\alpha) \setminus Z(\alpha)$
$E_{1/2}$	$\bigcup_{n\geq 1} A_{1/2}^{-n}(\{1/2\})$
E_0	$(\bigcup_{n\geq 1} A_{1/2}^{-n}(\{0\}))\setminus E_{1/2}$
$E_h^{(p)}(f)$	Iso-Hölder set of the function f

Symbols

$\overline{b}(\phi)$	Upper Boyd index of $\phi \in \mathcal{B}$
$\underline{b}(\phi)$	Lower Boyd index of $\phi \in \mathcal{B}$
$f \asymp g$	There exists a constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$
$\overline{s}(\phi)$	Upper Boyd index of $\phi \in \mathcal{B}^\infty$
$\underline{s}(\phi)$	Lower Boyd index of $\phi \in \mathcal{B}^\infty$
$\overline{s}(\sigma)$	Upper Boyd index of the admissible sequence σ
$\overline{\mathfrak{s}}(\sigma)$	Alternative upper Boyd index of the admissible sequence σ
$\underline{s}(\sigma)$	Lower Boyd index of the admissible sequence σ
$\underline{\mathfrak{s}}(\sigma)$	Alternative lower Boyd index of the admissible sequence σ
$\hookrightarrow$	Continuously embedded
$\mathcal{F}$	Fourier transform
$\mathcal{J}^s$	Bessel operator of order s
$\mathcal{J}^\phi$	Generalized Bessel operator with $phi \in \mathcal{B}$
$f * g$	Convolution product of f and g
$\\|T\\|_{X,Y}$	Operator norm of $T : X \to Y$
K	K-functional
J	J-functional
$\mathrm{trace}(f)$	$\lim_{t\to 0^+} f(t)$
q'	Conjugate exponent of q
$\mathcal{C}_K(\phi; \mathbf{A})$	X is of class $\mathcal{C}_K(\phi; \mathbf{A})$ if $K(t, a) \leq C\phi(t)\\|a\\|_X$ for all $a \in X$
$\mathcal{C}_J(\phi; \mathbf{A})$	X is of class $\mathcal{C}_J(\phi; \mathbf{A})$ if $\phi(t)\\|a\\|_X \leq CJ(t, a)$ for all $a \in \Delta(\mathbf{A})$
$\mathcal{C}(\phi; \mathbf{A})$	X is of class $\mathcal{C}(\phi; \mathbf{A})$ if it is both of class $\mathcal{C}_K(\phi; \mathbf{A})$ and $\mathcal{C}_J(\phi; \mathbf{A})$
ω	Haar measure on $\mathbb{P}^n_+$
$\wedge$	Exterior product (for differential forms)
$\rangle \cdot \langle$	Linear span
$\mathcal{F}(\mathbf{A})$	Condition on $a \in \sigma(\mathbf{A})$ in order to have the Equivalence Theorem for $n > 1$
$Ef(t)$	Elasticity of f at $t > 0$
$h_f^{(p)}(x)$	p-exponent of f at x
G	Golden ratio $\frac{\sqrt{5}+1}{2}$
g	$G^{-1} = \frac{\sqrt{5}-1}{2}$
$\tau(x)$	Irrationality exponent of x
$\tau^{(\alpha)}(x)$	α-irrationality exponent of x
$[x]_\alpha$	α-floor of x
$[(a_0, \varepsilon_0), \ldots]$	Representation of x when the α-continued fraction algorithm continues indefinitely
$\mathcal{A}$	$\mathbb{N} \times \{\pm 1\}$
$\mathcal{A}^*$	Set of all finite words over $\mathcal{A}$
$\mathcal{L}_n(\alpha)$	Set of all admissible words of length n

$\mathcal{L}(\alpha)$	$\bigcup_{n\geq 1}\mathcal{L}_n(\alpha)$
$\delta_n(x)$	Distance from x to the edges of the cell $\mathfrak{c}_n$
s_n	Sign of the cell $\mathfrak{c}_n$
$(F_k)_{k\in\mathbb{N}}$	Fibonacci sequence
$f = O(g)$	Big - O notation, i.e. f is at most the same order as the function g
$(\mathcal{H})$	Hypothesis satisfies by function $(0,\infty)\to(0,\infty)$ concerning continuity and monotony
$D_f^{(p)}$	p-spectrum of f
dim	Hausdorff dimension
$d(n)$	Number of divisors of the natural number n

Functions

Let $\phi \in \mathcal{B}$,

χ_A	The indicator function of A
$\bar{\phi}$	$\bar{\phi}(t) = \sup_{s>0}\phi(ts)/\phi(s)$ for $t>0$
ϕ_*	$\phi_*(t) = t/\phi(t)$ for $t>0$
$\check{\phi}$	$\check{\phi}(t) := 1/\phi(1/\cdot)$ for $t>0$
ϕ^*	$\phi^*(t) = t\,\phi(\frac{1}{t})$ for $t>0$
ϕ_α	$\phi_\alpha(t) = \phi(t^\alpha)^{1/\alpha}$ for $t>0$
a^*	Non-increasing rearrangement of $\lvert a\rvert$ in $(0,\infty)$
$D\phi$	Derivative of ϕ
$f^{(m)}$	Derivative of order m of f in the sense of the distribution theory, where f is an A-valued function on $(0,\infty)$
B	Brjuno function
B_α	Generalized Brjuno functions based on the α-continued fractions
$\mathfrak{B}$	Brjuno function $B_{1/2}$
A_α	Generalized Gauss map
$\psi_{(a,\varepsilon)}$	Möbius transformations $J(a,\varepsilon)\to\mathfrak{c}(a,\varepsilon) : t\mapsto\frac{1}{a+\varepsilon t}$
$\psi_{(a_1,\varepsilon_1)\ldots(a_n,\varepsilon_n)}$	Möbius transformations $\psi_{(a_1,\varepsilon_1)}\circ\psi_{(a_2,\varepsilon_2)}\circ\cdots\circ\psi_{(a_n,\varepsilon_n)}$
Γ	Gamma function
Ψ_α	Antiderivative of B_α s.t. $\Psi_\alpha(0)=0$
sgn	Sign function
ω_α	Modulus of continuity of Ψ_α
$\psi_{a,b}$	$\psi((x-b)/a)$, where ψ is a wavelet, $a>0$ and $b\in\mathbb{R}$
$W_f(b,a)$	Integral wavelet transform of f
B_θ	Brjuno function with power θ
W_α	Generalized Wilton functions based on the α-continued fractions
$\mathfrak{W}$	Wilton function $W_{1/2}$
b	Brjuno function without logarithmic singularities

$b_{\theta,z}$	b with a power θ and an additional geometrical term z
T	Thomae's function
T_θ	Generalized Thomae's function with parameter $\theta \in (0,2]$
$\mathfrak{T}$	Takagi function
M	Minkowski Question Mark function
S	Transformation for Engel Series
B_E	Brjuno function linked to Engel Series

Spaces

Generally, if X_ϕ is a space depending on $\phi \in \mathcal{B}$, then X_α is X_ϕ with $\phi(t) = t^\alpha$ and X is X_ϕ with $\phi(t) = t$.
Let $A \subseteq \mathbb{R}^d$, $E \subseteq (0, \infty)$, X be a Banach space,

$\bar{A}$	Topological closure of A
A°	Interior of A
∂A	Boundary of A
X'	Dual of X
$X^{(\alpha)}$	X equipped with semi-norm $\\|\cdot\\|^\alpha$, with $\alpha > 0$
$C(A)$	Space of continuous function on A
$C_b(A)$	Space of bounded continuous function on A
$C^k(A)$	Space of function which have k-th derivatives that are continuous on A
$C_b^k(A)$	Space of bounded functions in $C^k(A)$
$C^\infty(A)$	Space of infinitely differentiable function on A
$\Lambda^\phi(A)$	Uniform Hölder space of exponent $\phi \in \mathcal{B}$ on A
$\Lambda^\phi(x_0)$	Pointwise Hölder space of exponent $\phi \in \mathcal{B}$ at $x_0 \in \mathbb{R}^d$
$C_b^\phi(A)$	Space of bounded functions in $\Lambda^\phi(A)$
$L^p(A)$	Usual Lebesgue space L^p on A
$L^p(X)$	Space of sequences of functions $(f_j)_j$ such that $\\|\\|(f_j)_j\\|_X\\|_{L^p} < \infty$, with X a normed space of sequences
$L_*^p(E)$	Lebesgue space L^p on E with measure dt/t
$T_u^p(x_0)$	Calderón-Zygmund space of exponent $u > -d/p$ at $x_0 \in \mathbb{R}^d$
$T_\phi^p(x_0)$	Generalized Calderón-Zygmund space with weight $\phi \in \mathcal{B}$ at $x_0 \in \mathbb{R}^d$
$\ell_\phi^q(X)$	Space of sequences $(a_j)_j$ of X with $(\phi(2^j)\\|a_j\\|_X)_j \in \ell^q$
$c_{0,\psi}(X)$	Subspace of $\ell_\psi^\infty(X)$ such that $\lim_j \psi(2^j)\\|a_j\\|_X = 0$
$\mathcal{D}$	Space of test functions
$\mathcal{S}$	Schwartz space of rapidly decreasing functions
$\mathcal{S}'$	Space of tempered distributions
$B_{p,q}^\phi$	Generalized Besov space
$b_{p,\infty}^\phi$	Subspace of $B_{p,\infty}^\phi$ such that $\lim_j \phi(2^j)\\|\varphi_j * a\\|_{L^p} = 0$
$F_{p,q}^\phi$	Generalized Hardy-Sobolev space

H_p^s	Fractional Sobolev space
H_p^ϕ	Generalized (fractional) Sobolev space
M^p	Space of Fourier multipliers on L^p
Λ_ϕ^p	Lorentz space
$\mathfrak{F}(\mathbf{A})$	Banach space linked to the Complex Interpolation
A_ϕ	Minimal exact interpolation space of exponent ϕ with respect to $\mathbf{A}$ that contains A under assumptions
$\sigma(\mathbf{A})$	Subspace of $\Sigma(\mathbf{A})$ for which $\int \frac{K(t,a)}{\max t} d\omega(t) < \infty$
$\mathbf{A}_q^{\phi_1,\ldots,\phi_n}$	Space $K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$ when $\mathcal{F}(\mathbf{A})$ is satisfied

Categories

$\mathscr{N}$	Category of normed vector spaces
$\mathscr{B}$	Sub-category of $\mathscr{N}$ of Banach spaces
$\mathscr{C}_c$	Category of compatible couples $\mathbf{A} = (A_0, A_1)$ of a sub-category $\mathscr{C}$ of $\mathscr{N}$
Σ	Functor from $\mathscr{C}_c$ to $\mathscr{C}$
Δ	Functor from $\mathscr{C}_c$ to $\mathscr{C}$
K_q^ϕ	Functor of the generalized K-method
J_q^ϕ	Functor of the generalized J-method
X_q^ϕ	Functor of the generalized "espaces de moyennes" method
$TX_{q,m}^\phi$	Functor of the generalized "espaces de trace" method
$K_\infty^{0,\phi}$	Functor of the generalized continuous interpolation method
$X_\infty^{0,\phi}$	Functor of the generalized continuous "espaces de moyennes" method
$J_\infty^{0,\phi}$	Functor of the generalized continuous J-method
C_θ	Functor of the usual complex interpolation method
$K_q^{\phi_1,\ldots,\phi_n}$	Functor of the generalized K-method for several spaces
$J_q^{\phi_1,\ldots,\phi_n}$	Functor of the generalized J-method for several spaces

Contents

Part I

Generalized Interpolation Methods

Chapter 1

Notions of weights

The main aim of this part is to extend interpolation methods to develop more refined regularity spaces. A crucial aspect of this generalization is the introduction of Boyd functions [17, 18, 53, 60, 88], which serve as a replacement for the traditional functions $t \mapsto t^\theta$ typically used in these settings: a function $\phi : (0,\infty) \to (0,\infty)$ is a Boyd function if it is continuous, $\phi(1) = 1$ and

$$\overline{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty, \quad \forall t \in (0,\infty).$$

The lower and upper Boyd indices of a Boyd function ϕ are then defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively. For example, in the context of Brownian motion, Boyd functions enable us to account for the impact of the iterated logarithm, a phenomenon that classical Hölder spaces fail to capture. In the first Section of the Chapter, we present the basic properties of the Boyd functions to show that they can be decomposed into two parts, leading to a representation Theorem:

A function $\phi : (0,\infty) \to (0,\infty)$ is a Boyd function if and only if $\phi(1) = 1$ and there exist four bounded continuous functions $\eta_0, \xi_0 : (0,1] \to (0,\infty)$ and $\eta_\infty, \xi_\infty : [1,\infty) \to (0,\infty)$ such that

$$\phi(t) = e^{\eta_0(t) + \int_1^{1/t} \xi_0(s) \frac{ds}{s}}, \quad t \in (0,1],$$

$$\phi(t) = e^{\eta_\infty(t) + \int_1^{t} \xi_\infty(s) \frac{ds}{s}}, \quad t \in [1,\infty).$$

Another way to generalize spaces is to use admissible sequences: a sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. One also associates Boyd indices to such a sequence. Let $\underline{\sigma}_j :=$

$\inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively. We give the essential properties of these sequences and make some original remarks before investigating the relations between the two notions, with a special attention to the Boyd indices. It is in fact well-known that there is a connection between Boyd functions and admissible sequences. Many authors illustrate this link with the following example [3, 25, 90]; given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \sigma_0 & \text{if } t \in (0,1), \end{cases}$$

with $\sigma_0 = 1$, is a Boyd function. However, we easily show that with such a construction, we necessarily have $\underline{b}(\phi) \leq 0 \leq \overline{b}(\phi)$, so that even in the simplest case where $\sigma_j = 2^{sj}$ with $s > 0$ ($j \in \mathbb{N}$), $\underline{b}(\phi) < \underline{s}(\sigma) = s$. In other words, the Boyd indices are not preserved with this construction. We therefore present some methods for constructing Boyd functions starting from a given admissible sequence that preserve the Boyd indices, for instance :

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \dfrac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}],\ j \in \mathbb{N}_0. \end{cases}$$

or

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \dfrac{1}{\phi(1/t)} & \text{if } t \in (0,1), \end{cases}$$

That being said, two admissible sequences yield four Boyd indices, which means that there is no natural way to associate an admissible sequence to a Boyd function. Roughly speaking, an admissible sequence defines a Boyd function either on $(0,1]$ or $[1,\infty)$. As a consequence, the admissible sequences are best suited for pointwise spaces since only asymptotic conditions are usually needed, while the Boyd functions are more adapted to uniform spaces on open sets.

Next, we propose some smoother constructions of Boyd functions starting from an admissible sequence. By doing so, we are led to improve a result of Merucci in [88], where ξ is only differentiable:

If ϕ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in C^\infty(0,\infty)$, with

$$0 < \inf_{t>0} t \frac{|D\xi(t)|}{\xi(t)} \leq \sup_{t>0} t \frac{|D\xi(t)|}{\xi(t)} < \infty, \tag{1.1}$$

such that ξ is equivalent to ϕ, i.e., there exists a constant $C > 0$ such that $C^{-1}\xi \leq \phi \leq C\xi$. This final result is significant since Condition (1.1) is frequently utilized for substitutions of the form $s = f(t)$ in integrals with the measure dt/t.

In the second Section, we present some examples of Banach spaces, such as Lebesgue, Hölder, Besov, Sobolev or Calderón and Zygmund spaces, that are constructed using Boyd functions and we prove some results linked to those spaces which will be useful later.

The results established in this Chapter were published in [67].

1.1 Boyd functions and admissible sequences

1.1.1 Boyd functions

Boyd indices were first introduced by W. Matuszewska in [87] (see also Chapter 2 of [16] and Remark 1.1.15 to see connections between Boyd functions and O-regularly varying functions of [16]) in a general case and were then linked with Boyd functions in [17] where they investigate conditions under which the Hilbert transform defines a bounded linear operator from a given function space into itself. It turned out that this context of Boyd's functions was ideal for replacing the function $t \mapsto t^\theta$ appearing in certain functional spaces.

Definition 1.1.1. A function $\phi : (0, \infty) \to (0, \infty)$ is a Boyd function if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0, \infty)$. The set of Boyd functions is denoted by $\mathcal{B}$.

In this context, the so-called Boyd indices have many applications (see, for example, [17, 18, 53, 60, 88]).

Definition 1.1.2. The lower and upper Boyd indices of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

From the definition, it is easy to check that for a Boyd function ϕ, the indices $\underline{b}(\phi)$ and $\overline{b}(\phi)$ are two numbers such that $\underline{b}(\phi) \leq \overline{b}(\phi)$. Moreover, given $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that

$$C^{-1} t^{\overline{b}(\phi)+\varepsilon} \leq \phi(t) \leq C t^{\underline{b}(\phi)-\varepsilon}, \tag{1.2}$$

for any $t \leq R$. In the same way, we also have

$$C^{-1} t^{\underline{b}(\phi)-\varepsilon} \leq \phi(t) \leq C t^{\overline{b}(\phi)+\varepsilon}, \tag{1.3}$$

for any $t \geq R$. We have related inequalities for $\bar{\phi}$, that is $\bar{\phi}(t) \leq Ct^{\underline{b}(\phi)-\varepsilon}$ for $t \leq R$ and $\bar{\phi}(t) \leq Ct^{\overline{b}(\phi)+\varepsilon}$ for $t \geq R$.
Let us give some classical examples of Boyd functions.

Example 1.1.3. Let ψ be a continuous slowly varying function on $(0, \infty)$:

$$\lim_{t\to 0} \frac{\psi(ts)}{\psi(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{\psi(ts)}{\psi(t)} = 1$$

for any $s > 0$. For $\theta \in \mathbb{R}$, the function $t \mapsto t^\theta \psi(t)/\psi(1)$ is a Boyd function such that $\underline{b}(\phi) = \overline{b}(\phi) = \theta$ [61]. Such functions are known as Karamata regularly varying functions [16]. A standard choice for the slowly varying function is $\psi = (|\ln| + 1)^\gamma$, for $\gamma > 0$.

Such functions naturally appear when dealing with the law of the iterated logarithm [61], but logarithmic corrections are also commonly needed in interpolation theory [28]. One can deal with even more iterated logarithms [29]: set

$$L_0(t) = t, \quad L_1(t) = 1 + |\log t| \quad \text{and} \quad L_m(t) = 1 + |\log(L_{m-1}(t))| \quad \text{for} \quad m > 1.$$

Then, if $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{R}^{n+1}$, $\phi_\alpha : (0, \infty) \to (0, \infty)$, defined by

$$\phi_\alpha(t) = \prod_{j=0}^{n} L_j(t)^{\alpha_j},$$

is a Boyd function such that $\underline{b}(\phi) = \overline{b}(\phi) = \alpha_0$.

Example 1.1.4. Another example is given by $\phi : t \mapsto \exp(|\log t|^\alpha)$, $\alpha \in (0, 1]$, which has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function [96].

The following three examples will be useful in Section 3.4.

Example 1.1.5. Let $\phi \in \mathcal{B}$ and $p \in [1, \infty[$ such that

$$\int_0^1 \phi(s)^p \frac{ds}{s} < \infty,$$

then ψ defined by

$$\psi(t) = \int_0^t \phi(s)^p \frac{ds}{s}$$

for all $t > 0$ is a Boyd function since, for all $t > 0$, using the change of variables $u = vt$, one has

$$\bar{\psi}(t) = \sup_{s>0} \frac{\int_0^{ts} \phi(u)^p \frac{du}{u}}{\int_0^s \phi(l)^p \frac{dl}{l}} \leq \bar{\phi}(t)^p.$$

Example 1.1.6. Let $\phi \in \mathcal{B}$ and $p \in [1, \infty[$ such that

$$\int_1^\infty \phi(s)^p \frac{ds}{s} < \infty,$$

then ψ defined by

$$\psi(t) = \int_t^\infty \phi(s)^p \frac{ds}{s}$$

for all $t > 0$ is a Boyd function since we obtain once again $\bar{\psi} \leq \bar{\phi}^p$.

Example 1.1.7. Let $\phi \in \mathcal{B}$, then ψ defined by

$$\psi(t) = \frac{1}{t} \int_0^t \frac{1}{\phi(s)} ds$$

for all $t > 0$ is a Boyd function since, in this case, $\bar{\psi} \leq \overline{\frac{1}{\phi}}$.

For the remainder of this work, we set $I = (0, \infty)$.

Definition 1.1.8. Two functions $f : I \to I$ and $g : I \to I$ are said to be equivalent if there exists a constant $C > 0$ such that $C^{-1} g \leq f \leq Cg$ on I. If f and g satisfy such a relation, we will write $f \asymp g$. Since sequences can be seen as functions, we will also use this notion of equivalence for sequences.

If ϕ_1 and ϕ_2 are two Boyd functions such that $\phi_1 \asymp \phi_2$, then $\underline{b}(\phi_1) = \underline{b}(\phi_2)$ and $\overline{b}(\phi_1) = \overline{b}(\phi_2)$. Let $\mathcal{B}'$ denote the set of functions $f : I \to I$ that belong to $C^1(I)$ with $f(1) = 1$ and satisfy

$$0 < \inf_{t>0} t \frac{|Df(t)|}{f(t)} \leq \sup_{t>0} t \frac{|Df(t)|}{f(t)} < \infty. \tag{1.4}$$

One can show that $\mathcal{B}'$ is a subset of $\mathcal{B}$. If $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ (resp. $\overline{b}(\phi) < 0$), then there exists a non-decreasing bijection (resp. a non-increasing bijection) $\psi \in \mathcal{B}'$ such that $\phi \asymp \psi$ and $\psi^{-1} \in \mathcal{B}'$ (see also Proposition 1.1.29). Such a result is usually used for change-of-variable techniques (see, for example, [21, 88]).

Remark 1.1.9. The quantity $t\frac{|Df(t)|}{f(t)}$ appearing in (1.4) is called the elasticity or point elasticity of f at $t > 0$. It is denoted by $Ef(t)$ and has application in economy for example ([109]).

Let us give an example of a Boyd function that we will use in the sequel.

Example 1.1.10. Given $r, s \in \mathbb{R}$, let $\phi_{r,s} : I \to I$ be defined by

$$\phi_{r,s}(t) = \begin{cases} t^r & \text{if } t \leq 1, \\ t^s & \text{if } t \geq 1. \end{cases} \tag{1.5}$$

Let us suppose that $r \leq s$ (the reasoning is similar in the other case). Given $t > 0$, for $u \geq 1$, we have $\phi(tu)/\phi(u) = t^s$ if $tu \geq 1$ and $\phi(tu)/\phi(u) = t^r u^{r-s}$ otherwise. Now, if $u \in (0,1)$, we have $\phi(tu)/\phi(u) = t^r$ if $tu \leq 1$ and $\phi(tu)/\phi(u) = t^s u^{s-r}$ otherwise. As a consequence, $\bar{\phi}(t) \leq \max\{t^r, t^s\}$ and $\phi_{r,s}$ is a Boyd function. Moreover, we have $t^s \leq \bar{\phi}(t) \leq \max\{t^r, t^s\} = t^s$ for $t \geq 1$ and $t^r \leq \bar{\phi}(t) \leq \max\{t^r, t^s\} = t^r$ for $t \leq 1$, which implies $\underline{b}(\phi_{r,s}) = r$ and $\overline{b}(\phi_{r,s}) = s$. Of course, $\phi_{r,s}$ verifies inequalities of the form (1.4):

$$0 < \inf_{t \in (0,\infty)\setminus\{1\}} t \frac{|D\phi_{r,s}(t)|}{\phi_{r,s}(t)} \leq \sup_{t \in (0,\infty)\setminus\{1\}} t \frac{|D\phi_{r,s}(t)|}{\phi_{r,s}(t)} < \infty.$$

Here, we associate $\mathcal{B}$ to the space $\mathcal{B}^\infty \times \mathcal{B}^\infty$ (see Definition 1.1.11) to obtain a representation Theorem similar to the ones presented in [16, 64]. From a conceptual point of view, this decomposition of the Boyd functions into a cartesian product will allow us to consider that an element of $\mathcal{B}$ is defined by two germs, each being associated to an element of $\mathcal{B}^\infty$.

Definition 1.1.11. We will denote by $\mathcal{B}^\infty$ the set of continuous functions $\phi : [1,\infty) \to I$ such that $\phi(1) = 1$ and

$$0 < \underline{\phi}(t) := \inf_{s \geq 1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s \geq 1} \frac{\phi(ts)}{\phi(s)} < \infty, \tag{1.6}$$

for any $t \geq 1$.

Let us adopt the following conventions, by analogy with the admissible sequences. For $\phi \in \mathcal{B}^\infty$, we set

$$\underline{s}(\phi) := \lim_{t \to \infty} \frac{\log \underline{\phi}(t)}{\log t}$$

and

$$\overline{s}(\phi) := \lim_{t \to \infty} \frac{\log \bar{\phi}(t)}{\log t}.$$

If $\phi \in \mathcal{B}^\infty$, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C^{-1} t^{(\underline{s}(\phi) - \varepsilon)} \leq \underline{\phi}(t) \leq \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) \leq C t^{\overline{s}(\phi) + \varepsilon}, \tag{1.7}$$

for any $t, s \geq 1$. It is easy to check that we have $\phi \in \mathcal{B}^\infty$ if and only if $\psi : t \mapsto \phi(1/t)$ is a continuous function defined on $(0,1]$ such that $\psi(1) = 1$ and

$$0 < \inf_{s \leq 1} \frac{\psi(ts)}{\psi(s)} \leq \sup_{s \leq 1} \frac{\psi(ts)}{\psi(s)} < \infty.$$

For a short period of time, we will designate the space of such functions ψ by $\mathcal{B}^0$. In some way, for $\phi \in \mathcal{B}$, we will relate $\mathcal{B}^\infty$ to the sequences $(\phi(2^j))_{j \in \mathbb{N}}$ and $\mathcal{B}^0$ to $(\phi(2^{-j}))_{j \in \mathbb{N}}$.

Given $\phi \in \mathcal{B}$, we denote by ϕ_∞ the restriction of ϕ to $[1,\infty)$ and by ϕ_0 the restriction of ϕ to $(0,1]$. Of course, we have $\phi_\infty \in \mathcal{B}^\infty$ and $\phi_0 \in \mathcal{B}^0$. On the other hand, the converse is also true; more precisely, we have the following result.

Proposition 1.1.12. *The application*

$$\tau : \mathcal{B} \to \mathcal{B}^\infty \times \mathcal{B}^\infty \quad \phi \mapsto (t \mapsto \frac{1}{\phi_0(1/t)}, \phi_\infty)$$

is a bijection.

Proof. Let us consider the application that maps $(\phi_1, \phi_2) \in \mathcal{B}^\infty \times \mathcal{B}^\infty$ to

$$t \mapsto \phi(t) := \begin{cases} 1/\phi_1(1/t) & \text{if } t \in (0,1], \\ \phi_2(t) & \text{if } t \in [1,\infty). \end{cases}$$

We have to show that ϕ belongs to $\mathcal{B}$. We need to adjust the proof depending on the order between $\underline{s}(\phi_1)$ and $\underline{s}(\phi_2)$ on one hand and $\overline{s}(\phi_2)$ and $\overline{s}(\phi_1)$ on the other hand. This thus leads to four possibilities, which are very similar to each other. Let us handle the case where

$$\underline{s}(\phi_1) \leq \underline{s}(\phi_2) \leq \overline{s}(\phi_2) \leq \overline{s}(\phi_1).$$

The idea is to use (1.7). Let $t > 0$; we want to show that

$$\overline{\phi}(t) = \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty.$$

If we have $s \geq 1$ and $ts \geq 1$, then $\phi(ts)/\phi(s) \leq \overline{\phi_2}(t)$. If $s \leq 1$ and $ts \leq 1$, we have

$$\frac{\phi(ts)}{\phi(s)} = \Big(\frac{\phi_1(\frac{1}{ts})}{\phi_1(1/s)}\Big)^{-1} \leq 1/\underline{\phi_1}(1/t).$$

For $s \geq 1$ and $ts \leq 1$ we get

$$\frac{\phi(ts)}{\phi(s)} = \big(\phi_1(\frac{1}{ts})\, \phi_2(s)\big)^{-1} \leq C \frac{(ts)^{\underline{s}(\phi_1)-\varepsilon}}{s^{\underline{s}(\phi_2)-\varepsilon}} \leq C t^{\underline{s}(\phi_1)-\varepsilon},$$

for some $\varepsilon > 0$. Finally, if $s \leq 1$ and $ts \geq 1$, we can write

$$\frac{\phi(ts)}{\phi(s)} = \phi_2(ts)\, \phi_1(1/s) \leq C \frac{(ts)^{\overline{s}(\phi_2)+\varepsilon}}{s^{\overline{s}(\phi_1)+\varepsilon}} \leq C t^{\overline{s}(\phi_1)+\varepsilon},$$

for some $\varepsilon > 0$. The remaining cases can be treated in the same way. □

To obtain a representation Theorem for the Boyd functions, we first need a representation Theorem for the elements of $\mathcal{B}^\infty$.

Theorem 1.1.13. *A function $\phi : [1,\infty) \to I$ belongs to $\mathcal{B}^\infty$ if and only if $\phi(1) = 1$ and there exist two bounded continuous functions $\eta, \xi : [1,\infty) \to I$ such that*

$$\phi(t) = e^{\eta(t) + \int_1^t \xi(s) \frac{ds}{s}}. \tag{1.8}$$

Proof. To prove this result, we rely on what was done in [16] for closely related function classes (see Remark 1.1.15). Suppose that $\phi \in \mathcal{B}^\infty$ and define $H(t) := \log \phi(t)$ for $t \geq 1$. Let us set $\psi = \tau^{-1}(\phi, \phi)$, where τ is defined in Proposition 1.1.12. There exists a constant C such that

$$H(t) = C + \int_t^{2t} \frac{H(t) - H(s)}{\log 2} \frac{ds}{s} + \int_1^t \frac{H(2s) - H(s)}{\log 2} \frac{ds}{s}.$$

Now, set

$$\eta(t) := C + \int_t^{2t} \frac{H(t) - H(s)}{\log 2} \frac{ds}{s} \quad \text{and} \quad \xi(s) := \frac{H(2s) - H(s)}{\log 2}.$$

Obviously, η and ξ are continuous functions. Let us show that η and ξ are bounded. For all $s \geq 1$, one has

$$\xi(s) = \frac{H(2s) - H(s)}{\log 2} = \frac{\log\big(\phi(2s)/\phi(s)\big)}{\log 2},$$

so that

$$\frac{\log \underline{\phi}(2)}{\log 2} \leq \xi(s) \leq \frac{\log \overline{\phi}(2)}{\log 2}.$$

For all $t \geq 1$, by using the change of variables $s = tu$, one gets

$$\begin{aligned}
\int_t^{2t} \frac{H(t) - H(s)}{\log 2} \frac{ds}{s} &= \int_t^{2t} \frac{\log\big(\phi(t)/\phi(s)\big)}{\log 2} \frac{ds}{s} \\
&= \int_t^{2t} \frac{\log\big(\psi(t)/\psi(s)\big)}{\log 2} \frac{ds}{s} \\
&\leq \int_t^{2t} \frac{\log\big(\overline{\psi}(t/s)\big)}{\log 2} \frac{ds}{s} \leq C
\end{aligned}$$

and

$$\begin{aligned}
\int_t^{2t} \frac{H(t) - H(s)}{\log 2} \frac{ds}{s} &= \int_t^{2t} \frac{\log\big(\psi(t)/\psi(s)\big)}{\log 2} \frac{ds}{s} \\
&\geq \int_t^{2t} \frac{\log\big(1/\overline{\psi}(s/t)\big)}{\log 2} \frac{ds}{s} \geq C.
\end{aligned}$$

Now suppose that $\phi(1) = 1$ and that there exist two bounded continuous functions $\eta, \xi : [1, \infty) \to I$ such that

$$\phi(t) = e^{\eta(t) + \int_1^t \xi(s) \frac{ds}{s}}.$$

Then ϕ is continuous and if $t, s \geq 1$, we have

$$\frac{\phi(ts)}{\phi(s)} = e^{\eta(ts) - \eta(s) + \int_s^{ts} \xi(u) \frac{du}{u}},$$

so that ϕ belongs to $\mathcal{B}^\infty$. □

In [64], they derive the Boyd indices from the functions ξ in Theorem 1.1.13:

Proposition 1.1.14. *Let $\phi \in \mathcal{B}^\infty$, then*

$$\underline{b}(\phi) = \inf \sup_{t\geq 1} \xi(t),$$

the infimum being taken on all representations (1.8) *of ϕ and*

$$\overline{b}(\phi) = \sup \inf_{t\geq 1} \xi(t),$$

the supremum being taken on all representations (1.8) *of ϕ.*

Remark 1.1.15. This problem is partially addressed in [16, 64]. If we ask η and ξ to be measurable instead of continuous in Theorem 1.1.13 (the proof can be easily adapted), the spaces $\mathfrak{B}^\infty$ of functions ϕ satisfying (1.6) lies inbetween two spaces for which we have similar representation Theorems. Let us denote by SV the set of slowly varying functions [16], for which one requires $\lim_{t\to\infty} \xi(t) = 0$ and $\lim_{t\to\infty} \eta(t) = C$ for a constant C and by OR the set of the functions ϕ for which one only requires η and ξ to be bounded on $[t_0, \infty)$ for some $t_0 \geq 1$ in Theorem 1.1.13. We obviously have $SV \subsetneq \mathcal{B}^\infty \subsetneq OR$.

Corollary 1.1.16. *A function $\phi : I \to I$ belongs to $\mathcal{B}$ if and only if $\phi(1) = 1$ and there exist four bounded continuous functions $\eta_0, \xi_0 : (0,1] \to I$ and $\eta_\infty, \xi_\infty : [1,\infty) \to I$ such that*

$$\phi(t) = \begin{cases} e^{\eta_0(t) + \int_1^{1/t} \xi_0(s) \frac{ds}{s}} & \text{if } t \in (0,1], \\ e^{\eta_\infty(t) + \int_1^{t} \xi_\infty(s) \frac{ds}{s}} & \text{if } t \in [1,\infty). \end{cases}$$

1.1.2 Admissible sequences

Another approach for providing generalized spaces is proposed in [37]; it relies on the so-called admissible sequences.

Definition 1.1.17. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j.

One also associates Boyd indices to such a sequence.

Definition 1.1.18. Given an admissible sequence σ, let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively.

Concerning the admissible sequences, we have inequalities similar to (1.7): if σ is an admissible sequence, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C^{-1}2^{(\underline{s}(\sigma)-\varepsilon)j} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C2^{(\overline{s}(\sigma)+\varepsilon)j}, \tag{1.9}$$

for any $j, k \in \mathbb{N}$.

Remark 1.1.19. The previous inequalities are not valid for $\varepsilon = 0$. For example, if we set $\sigma_j = 2^{js}|\log 2^{-j}|$, the corresponding admissible sequence σ is such that $\overline{\sigma}_j = 2^{js}(1+j)$ and $\overline{s}(\sigma) = s$. Therefore, for any ε, there exists a constant $C > 0$ such that $\overline{\sigma}_j \leq C2^{(s+\varepsilon)j}$, but we cannot have $\overline{\sigma}_j \leq C2^{sj}$.

The following example is taken from [64] and is due to [54]. It shows that an admissible sequence does not necessarily have a fixed main order and their upper and lower Boyd indices do not necessarily coincide as remarked in [90]. Thus, it generalizes the slowly varying functions of Karamata [16].

Example 1.1.20. Consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define the admissible sequence σ by

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1}, \\ 2^{j_{2n}} 4^{j-j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

The sequence oscillates between $(j)_j$ and $(2^j)_j$ and we have $\underline{s}(\sigma) = 0$ and $\overline{s}(\sigma) = 1$.

From the inequalities (1.9), we know that an admissible sequence σ is oscillating between $C^{-1}2^{(\underline{s}(\sigma)-\varepsilon)j}$ and $C2^{(\overline{s}(\sigma)+\varepsilon)j}$. One can wonder if the sequence actually oscillates between these 2 quantities or if it can actually vary between "smaller quantities". In fact, it can oscillate between any value, as the following example shows.

Example 1.1.21. Let $\sigma_0 = 1$, $\alpha > 0$ and σ be defined by

$$\sigma_{j+1} := \begin{cases} \sigma_j & \text{if } j_{2n} \leq j \leq j_{2n+1}, \\ \sigma_j 2^{\alpha} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

We have $\underline{s}(\sigma) = 0$, $\overline{s}(\sigma) = 1$ and for all $\varepsilon > 0$, there exists $C > 0$ such that $\sigma_j \leq C\, 2^{j\varepsilon}$ for all j.

Remark 1.1.22. There exist alternative definitions of the Boyd indices. For example, in [20], the following quantities are proposed:

$$\underline{\mathfrak{s}}(\sigma) = \liminf_j \frac{\log \sigma_j}{\log 2^j} \quad \text{and} \quad \overline{\mathfrak{s}}(\sigma) = \limsup_j \frac{\log \sigma_j}{\log 2^j}.$$

While presenting the advantage of being simpler to consider than Definition 1.1.18, since one does not deal with the sequences $(\underline{\sigma}_j)_{j\in\mathbb{N}}$ and $(\overline{\sigma}_j)_{j\in\mathbb{N}}$, these definitions are less precise concerning the subtle behavior of σ. For example, if we consider the sequence given in Example 1.1.21, we directly get $\underline{\mathfrak{s}}(\sigma) = \overline{\mathfrak{s}}(\sigma) = 0$, which only reveals that $\sigma = o((2^j)_{j\in\mathbb{N}})$. To address this problem, other indices must be considered beside $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$ and they usually involve the quantities $\underline{\sigma}_j$ and $\overline{\sigma}_j$. For example, in [20], the quantities $\liminf_j \sigma_{j+1}/\sigma_j$ and $\limsup_j \sigma_{j+1}/\sigma_j$ are also taken into account.

Of course, $\overline{s}(\sigma) < 0$ implies $\sigma_j \to 0$ and $\underline{s}(\sigma) > 0$ implies $\sigma_j \to \infty$. It is easy to check that if ϕ is a Boyd function then $(\phi(2^j))_{j\in\mathbb{N}}$ and $(\phi(2^{-j}))_{j\in\mathbb{N}}$ are both admissible sequences. On the other hand, from an admissible sequence $(\sigma_j)_{j\in\mathbb{N}}$ and up to a normalizing constant, one can define a function $\phi \in \mathcal{B}^\infty$ such that $\phi(2^j) = \sigma_j$ for all $j \in \mathbb{N}$. As a consequence, Theorem 1.1.13 implies the following result.

Corollary 1.1.23. *If σ is an admissible sequence, then there exist two bounded continuous functions $\eta, \xi : [1, \infty) \to I$ such that*

$$\sigma_j = e^{\eta(2^j) + \int_1^{2^j} \xi(s) \frac{ds}{s}},$$

for all $j \in \mathbb{N}$.

1.1.3 Relation between the two concepts

The analogy between Definition 1.1.2 and Definition 1.1.18 is clear by remarking that we have, for a Boyd function ϕ,

$$\underline{b}(\phi) = \lim_{t\to\infty} \frac{\log \underline{\phi}(t)}{\log t},$$

where we have set

$$\underline{\phi}(t) := \inf_{s>0} \frac{\phi(st)}{\phi(s)}.$$

For example, as expected, the K-method of interpolation can be generalized using admissible sequences in order to obtain an interpolation method similar to the one using Boyd functions [77]. Of course, these techniques (relying on either the Boyd functions or the admissible sequences) are not limited to the Besov spaces; they have been applied to the Triebel-Lizorkin spaces, Lorentz spaces, the pointwise Hölder spaces and the T_u^p spaces of Calderón-Zygmund among others [27, 37, 44, 63, 78, 88].

It is well known that there is a connection between Boyd functions and admissible sequences. Many authors illustrate this link with the following example [3, 25, 90]; given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \sigma_0 & \text{if } t \in (0, 1), \end{cases} \tag{1.10}$$

with $\sigma_0 = 1$, is a Boyd function. For instance, the generalized Besov space associated to ϕ_σ is the generalized Besov space associated to σ [3]. However, we easily show that with such a construction, we necessarily have $\underline{b}(\phi) \leq 0 \leq \overline{b}(\phi)$, so that even in the simplest case where $\sigma_j = 2^{sj}$ with $s > 0$ ($j \in \mathbb{N}$), $\underline{b}(\phi) < \underline{s}(\sigma) = s$. In other words, the Boyd indices are not preserved with this construction. Of course, with this sequence, $\phi_\sigma(t)$ is comparable to the function ϕ such that $\phi(t) = t^s$ for $t \geq 1$ and $\phi(t) = 1$ for $t < 1$ (more rigorously, they are equivalent; see Definition 1.1.8). Starting from the fundamental Boyd function $t \mapsto t^s$, one can impose the supplementary condition $1/\phi(t) = \phi(1/t)$ to the Boyd functions in Definition 1.1.1. This is done in [21], where the authors use the construction (1.10) for $t \geq 1$ but impose $\phi(t) = 1/\phi(1/t)$ for $t \in (0, 1)$. To quote their own words, "somehow unexpectedly, it turns out that the lower and upper Boyd indices of any such interpolating function do coincide with the corresponding indices of the starting sequence".

In this Section, we investigate the relations between the Boyd functions and the admissible sequences. By doing so, we underline the fundamental differences between Definition 1.1.2 and Definition 1.1.18. We showed that a Boyd function can be identified to a couple of germs, while a germ can be associated to an admissible sequence. That being said, two admissible sequences yield four Boyd indices, which means that there is no natural way to associate an admissible sequence to a Boyd function. Roughly speaking, an admissible sequence defines a Boyd function either on $(0, 1]$ or $[1, \infty)$. As a consequence, the admissible sequences are best suited for pointwise spaces since only asymptotic conditions are usually needed, while the Boyd functions are more adapted to uniform spaces on open sets. In particular, there is a subtle difference in the generalized interpolation theories in [77] since the method relying on the admissible sequences corresponds to the method based on the Boyd functions for a specific class of functions only.

Proposition 1.1.24. *If $\phi \in \mathcal{B}$ and $\sigma_j = \phi(2^j)$ or $\sigma_j = 1/\phi(2^{-j})$ then we have $\underline{b}(\phi) \leq \underline{s}(\sigma) \leq \overline{s}(\sigma) \leq \overline{b}(\phi)$.*

Proof. For any $j \in \mathbb{N}$, let $\sigma_j = \phi(2^j)$; we have

$$\overline{\sigma}_j \leq \sup_{k \geq 1} \frac{\phi(2^j 2^k)}{\phi(2^k)} \leq \sup_{s > 0} \frac{\phi(2^j s)}{\phi(s)} = \bar{\phi}(2^j)$$

and so $\overline{s}(\sigma) \leq \lim_j \log \bar{\phi}(2^j)/\log 2^j \leq \overline{b}(\phi)$. From there, we directly get $\underline{b}(\phi) = -\overline{b}(1/\phi) \leq -\overline{s}(1/\sigma) = \underline{s}(\sigma)$. For the other case, it suffices to consider $\psi \in \mathcal{B}$ defined by $\psi(t) = 1/\phi(1/t)$. □

As a consequence, if $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then $\underline{b}(\phi) = \underline{s}(\sigma)$ and $\overline{b}(\phi) = \overline{s}(\sigma)$ imply $\underline{s}(\sigma) \leq \underline{s}(\theta)$ and $\overline{s}(\theta) \leq \overline{s}(\sigma)$. In fact, we have some kind of equivalence, as stated by the following result.

Proposition 1.1.25. *If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$, then*

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Proof. Let $\varepsilon > 0$, $j \in \mathbb{N}$ and suppose that $\overline{s}(\sigma) \geq \overline{s}(\theta)$ (what follows is similar for the converse inequality). If t belongs to $[2^n, 2^{n+1})$ for some $n \in \mathbb{N}_0$, we have

$$\frac{\phi(2^j t)}{\phi(t)} \leq \phi(2^j 2^n)\bar{\phi}(t2^{-n})\frac{\bar{\phi}(2^n/t)}{\phi(2^n)} \leq C\frac{\sigma_{j+n}}{\sigma_n} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

Now, if t belongs to $(2^{-n-1}, 2^{-n}]$ for some $n \in \mathbb{N}_0$, since

$$\frac{\phi(2^j t)}{\phi(t)} \leq \phi(2^j 2^{-n})\bar{\phi}(t2^n)\frac{\bar{\phi}(2^{-n}/t)}{\phi(2^{-n})},$$

if $j \geq n$, we can write

$$\frac{\phi(2^j t)}{\phi(t)} \leq C\sigma_{j-n}/\phi(2^{-n}) \leq C2^{(\overline{s}(\sigma)+\varepsilon)(j-n)}2^{(\overline{s}(\theta)+\varepsilon)n} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

On the other hand, for $j \leq n$, we have

$$\frac{\phi(2^j t)}{\phi(t)} \leq C\frac{1/\phi(2^{-n})}{1/\phi(2^{-(n-j)})} \leq C2^{(\overline{s}(\theta)+\varepsilon)j} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

In any case, we get

$$\overline{b}(\phi) \leq \lim_j \frac{\log C}{\log 2^j} + \frac{\log 2^{j(\overline{s}(\sigma)+\varepsilon)}}{\log 2^j} = \overline{s}(\sigma) + \varepsilon.$$

Using Proposition 1.1.24, we get $\overline{b}(\phi) = \overline{s}(\sigma)$. If we now assume that $\underline{s}(\sigma) \leq \underline{s}(\theta)$, we can obtain $\underline{b}(\theta) = \underline{s}(\sigma)$ in the same way, which ends the proof. □

We can now bind the indices of a Boyd function to the indices of its "components" in $\mathcal{B}^\infty$. Let τ be the mapping defined in Proposition 1.1.12 and denote by τ_1 and τ_2 its two components in $\mathcal{B}^\infty$.

Corollary 1.1.26. *If ϕ belongs to $\mathcal{B}$, then we have $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ and $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.*

1.1.4 Some elementary examples of a Boyd function obtained from one admissible sequence

Let us give some methods for constructing Boyd functions starting from a given admissible sequence that preserve the Boyd indices.

Given an admissible sequence σ, we are naturally led to define a function $\phi_\sigma \in \mathcal{B}$ such that $\phi_\sigma(2^j) = \sigma_j$ and $\phi_\sigma(2^{-j}) = 1/\sigma_j$ for $j \in \mathbb{N}_0$ with $\sigma_0 = 1$ in order to preserve the Boyd indices. If, in the definition of ϕ_σ, we connect the elements σ_j on $[1, \infty)$ and $1/\sigma_j$ on $(0, 1]$ with straight lines, we obtain

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \dfrac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}],\ j \in \mathbb{N}_0. \end{cases}$$

We obviously have $\phi_\sigma \in \mathcal{B}$: since σ is an admissible sequence, we have $\bar{\phi}_\sigma(t) \le C(t)$.

Another possibility, consists in using the construction proposed in [21], that is

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \\ \dfrac{1}{\phi(1/t)} & \text{if } t \in (0, 1), \end{cases}$$

and it is now clear that the Boyd indices of σ and this function ϕ_σ do coincide.

Other constructions can be obtained by setting $\phi_\sigma(2^j) = \sigma_j$ for $j \in \mathbb{N}$ and $\phi_\sigma(t) = t^s$ for $t \in (0, 1]$, where s satisfies $\underline{s}(\sigma) \le s \le \overline{s}(\sigma)$. For example, we can define

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ t^s & \text{if } t \in (0, 1). \end{cases} \tag{1.11}$$

On the other hand, if s is chosen so that $s < \underline{s}(\sigma)$ (resp. so that $s > \overline{s}(\sigma)$) in (1.11), then we have $\underline{b}(\phi_\sigma) = s$ (resp. $\overline{b}(\phi_\sigma) = s$). This is what happens in the construction (1.10) with $s = 0$.

Given an admissible sequence σ, it can be useful to derive a regular function $\phi \in \mathcal{B}$ such that $\phi(2^j) = \sigma_j$ for all $j \in \mathbb{N}$ and

$$0 < \inf_{t>1} E_\phi(t) \quad \text{or} \quad \sup_{t>1} E_\phi(t) < \infty \tag{1.12}$$

(see [88] for example). This can be done using smooth transition functions. For example, let

$$f(t) = \begin{cases} t^2 & \text{if } t \ge 0, \\ 0 & \text{else,} \end{cases}$$

to define

$$g : t \mapsto \frac{f(t)}{f(t) + f(1-t)} \chi_{[0,1]}(t) + \chi_{(1,\infty)}(t). \tag{1.13}$$

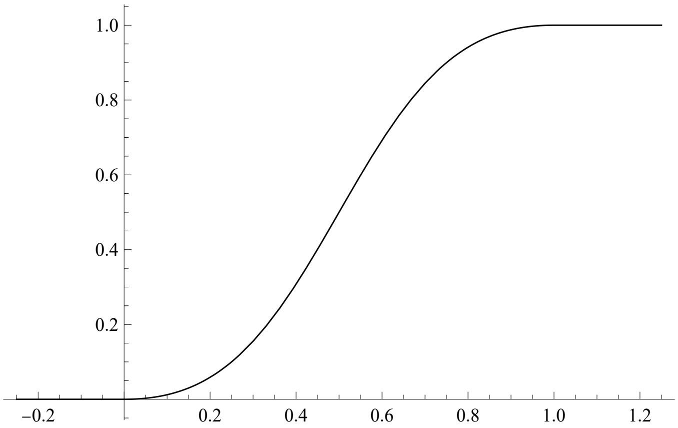


Figure 1.1: The function g defined by (1.13).

We will use this function to connect the points $(2^j, \sigma_j)$ and $(2^{j+1}, \sigma_{j+1})$ $(j \in \mathbb{N})$. Remark that one can replace t^2 by $e^{-1/t}, t^n$ or $\cosh(t)$ for example in the definition of f. However, with such a construction, the resulting Boyd function has a vanishing derivative at 2^j for every $j \in \mathbb{N}$. If σ is strictly increasing, we can avoid this pitfall by applying a rotation; the angle α must be chosen sufficiently small so that the resulting curve can be associated to a function. This threshold obviously depends on both the function f and the admissible sequence σ. Let us make this construction explicitly. We will work with the axes (t', s') such that

$$\begin{pmatrix} t' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}.$$

For $j \in \mathbb{N}$, we set

$$\begin{cases} t'_j = 2^j \cos\alpha + \sigma_j \sin\alpha, \\ s'_j = -2^j \sin\alpha + \sigma_j \cos\alpha, \end{cases}$$

$$\xi^{(j)}(t') = \frac{t' - t'_j}{t'_{j+1} - t'_j}$$

and

$$\tau^{(j)}(t') = s'_j + (s'_{j+1} - s'_j)t'$$

to consider the curve

$$s' = \tau^{(j)}(g(\xi^{(j)}(t')))$$

on $[t'_j, t'_{j+1}]$, which gives rise to

$$s'(s) = \tau^{(j)}(g(\xi^{(j)}(t'(t))))$$

on the original Euclidean plane. Explicitly, we have

$$\begin{aligned}-t\sin\alpha + s\cos\alpha = {}& -2^j\sin\alpha + \sigma_j\cos\alpha \\ & + (-2^{j+1}\sin\alpha + \sigma_{j+1}\cos\alpha + 2^j\sin\alpha - \sigma_j\cos\alpha) \\ & \times \frac{1}{1+\left(\frac{t\cos\alpha + s\sin\alpha - 2^{j+1}\cos\alpha - \sigma_{j+1}\sin\alpha}{t\cos\alpha + s\sin\alpha - 2^j\cos\alpha - \sigma_j\sin\alpha}\right)^2},\end{aligned}$$

for $t \in [2^j, 2^{j+1}]$ with $j \in \mathbb{N}_0$ and $\sigma_0 = 1$. Let $\eta_j^{(\alpha)}$ be the function $t \mapsto s$ on $[2^j, 2^{j+1}]$. For $\alpha = 0$, we explicitly get

$$\eta_j^{(0)}(t) = \sigma_j + \frac{\sigma_{j+1} - \sigma_j}{1 + \left(\frac{t-2^{j+1}}{t-2^j}\right)^2}$$

on $(2^j, 2^{j+1})$. If $\alpha > 0$ is small enough, we get a function $\eta_j^{(\alpha)}$ whose explicit form is

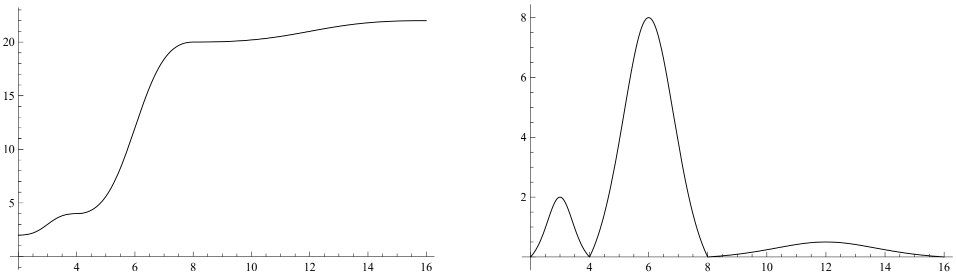


Figure 1.2: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

much more complicated. However, its derivative never vanishes on $[2^j, 2^{j+1}]$. Now, we can

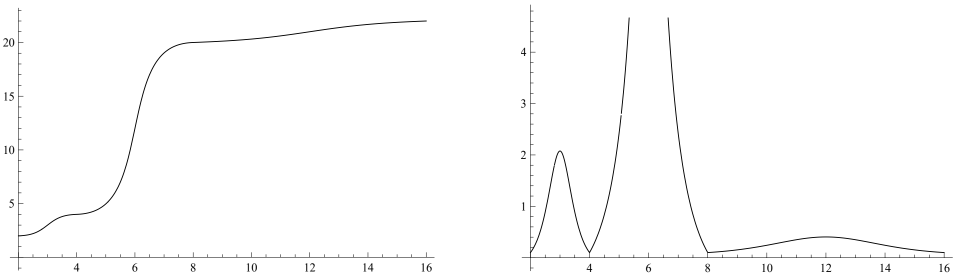


Figure 1.3: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0.1$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1). \end{cases}$$

It is easy to show that the first inequality (resp. the second inequality) in (1.12) is satisfied if $\sigma_j \leq C2^j$ (resp. $\sigma_j \geq C\,2^j$) for every $j \in \mathbb{N}$ since in this case, $\phi(t) \leq C't$ implies

$$t|D\phi(t)|/\phi(t) \geq \alpha/C' > 0$$

for all $t > 1$.

Remark 1.1.27. Using smooth transition functions is not the only approach; we could also use regularizing functions or the Fabius function, for example.

1.1.5 Constructing a regular Boyd function from an admissible sequence

We can improve the previous construction of a regular Boyd function by developing some considerations given in [88]: it is possible, under some general conditions, to obtain an infinitely differentiable function that belongs to $\mathcal{B}'$. By doing so, we improve the original result where the obtained function is only continuously differentiable.

Proposition 1.1.28. *If σ is an admissible sequence such that either $\underline{s}(\sigma) > 0$ or $\overline{s}(\sigma) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $(\xi(2^j))_j \asymp \sigma$.*

Proof. First, we construct $\phi \in \mathcal{B}$ by

$$\phi(t) = \begin{cases} \sigma_j + \frac{\sigma_{j+1}-\sigma_j}{1+\left(\frac{t-2^{j+1}}{t-2^j}\right)^2} & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1). \end{cases}$$

Let $a \in \mathbb{R}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < a < \infty$. For all $s > 0$, let us set

$$\xi_1(s) = \int_0^\infty \min\{1, \left(\frac{s}{t}\right)^a\}\, \phi(t)\, \frac{dt}{t}.$$

It then suffices to take $\xi = \xi_1/\xi_1(1)$ to conclude. □

As a consequence, we can improve the original result in [88], where the function ξ can be supposed differentiable. Indeed, we can assume that it is infinitely differentiable.

Proposition 1.1.29. *If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $\xi \asymp \phi$.*

1.1.6 Some properties of Boyd functions

The next four propositions are proven in [66].

Proposition 1.1.30. *Let $\phi \in \mathcal{B}$, then $1/\phi, \phi(1/\cdot) \in \mathcal{B}$ and*

$$\overline{\frac{1}{\phi}}(t) = \overline{\phi\left(\frac{1}{\cdot}\right)}(t) = \overline{\phi}\left(\frac{1}{t}\right) \qquad \forall t > 0.$$

Moreover, one has

$$\underline{b}(1/\phi) = -\overline{b}(\phi) \quad \text{and} \quad \overline{b}(1/\phi) = -\underline{b}(\phi).$$

Proposition 1.1.31. *Let $\phi_1, \phi_2 \in \mathcal{B}$, then $\phi_1\phi_2, \phi_1/\phi_2 \in \mathcal{B}$ and*

$$\overline{\phi_1\phi_2}(t) \leq \overline{\phi_1}(t)\,\overline{\phi_2}(t) \quad \text{and} \quad \overline{\phi_1/\phi_2}(t) \leq \overline{\phi_1}(t)\,\overline{\phi_2}(1/t) \quad \forall t > 0.$$

Moreover, one has

$$\underline{b}(\phi_1\phi_2) \leq \underline{b}(\phi_1) + \underline{b}(\phi_2) \quad \text{and} \quad \overline{b}(\phi_1\phi_2) \leq \overline{b}(\phi_1) + \overline{b}(\phi_2).$$

Proposition 1.1.32. *If $a \in \mathbb{R}$ and $\phi \in \mathcal{B}$, then $\phi^a \in \mathcal{B}$ and for all $t > 0$,*

$$\overline{\phi^a}(t) = \begin{cases} \overline{\phi}^{\,a}(t) & \text{if } a \geq 0, \\ \overline{\phi}^{\,|a|}(1/t) & \text{if } a < 0. \end{cases}$$

Moreover, if $a > 0$,

$$\underline{b}(\phi^a) = a\,\underline{b}(\phi) \quad \text{and} \quad \overline{b}(\phi^a) = a\,\overline{b}(\phi),$$

and if $a < 0$,

$$\underline{b}(\phi^a) = a\,\overline{b}(\phi) \quad \text{and} \quad \overline{b}(\phi^a) = a\,\underline{b}(\phi).$$

If $q \in [1, \infty)$ and $E \subseteq (0, \infty)$, we denote by $L^q_*(E)$ the Lebesgue space L^q on E with measure dt/t.

Proposition 1.1.33.

- $\underline{b}(\phi) > 0 \Leftrightarrow \bar{\phi} \in L^1_*(0,1) \Leftrightarrow \lim_{t \to 0^+} \bar{\phi}(t) = 0,$
- $\overline{b}(\phi) < 0 \Leftrightarrow \bar{\phi} \in L^1_*(1,\infty) \Leftrightarrow \lim_{t \to \infty} \bar{\phi}(t) = 0,$
- $\underline{b}(\phi) > 0 \Rightarrow \phi \in L^\infty(0,1),$
- $\overline{b}(\phi) < 0 \Rightarrow \phi \in L^\infty(1,\infty).$

Proposition 1.1.34. *Let u, v, ϕ be functions from $(0,\infty)$ to $(0,\infty)$; if $u \asymp v$ and $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then $\phi \circ u \asymp \phi \circ v$.*

Proof. Let us suppose that $\underline{b}(\phi) > 0$ and let $\xi \in \mathcal{B}'$ be such that $C_1\phi \leq \xi \leq C_2\phi$ for two constants $C_1, C_2 > 0$. Now, let $C_1', C_2' > 0$ be two constants such that $C_1'v \leq u \leq C_2'v$. It is easy to check that we have

$$\frac{C_1}{C_2\bar{\xi}(C_2')}\,\phi \circ u \leq \phi \circ v \leq \frac{C_2\bar{\xi}(1/C_1')}{C_1}\,\phi \circ u.$$

The case $\overline{b}(\phi) < 0$ can be treated in the same way. □

Proposition 1.1.35. *Let $\phi_1, \phi_2 \in \mathcal{B}$ be such that $\underline{b}(\phi_2) > 0$. If $\underline{b}(\phi_1) > 0$, then $\phi_1 \circ \phi_2$ belongs to $\mathcal{B}$ and $\underline{b}(\phi_1 \circ \phi_2) > 0$. If $\overline{b}(\phi_1) < 0$, then $\phi_1 \circ \phi_2 \in \mathcal{B}$ and $\overline{b}(\phi_1 \circ \phi_2) < 0$.*

Proof. Let us suppose that $\underline{b}(\phi_1) > 0$ and let $\xi, \eta \in \mathcal{B}'$ be such that $\xi \asymp \phi_1$ and $\eta \asymp \phi_2$. For $t > 0$, one has

$$\overline{\phi_1 \circ \phi_2}(t) \leq C\bar{\xi}(\bar{\eta}(t)),$$

so that $\overline{\phi_1 \circ \phi_2}(t)$ tends to 0 as t tends to 0^+. As a consequence, we have $\underline{b}(\phi_1 \circ \phi_2) > 0$. The case $\overline{b}(\phi_1 \circ \phi_2) < 0$ can be treated in the same way. □

1.2 Examples of Generalized Spaces

We present here some examples of Banach spaces that are constructed using Boyd functions and we prove some results linked to those spaces which will be useful later. Many results related to these spaces can be generalized, leading to more precise statements, particularly regarding regularity [27, 66, 78, 88]. Other spaces will be considered using interpolation in Chapters 2 and 3.

1.2.1 Lebesgue Spaces

Let $q \in [1, \infty]$ and $\phi \in \mathcal{B}$; if X is a Banach space, the space $\ell^q_\phi(X)$ consists of all sequences $(a_j)_j$ of X such that

$$(\phi(2^j)\|a_j\|_X)_j \in \ell^q.$$

This space is equipped with the norm

$$\|(a_j)_j\|_{\ell^q_\phi(X)} := \|(\phi(2^j)\|a_j\|_X)_j\|_{\ell^q}.$$

If $\phi(t) = t^s$ with $s \in \mathbb{R}$, we will write $\ell^q_s(X)$ instead of $\ell^q_\phi(X)$; moreover, we set $\ell^q_\phi := \ell^q_\phi(\mathbb{C})$. If X is a normed space of sequences, then one can define $L^p(X)$ as the space of sequences of functions $(f_j : \mathbb{R}^d \to \mathbb{C})_j$ such that

$$\|(f_j)_j\|_{L^p(X)} := \|\|(f_j)_j\|_X\|_{L^p} < \infty.$$

1.2.2 Global Hölder Spaces

$C_b(\mathbb{R}^d)$ is the space of bounded continuous functions in $\mathbb{R}^d$, with the L^∞ norm.
$C^1_b(\mathbb{R}^d)$ is the subspace of continuously differentiable functions with bounded derivatives, with the norm $\|a\|_\infty + \sum_{k=1}^d \|D_k f\|_\infty$.
For $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$, $C^\phi_b(\mathbb{R}^d)$ is the space of the so-called bounded and uniformly ϕ-Hölder continuous functions, equipped with the norm

$$\|f\|_{C^\phi_b(\mathbb{R}^d)} := \|f\|_\infty + |f|_{C^\phi} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\phi(|x - y|)}.$$

1.2.3 Besov and Hardy-Sobolev Spaces

Let φ be a function in $\mathcal{S}$ with support in $\{x \in \mathbb{R}^d : 2^{-1} \leq |x| \leq 2\}$ such that $\varphi(x) > 0$ for x satisfying $2^{-1} < |x| < 2$ and $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1$ for $x \neq 0$. Let us choose functions ψ_0 and Φ_j $(j \in \mathbb{Z})$ in $\mathcal{S}$ such that

$$\mathcal{F}\Phi_j(x) = \varphi(2^{-j}x) \quad \text{and} \quad \mathcal{F}\psi_0(x) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}x),$$

where $\mathcal{F}$ denotes the Fourier transform. We then set $\varphi_0 = \psi_0$ and $\varphi_j = \Phi_j$ for all $j \in \mathbb{N}$; $(\varphi_j)_j$ is called a Paley-Littlewood system of test functions. Let $\phi \in \mathcal{B}$ and $(\varphi_j)_j$ be such a system. For $p, q \in [1, \infty]$, we define the generalized Besov Space by

$$B^{\phi}_{p,q} := \{f \in \mathcal{S}' : (\varphi_j * f)_j \in \ell^q_\phi(L^p)\}.$$

This space is equipped with the norm

$$\|f\|_{B^{\phi}_{p,q}} := \|(\varphi_j * f)_j\|_{\ell^q_\phi(L^p)} = \|(\|\varphi_j * f\|_{L^p})_j\|_{\ell^q_\phi}.$$

In the same way, we define the generalized Hardy-Sobolev Space (or Triebel-Lizorkin Space) by

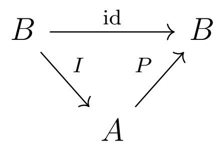
$$F^{\phi}_{p,q} := \{f \in \mathcal{S}' : (\varphi_j * f)_j \in L^p(\ell^q_\phi)\}.$$

This space is equipped with the norm

$$\|f\|_{F^{\phi}_{p,q}} := \|(\varphi_j * f)_j\|_{L^p(\ell^q_\phi)} = \|(\|(\varphi_j * f)_j\|_{\ell^q_\phi})\|_{L^p}.$$

First, we use the notion of retract.

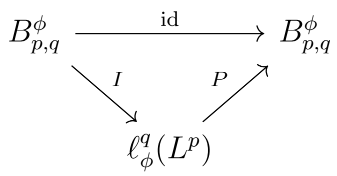
Definition 1.2.1. Let A and B be two objects in a given category, we say that B is a retract of A if there exist morphisms P and I such that the diagram

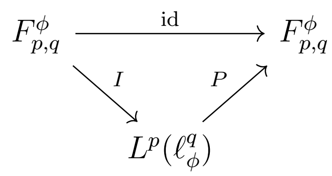


is commutative, i.e. $P \circ I = \text{id}$.

Theorem 1.2.2. *For $\phi \in \mathcal{B}$,*

- *$B^{\phi}_{p,q}$ is a retract of $\ell^q_\phi(L^p)$ for $p, q \in [1, \infty]$,*
- *$F^{\phi}_{p,q}$ is a retract of $L^p(\ell^q_\phi)$ for $p, q \in (1, \infty)$.*





Proof. The two points can be obtained in a similar way; let us show the first one. Let $I : \mathcal{S}' \to \mathcal{S}'^{\mathbb{N}_0}$ be defined as $I(f) = (\varphi_j * f)_j$ and set $P(\alpha) = \sum_{j\in\mathbb{N}_0} \tilde{\varphi}_j * \alpha_j$, for $\alpha \in \mathcal{S}'^{\mathbb{N}_0}$, where $\tilde{\varphi}_0 = \varphi_0 + \varphi_1$ and $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. Since $\widetilde{\varphi}_j * \varphi_j = \varphi_j$ for all $j \in \mathbb{N}_0$, we have $P \circ I = \mathrm{id}$ on $\mathcal{S}'$ and

$$\|f\|_{B^{\phi}_{p,q}} = (\sum_{j\in\mathbb{N}_0} \phi(2^j)^q \|\varphi_j * f\|^q_{L^p})^{1/q} = (\sum_{j\in\mathbb{N}_0} \phi(2^j)^q \|(I(f))_j\|^q_{L^p})^{1/q} = \|I(f)\|_{\ell^q_\phi(L^p)}.$$

so that $I : B^{\phi}_{p,q} \to \ell^q_\phi(L^p)$ is continuous. Now, for $\alpha \in \ell^q_\phi(L^p)$, we have

$$\begin{aligned}
\|P(\alpha)\|_{B^{\phi}_{p,q}} &= (\sum_{j\in\mathbb{N}_0} \phi(2^j)^q \|\varphi_j * \sum_{k\in\mathbb{N}_0} \tilde{\varphi}_k * \alpha_k\|^q_{L^p})^{1/q} \\
&= (\sum_{j\in\mathbb{N}_0} \phi(2^j)^q \|\varphi_j * \alpha_j\|^q_{L^p})^{1/q} \\
&\leq C(\sum_{j\in\mathbb{N}_0} \phi(2^j)^q \|\alpha_j\|^q_{L^p})^{1/q} \\
&= C\|\alpha\|_{\ell^q_\phi(L^p)},
\end{aligned}$$

so that $P : l^q_\phi(L^p) \to B^{\phi}_{p,q}$ is continuous. Finally, $B^{\phi}_{p,q}$ is a retract of $l^q_\phi(L^p)$. □

1.2.4 Sobolev Spaces

Let $\mathcal{J}$ be the Bessel operator of order s :

$$\mathcal{J}^s f = \mathcal{F}^{-1}\left((1 + |\cdot|^2)^{s/2}\mathcal{F}f\right),$$

for $f \in \mathcal{S}'$ and $s \in \mathbb{R}$. Given $s \in \mathbb{R}$ and $p \in [1, \infty]$, the fractional Sobolev space H^s_p is defined by

$$H^s_p = \{f \in \mathcal{S}' : \|\mathcal{J}^s f\|_{L^p} < \infty\}.$$

Definition 1.2.3. We denote by $\mathcal{B}''$ the set of functions $\phi \in \mathcal{B}$ which are C^∞ on $[1, \infty)$ and such that for all $m \in \mathbb{N}$, $t^m|\phi^{(m)}(t)| \leq C_m\, \phi(t)$ is satisfied for all $t \in [1, \infty)$, where C_m is a constant depending on m.

For example, if $\phi(t) = t^\theta$ with $\theta \in \mathbb{R}\backslash\mathbb{Z}$, then $\phi \in \mathcal{B}''$ with $C_m = |\prod_{j=0}^{m-1}(\theta - j)|$. In general, given $\theta, \gamma \in \mathbb{R}$, it is easy to check that $\phi(t) := t^\theta(1 + |\log t|)^\gamma$ belongs to $\mathcal{B}''$. One directly checks that given $\phi \in \mathcal{B}''$ and $f \in \mathcal{S}'$, $\phi(\sqrt{1 + |\cdot|^2})f$ belongs to $\mathcal{S}'$. Therefore, given $\phi \in \mathcal{B}''$, we can define the generalized Bessel operator $\mathcal{J}^\phi$ on $\mathcal{S}'$ as

$$\mathcal{J}^\phi f = \mathcal{F}^{-1}(\phi(\sqrt{1 + |\cdot|^2})\mathcal{F}f).$$

It is clear that $\mathcal{J}^\phi$ is a linear bijective operator from $\mathcal{S}'$ to $\mathcal{S}'$ such that $(\mathcal{J}^\phi)^{-1} = \mathcal{J}^{1/\phi}$ and $\mathcal{J}^\phi(\mathcal{S}) = \mathcal{S}$. From there, the generalized (fractional) Sobolev space H^ϕ_p is defined by

$$H^\phi_p = \{f \in \mathcal{S}' : \|\mathcal{J}^\phi f\|_{L^p} < \infty\}.$$

For $\phi(t) = t^s$, we obviously have $H_p^\phi = H_p^s$. Let us recall (see [15] for example) that for $p \in [1,\infty]$, the space of Fourier multipliers on L^p is defined by

$$M^p := \{f \in \mathcal{S}' : \forall g \in \mathcal{S}, \|(\mathcal{F}^{-1}f) * g\|_{L^p} \leq C\|g\|_{L^p}\},$$

equipped with the norm

$$\|f\|_{M^p} := \sup\{\|(\mathcal{F}^{-1}f) * g\|_{L^p}, g \in \mathcal{S}, \|g\|_{L^p} \leq 1\}.$$

The following lemma is proved in [15] for example.

Lemma 1.2.4. *Let $N \in \mathbb{N}$, if $f \in \mathcal{S}'$ is such that $f \in L^2$ and $D^\alpha f \in L^2$ for $|\alpha| = N > d/2$, then $f \in M^p$ and*

$$\|f\|_{M^p} \leq C\|f\|_{L^2}^{1-\frac{d}{2N}} \left(\sup_{|\alpha|=N} \|D^\alpha f\|_{L^2} \right)^{\frac{d}{2N}}. \tag{1.14}$$

Proposition 1.2.5. *Let $f \in \mathcal{S}'$, $\phi \in \mathcal{B}''$, $p \in [1,\infty]$ and $(\varphi_j)_j$ be a Paley-Littlewood system of test functions; if $\varphi_j * f$ belongs to L^p for all $j \in \mathbb{N}_0$, then*

$$\|\mathcal{J}^\phi \varphi_j * f\|_{L^p} \leq C\,\phi(2^j)\,\|\varphi_j * f\|_{L^p},$$

for all $j \in \mathbb{N}_0$.

Proof. Since $\varphi_j * f = \sum_{l=-1}^{1} \varphi_{j+l} * \varphi_j * f$, one has

$$\|\mathcal{J}^\phi \varphi_j * f\|_{L^p} \leq \sum_{l=-1}^{1} \|\mathcal{F}\mathcal{J}^\phi \varphi_{j+l}\|_{M^p} \|\varphi_j * f\|_{L^p}$$

and thus it is sufficient to prove that $\|\mathcal{F}\mathcal{J}^\phi \varphi_{j+l}\|_{M^p} \leq C\phi(2^j)$, that is

$$\|\phi(2^{j+l}\sqrt{2^{-2(j+l)} + |\cdot|^2})\varphi(\cdot)\|_{M^p} \leq C\phi(2^j).$$

Now, for $\alpha \in \mathbb{N}^d$ and $g = \phi(2^{j+l}\sqrt{2^{-2(j+l)} + |\cdot|^2})\varphi(\cdot)$, $|D^\alpha g(x)|$ is bounded by terms of the form

$$C2^{(j+l)\alpha_1}|\phi^{(\alpha_1)}(2^{j+l}\sqrt{2^{-2(j+l)} + |x|^2})|$$

in finite number, with $0 \leq \alpha_1 \leq |\alpha|$. Since $\phi \in \mathcal{B}^\infty$, $|D^\alpha g(x)|$ is thus bounded by a finite sum of terms

$$C\phi(2^j)\bar{\phi}(\sqrt{2^{-2(j+l)} + |x|^2})(2^{-2(j+l)} + |x|^2)^{-\alpha_1/2}.$$

From inequality (1.14), we thus have $\|g\|_{M^p} \leq C\phi(2^j)$, as required. □

Lemma 1.2.6. *For $\phi \in \mathcal{B}''$ such that $\underline{b}(\phi) > 0$, $\mathcal{J}^{1/\phi}$ maps L^p into L^p continuously.*

Proof. By Proposition 1.2.5, one has

$$\begin{aligned}\|\mathcal{J}^{1/\phi} f\|_{L^p} &\leq C \sum_{j\in\mathbb{N}_0} (1/\phi)(2^j)\, \|\varphi_j * f\|_{L^p} \\ &\leq C(\int_0^1 \bar{\phi}(t)\frac{dt}{t})\|f\|_{L^p} \leq C\|f\|_{L^p}.\end{aligned}$$

□

Proposition 1.2.7. *Let $\phi_0, \phi_1 \in \mathcal{B}''$, $\phi = \phi_0/\phi_1$ and $p \in [1,\infty]$. If $\underline{b}(\phi) > 0$, then $H^{\phi_0}_p \hookrightarrow H^{\phi_1}_p$.*

Proof. Lemma 1.2.6 implies $\|\mathcal{J}^{\phi_1} f\|_{L^p} \leq C\|\mathcal{J}^{\phi_0} f\|_{L^p}$. □

1.2.5 Pointwise Spaces

The T^p_u spaces were initially introduced by Calderón and Zygmund [26] : given a point $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and a number $u \geq -d/p$, $T^p_u(x_0)$ denotes the class of functions f in $L^p(\mathbb{R}^d)$ for which there exists a polynomial P of degree strictly less than u with the property that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\, r^u \quad \forall r > 0.$$

They allow to determine the pointwise regularity of functions that are not locally bounded (see for example [48, 105, 66]). If $p = \infty$, it is the classical pointwise Hölder space, denoted by $\Lambda^u(x_0)$.
Let us introduce the concept of p-exponent of a function. As spaces $(T^p_u(x_0))_u$ are embedded [79], we can define the p-exponent of a function at a point x_0.

Definition 1.2.8. The p-exponent of a function $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ at $x_0 \in \mathbb{R}^d$ is defined by

$$h^{(p)}_f(x_0) = \sup\{u > \frac{-d}{p} : f \in T^p_u(x_0)\}.$$

Remark 1.2.9. The equality $u = h^{(p)}_f(x_0)$ does not necessarily imply that $f \in T^p_u(x_0)$. In fact, from the prevalent point of view, most functions f do not belong to $T^p_{h^{(p)}_f(x_0)}(x_0)$, as a logarithmic correction is required [80].

In order to characterize the size of the sets of points sharing the same p-exponent, we adopt the following definition.

Definition 1.2.10. Given $p \in [1,\infty]$, let $E^{(p)}_h(f)$ be the iso-Hölder set of $f \in L^p_{\text{loc}}(\mathbb{R})$ related to h:

$$E^{(p)}_h(f) = \{x : h^{(p)}_f(x) = h\}.$$

The p-spectrum of f is the function

$$D^{(p)}_f \;:\; [\frac{-1}{p}, \infty] \to \mathbb{R}\cup\{-\infty\} \;:\; h \mapsto \dim E^{(p)}_h(f),$$

where $\dim A$ denotes the Hausdorff dimension of the set A, with the convention $\dim \emptyset = -\infty$.

Relatively few p-spectra have been determined for $p < \infty$. Let us mention [48] and [105], where the 2-exponents of trigonometric series which are not locally bounded are computed. Generic results are obtained in [80].
In [78], they generalize T_u^p spaces using Boyd functions, for the purpose of applications in many financial models that are derived from the Brownian motion for example. They also give a generalization of Whitney extension Theorem and connections with operators and elliptic partial differential equations. In [66], they give a generalization of Rademacher's Theorem.

Definition 1.2.11. Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$ and $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > -\frac{d}{p}$.
A function $f \in L^p(\mathbb{R}^d)$ belongs to the space $T_\phi^p(x_0)$ if there exists a polynomial P with degree strictly smaller than $\underline{b}(\phi)$ and a constant $C > 0$ such that

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq C\, \phi(r) \quad \forall r > 0. \tag{1.15}$$

The condition $\underline{b}(\phi) > -\frac{d}{p}$ is here to ensure that the spaces are not degenerate. One can easily check that the polynomial appearing in (1.15) is unique. We set

$$\Lambda^\phi(x_0) = T_\phi^\infty(x_0).$$

Let $f \in T_\phi^p(x_0)$ and P be the polynomial relating to f, then P can be written as

$$P(x) = \sum_{|\alpha| < \underline{b}(\phi)} \frac{D^\alpha P(x_0)}{|\alpha|!} (x - x_0)^\alpha.$$

We set

$$|f|_{T_\phi^p(x_0)} = \sup_{r>0} \phi(r)^{-1} r^{-d/p} \|f - P\|_{L^p(B(x_0,r))},$$

and

$$\|f\|_{T_\phi^p(x_0)} = \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\alpha| < \underline{b}(\phi)} \frac{|D^\alpha P(x_0)|}{|\alpha|!} + |f|_{T_\phi^p(x_0)}.$$

It is then straightforward that $(T_\phi^p(x_0), \|\cdot\|_{T_\phi^p(x_0)})$ is a Banach space. Following the idea of article [79], we can generalize those spaces :

Definition 1.2.12. Let $x_0 \in \mathbb{R}^d$, $p, q \in [1, \infty]$ and $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > -\frac{d}{p}$.
A function $f \in L^p(\mathbb{R}^d)$ belongs to the space $T_\phi^{p,q}(x_0)$ if there exists a polynomial P with degree strictly smaller than $\underline{b}(\phi)$ and a constant $C > 0$ such that

$$r \mapsto \frac{r^{-d/p}}{\phi(r)} \|f - P\|_{L^p(B(x_0,r))} \in L_*^q.$$

We set

$$|f|_{T_\phi^{p,q}(x_0)} = \|\phi(r)^{-1} r^{-d/p} \|f - P\|_{L^p(B(x_0,r))}\|_{L_*^q}$$

and

$$\|f\|_{T_\phi^{p,q}(x_0)} = \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\alpha| < \underline{b}(\phi)} \frac{|D^\alpha P(x_0)|}{|\alpha|!} + |f|_{T_\phi^{p,q}(x_0)}.$$

Chapter 2

Generalized Interpolation Spaces

The origin of the theory of interpolation can be traced back to Marcinkiewicz [84] and the Riesz-Thorin Theorem [100, 113], which states that if a linear function is continuous on L^p and L^q, then it is also continuous on L^r for r between p and q. Later, as it was shown that Sobolev spaces were constituted of functions that have a non-integer order of differentiability [1, 76, 110], various techniques were conceived to generate similar spaces. Among them were the interpolation methods. Let us briefly recall the basic definitions. Let A, A_0, A_1, B, B_0 and B_1 be Banach spaces. The pair (A, B) is called an interpolation pair if we have

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1 \quad \text{and} \quad B_0 \cap B_1 \hookrightarrow B \hookrightarrow B_0 + B_1,$$

where $\hookrightarrow$ is the symbol for the continuous embedding and if any linear operator $T : A_0 + A_1 \to B_0 + B_1$ which maps continuously A_0 to B_0 and A_1 to B_1 also maps A to B continuously. Moreover, (A, B) is said to be of exponent $\theta \in [0, 1]$ if there exists a constant $C > 0$ such that

$$\|T\|_{A,B} \leq C \|T\|^{1-\theta}_{A_0,B_0} \|T\|^{\theta}_{A_1,B_1}, \tag{2.1}$$

for any operator T as above, where $\|T\|_{X,Y}$ is the norm of $T : X \to Y$.

The real interpolation methods [1, 15, 60, 76, 110, 114] have been generalized using a function parameter (see [23, 30, 42, 50, 83, 88, 97, 98, 99] and references therein). Most of the time, these authors start with the K-method. Let A_0 and A_1 be two Banach spaces continuously embedded into a Hausdorff topological vector space so that $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined Banach spaces. One defines the K-functional by

$$K(t, a) := \inf_{a=a_0+a_1} \{\|a_1\|_{A_0} + t\|a_1\|_{A_1}\},$$

for $t > 0$ and $a \in A_0 + A_1$. Given $0 < \theta < 1$ and $q \in [1, \infty]$, a belongs to the interpolation space $K^{\theta}_q(A_0, A_1)$ if $a \in A_0 + A_1$ and

$$(2^{-j\theta} K(2^j, a))_{j \in \mathbb{Z}} \in \ell^q. \tag{2.2}$$

The generalized version is obtained by replacing the sequence $(2^{-j\theta})_{j\in\mathbb{Z}}$ appearing in (2.2) with a Boyd function. Similar definitions have been proposed in [62, 77] using admissible sequences and the relations between these techniques have been studied in Chapter 1.

The Boyd functions form a natural apparatus for studying function spaces [3, 37, 44, 62, 77, 78, 91, 92] and interpolation methods with a function parameter provide an interesting tool in this context [27, 42, 88, 99]. For example, they lead to a definition of the Besov spaces of generalized smoothness based on the usual Sobolev spaces [77]. Other examples are given in [88].

In the first Section of the Chapter, we begin by replacing $t \mapsto t^\theta$ in (2.1) with a Boyd function ϕ, which leads to (2.3), as a starting point. Next, we explore the basic properties of this theory and we show that this generalized approach still allows a functorial interpretation. We are thus naturally led to consider results such as the Aronszajn-Gagliardo Theorem, establishing that any interpolation space of a given Banach couple can be realized as the value of a minimal or maximal interpolation functor on the category of all Banach couples [5]. After verifying some properties on the generalized K-method, one may proceed to the J-method, which is defined in a similar way: namely, for $t > 0$ and $a \in A_0 \cap A_1$, one sets

$$J(t,a) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

This time, one considers

$$(2^{-j\theta} J(2^j, b_j))_{j\in\mathbb{Z}} \in \ell^q,$$

with $a = \sum_{j\in\mathbb{Z}} b_j$ and $b_j \in A_0 \cap A_1$ (for all j), the convergence being in $A_0 + A_1$. This approach can be generalized in the same way and one can show that both methods give rise to the same spaces, in the sense of Theorem 2.4.3 (with equivalence of norms). Furthermore, we establish a reiteration Theorem, compactness Theorems and results about retracts, reinforcing the robustness of these methods within this analytical framework. We also observe that generalizing the complex interpolation method C_θ using Boyd's functions appears to be challenging, as only the exponent θ is involved, rather than the function $t \mapsto t^\theta$. We still obtain connections with the generalized real interpolation method.

The results established in this Section were published in [68].

2.1 Categories and Interpolation

We present here a generalization of the interpolation spaces of real exponent using Boyd functions. We shall reserve $\mathscr{N}$ for the category of all normed vector spaces (the objects of $\mathscr{N}$ are normed vector spaces and the morphisms are the bounded linear operators) and $\mathscr{B}$ for the sub-category of all Banach spaces.

Let us first briefly recall the basic theory of interpolation (see [15, 23] for example). If $(A_0, \|\cdot\|_{A_0})$ and $(A_1, \|\cdot\|_{A_1})$ are two normed topological vector spaces, A_0 and A_1 are compatible if they are both subspaces of a Hausdorff topological vector space. In this case, $A_0 \cap A_1$ is a normed vector space for the norm

$$\|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

and $A_0 + A_1$ is a normed vector space for the norm

$$\|a\|_{A_0+A_1} := \inf_{a=a_0+a_1} \{\|a_0\|_{A_0}, \|a_1\|_{A_1}\}.$$

Moreover, if A_0 and A_1 are both complete, so are $A_0 \cap A_1$ and $A_0 + A_1$. Let $\mathscr{C}$ be a subcategory of $\mathscr{N}$ and denote by $\mathscr{C}_c$ a category of compatible couples $\mathbf{A} = (A_0, A_1)$ (such that $A_0 \cap A_1$ and $A_0 + A_1$ are in $\mathscr{C}$). The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathscr{C}_c$ are bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that both $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are morphisms in $\mathscr{C}$. The two basic functors Σ and Δ from $\mathscr{C}_c$ to $\mathscr{C}$ are defined as follows: $\Sigma(T) = \Delta(T) = T$ and

$$\Delta(\mathbf{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\mathbf{A}) = A_0 + A_1.$$

In the sequel, $\mathscr{C}$ will stand for any subcategory of $\mathscr{N}$ such that $\mathscr{C}$ is closed under the operations sum and intersection, while $\mathscr{C}_c$ will stand for the category of all compatible couples $\mathbf{A}$ of spaces in $\mathscr{C}$. Given a couple $\mathbf{A} = (A_0, A_1)$ in $\mathscr{C}_c$, a space $A \in \mathscr{C}$ is an intermediate space between A_0 and A_1 (or with respect to $\mathbf{A}$) if

$$\Delta(\mathbf{A}) \hookrightarrow A \hookrightarrow \Sigma(\mathbf{A}).$$

Such a space A is called an interpolation space between A_0 and A_1 (or with respect to $\mathbf{A}$) if in addition $T : \mathbf{A} \to \mathbf{A}$ implies $T : A \to A$. Now, if $\mathbf{B}$ is another couple in $\mathscr{C}_c$, two spaces A and B in $\mathscr{C}$ are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$ if A and B are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$ respectively and if $T : \mathbf{A} \to \mathbf{B}$ implies $T : A \to B$. These interpolation spaces are uniform interpolation spaces if

$$\|T\|_{A,B} \leq C \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$$

always holds for some constant $C > 0$. If $C = 1$ in the previous inequality, A and B will be called exact interpolation spaces. Of course, in the case $B = A$, we will omit any reference to the second interpolation space B; in particular, we set $\|T\|_X := \|T\|_{X,X}$. An interpolation functor (or interpolation method) on $\mathscr{C}$ is a functor $F : \mathscr{C}_c \to \mathscr{C}$ such that if $\mathbf{A}$ and $\mathbf{B}$ are two couples in $\mathscr{C}_c$, then $F(\mathbf{A})$ and $F(\mathbf{B})$ are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$, with $F(T) = T$ for all $T : \mathbf{A} \to \mathbf{B}$. Naturally, the descriptive terms related to the interpolation spaces can be transposed to the interpolation functors; we shall say that F is a uniform (exact) interpolation functor if $F(\mathbf{A})$ and $F(\mathbf{B})$ are uniform (exact) interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$.
Given $\phi \in \mathcal{B}$, we will denote by ϕ_* the function explicitly defined by $\phi_*(t) = t/\phi(t)$ for $t > 0$.

Definition 2.1.1. Let $\mathbf{A} = (A_0, A_1)$ and $\mathbf{B} = (B_0, B_1)$ be two couples in $\mathscr{C}_c$; two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$ respectively are of exponent $\phi \in \mathcal{B}$ if, for any $T : \mathbf{A} \to \mathbf{B}$,

$$\|T\|_{A,B} \leq C\bar{\phi}_*(\|T\|_{A_0,B_0})\bar{\phi}(\|T\|_{A_1,B_1}) \tag{2.3}$$

always holds for some constant $C > 0$. If $C = 1$, we say that A and B are exact of exponent ϕ.

Most of the time, we will assume $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, which corresponds to the classical assumption $0 < \theta < 1$ in (2.2) for example. The extreme cases (0 and 1) are treated in Section 3.4.

Let us remark that A and B are of exponent ϕ if and only if they are of exponent $\check{\phi} := 1/\phi(1/\cdot)$. Similarly, we say that F is (exact) of exponent ϕ if $F(\mathbf{A})$ and $F(\mathbf{B})$ are (exact) of exponent ϕ. Obviously, Δ and Σ are exact interpolation functors on any subcategory of $\mathscr{N}$. One can also note that, thanks to Proposition 1.1.30, two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$ respectively are of exponent $\phi \in \mathcal{B}$ if and only if

$$\|T\|_{A,B} \leq C\bar{\phi}^*(\|T\|_{A_0,B_0})\bar{\phi}(\|T\|_{A_1,B_1})$$

always holds for some constant $C > 0$, where ϕ^* is the Boyd function explicitly defined by $\phi^*(t) := t\,\phi(\frac{1}{t})$ for $t > 0$.

Theorem 2.1.2. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, A be an interpolation space with respect to $\mathbf{A}$ and define A_ϕ as the space of all $x \in \Sigma(\mathbf{A})$ admitting a representation of the form*

$$x = \sum_j T_j a_j,$$

with $T_j : \mathbf{A} \to \mathbf{A}$ and $a_j \in A$ for all j, the convergence being in $\Sigma(\mathbf{A})$, equipped with the norm

$$\|x\|_{A_\phi} := \inf_{x=\sum_j T_j a_j} \sum_j \bar{\phi}_*(\|T_j\|_{A_0})\bar{\phi}(\|T_j\|_{A_1})\|a_j\|_A.$$

If $A_\phi \hookrightarrow \Sigma(\mathbf{A})$, then A_ϕ is a minimal exact interpolation space of exponent ϕ with respect to $\mathbf{A}$ that contains A.

Proof. Given $a \in A$, if T_1 is the identity map, $a_1 = a$, $T_j = 0$ and $a_j = 0$ for $j > 1$, we have $A \hookrightarrow A_\phi$. Therefore, A_ϕ is an intermediate space with respect to $\mathbf{A}$ and thus an interpolation space with respect to $\mathbf{A}$. Now, for $\sum_j T_j a_j \in A_\phi$ and $T : \mathbf{A} \to \mathbf{A}$, we have

$$\|T\sum_j T_j a_j\|_{A_\phi} \leq \bar{\phi}_*(\|T\|_{A_0})\bar{\phi}(\|T\|_{A_1})\sum_j \bar{\phi}_*(\|T_j\|_{A_0})\bar{\phi}(\|T_j\|_{A_1})\|a_j\|_A.$$

so that

$$\|T\|_{A_\phi} \leq \bar{\phi}_*(\|T\|_{A_0})\bar{\phi}(\|T\|_{A_1}).$$

As a consequence, A_ϕ is exact of exponent ϕ. Finally, let B be an exact interpolation space of exponent ϕ with respect to $\mathbf{A}$ that contains A. For $\sum_j T_j a_j \in A_\phi$, we directly get

$$\|\sum_j T_j a_j\|_B \leq \sum_j \bar{\phi}_*(\|T_j\|_{A_0})\bar{\phi}(\|T_j\|_{A_1})\|a_j\|_B,$$

so that $\|\sum_j T_j a_j\|_B \leq C\|\sum_j T_j a_j\|_{A_\phi}$. □

2.2 Aronszajn-Gagliardo-type Theorems

In 1965, Aronszajn and Gagliardo showed that any interpolation space of a given Banach couple could be realized as the value of a minimal or maximal interpolation functor on the category of all Banach couples [5]. Later, connections between important interpolation methods and this result were identified [23, 50], highlighting the importance of this Theorem.

Theorem 2.2.1. *Let A be an interpolation space of exponent ϕ with respect to $\mathbf{A}$, where $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$. If $\mathbf{X}$ is a given couple, the set $F(\mathbf{X})$ consists of those $x \in \Sigma(\mathbf{X})$ which admit a representation $x = \sum_j T_j a_j$ (with convergence in $\Sigma(\mathbf{X})$), with $T_j : \mathbf{A} \to \mathbf{X}$ and $a_j \in A$ for all j. Define*

$$N_\phi(\sum_j T_j a_j) := \sum_j \bar{\phi}_*(\|T_j\|_{A_0,X_0})\bar{\phi}(\|T_j\|_{A_1,X_1})\,\|a_j\|_A,$$

so that $F(\mathbf{X})$ can be equipped with the norm

$$\|x\|_{F(\mathbf{X})} := \inf_{x=\sum_j T_j a_j} N_\phi(\sum_j T_j a_j).$$

If $F(\mathbf{X}) \hookrightarrow \Sigma(\mathbf{X})$ for all couples $\mathbf{X}$, then F gives a minimal interpolation functor which is exact of exponent ϕ such that $F(\mathbf{A}) = A$.

Proof. The classical proof can be adapted without any difficulty to this context (see [15] for example): let ψ be a bounded linear functional on $\Sigma(\mathbf{A})$ such that $\psi(a_1) = 1$ for some fixed $a_1 \in A$. Let us show that $F(\mathbf{X})$ is an intermediate space with respect to $\mathbf{X}$. Let us first show that $\Delta(\mathbf{X}) \hookrightarrow F(\mathbf{X})$. For a fixed $x \in \Delta(\mathbf{X})$, let us set $T_1 a = \psi(a)x$ for $a \in A$ so that we have

$$\|T_1 a\|_{X_k} \leq C\|a\|_{\Sigma(\mathbf{A})}\|x\|_{X_k}$$

Since $x = \sum_j T_j a_j$ with $T_j = 0$ and $a_j = 0$ for $j \neq 1$, we have

$$\|x\|_{F(\mathbf{X})} \leq C\bar{\phi}_*(\|T_1\|_{A_0,X_0})\bar{\phi}(\|T_1\|_{A_1,X_1})\,\|a_1\|_A,$$

which gives the continuous inclusion. Therefore, $F(\mathbf{X})$ is an intermediate space with respect to $\mathbf{X}$ and thus an interpolation space with respect to $\mathbf{X}$. Let us prove that F is an exact interpolation functor of exponent ϕ. Let $T : \mathbf{X} \to \mathbf{Y}$; for $x = \sum_j T_j a_j \in F(\mathbf{X})$, we have $Tx = \sum_j TT_j a_j \in F(\mathbf{Y})$ and

$$\begin{aligned}\|Tx\|_{F(\mathbf{Y})} &\leq \sum_j \bar{\phi}_*(\|TT_j\|_{A_0,Y_0})\bar{\phi}(\|TT_j\|_{A_1,Y_1})\,\|a_j\|_A \\ &\leq \bar{\phi}_*(\|T\|_{X_0,Y_0})\bar{\phi}(\|T\|_{X_1,Y_1})\sum_j \bar{\phi}_*(\|T_j\|_{A_0,X_0})\bar{\phi}(\|T_j\|_{A_1,X_1})\,\|a_j\|_A,\end{aligned}$$

which implies
$$\|T\|_{F(\mathbf{X}),F(\mathbf{Y})} \leq \bar{\phi}_*(\|T\|_{X_0,Y_0})\bar{\phi}(\|T\|_{X_1,Y_1}).$$
Let us show that $F(\mathbf{A}) = A$. For $a = \sum_j T_j a_j \in F(\mathbf{A})$, we have

$$\|a\|_A \leq \sum_j \|T_j a_j\|_A \leq \sum_j \|T_j\|_A \|a_j\|_A \leq C \sum_j \bar{\phi}_*(\|T\|_{A_0})\bar{\phi}(\|T\|_{A_1})\, \|a\|_A,$$

so that $\|a\|_A \leq C\|a\|_{F(\mathbf{A})}$. For $a \in A$, let $a = \sum_j T_j a_j$, with

$$\begin{cases} T_j = 0,\ a_j = 0 & \text{for } j \neq 1, \\ T_1 = \mathrm{id},\ a_1 = a, \end{cases}$$

which implies $a \in F(\mathbf{A})$. The continuous inclusion follows from

$$\|a\|_{F(\mathbf{A})} \leq \sum_j \bar{\phi}_*(\|T_j\|_{A_0})\bar{\phi}(\|T_j\|_{A_1})\, \|a_j\|_A = \|a\|_A.$$

Finally, let us prove that F is minimal. Let G be an exact interpolation functor of exponent ϕ such that $G(\mathbf{A}) = A$. For $\sum_j T_j a_j \in F(\mathbf{X})$, we have

$$\|\sum_j T_j a_j\|_{G(\mathbf{X})} \leq \sum_j \|T_j a_j\|_{G(\mathbf{X})} \leq C \sum_j \bar{\phi}_*(\|T_j\|_{A_0,X_0})\bar{\phi}(\|T_j\|_{A_1,X_1})\, \|a_j\|_A,$$

so that $\|\sum_j T_j a_j\|_{G(\mathbf{X})} \leq C\|\sum_j T_j a_j\|_{F(\mathbf{X})}$. □

Theorem 2.2.2. *Let A be an interpolation space of exponent ϕ with respect to $\mathbf{A}$, where $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$. If $\mathbf{X}$ is a given couple, the set $F(\mathbf{X})$ consists of those $x \in \Sigma(\mathbf{X})$ such that $Tx \in A$ for all $T : \mathbf{X} \to \mathbf{A}$ with norm*

$$\|x\|_{F(\mathbf{X})} := \sup\{\|Tx\|_A : \bar{\phi}_*(\|T\|_{X_0,A_0})\bar{\phi}(\|T\|_{X_1,A_1}) \leq 1\}.$$

If $\Delta(\mathbf{X}) \hookrightarrow F(\mathbf{X})$ for all couples $\mathbf{X}$, then F gives a maximal interpolation functor which is exact of exponent ϕ such that $F(\mathbf{A}) = A$.

Proof. Let $x \in F$ and $f \in \Sigma(\mathbf{X})'$ such that $\|f\| = 1$ and $f(x) = \|x\|_{\Sigma(\mathbf{X})}$. If C_A is the constant corresponding to the continuous inclusion $\Delta(\mathbf{A}) \hookrightarrow A$, assuming that $\Delta(\mathbf{A}) \neq \{0\}$, for a given $\varepsilon > 0$, there exists an element $a \in \Delta(\mathbf{A})$ with $\|a\|_{\Delta(\mathbf{A})} = 1$ such that $\|a\|_A > C_A - \varepsilon$.
Consider the operator $P : \mathbf{X} \to \mathbf{A}$ defined by $P(y) = f(y)a$. One has

$$\bar{\phi}_*(\|P\|_{X_0,A_0})\bar{\phi}(\|P\|_{X_1,A_1}) \leq \bar{\phi}_*(\|a_0\|_{A_0})\bar{\phi}(\|a_1\|_{A_1}) \leq 1.$$

Therefore,
$$C_A\|x\|_{\Sigma(\mathbf{X})} \leq \|x\|_{F(\mathbf{X})}.$$

As a consequence, $F(\mathbf{X})$ is an interpolation space with respect to $\mathbf{X}$. Let us prove that F is an exact interpolation functor of exponent ϕ. For $S : \mathbf{X} \to \mathbf{Y}$, we have

$$\begin{aligned}\|Sx\|_{F(\mathbf{Y})} &= \sup\{\|TSx\|_A : \bar{\phi}_*(\|T\|_{Y_0,A_0})\bar{\phi}(\|T\|_{Y_1,A_1}) \leq 1\} \\ &\leq \bar{\phi}_*(\|S\|_{X_0,Y_0})\bar{\phi}(\|S\|_{X_1,Y_1})\|Sx\|_{F(\mathbf{X})}.\end{aligned}$$

Let us show that $F(\mathbf{A}) = A$. On the one hand, using the identity operator, one has

$$\|a\|_A \leq \|a\|_{F(\mathbf{A})}.$$

On the other hand, if $a \in A$ and $T : \mathbf{A} \to \mathbf{A}$,

$$\|Ta\|_A \leq C\bar{\phi}_*(\|T\|_{A_0,A_0})\bar{\phi}(\|T\|_{A_1,A_1})\|a\|_A$$

and thus

$$\|a\|_{F(\mathbf{A})} \leq C\|a\|_A.$$

Finally, let us prove that F is maximal. Let G be an exact interpolation functor of exponent ϕ such that $G(\mathbf{A}) = A$. If $T : \mathbf{X} \to \mathbf{A}$ is such that $\bar{\phi}_*(\|T\|_{X_0,A_0})\bar{\phi}(\|T\|_{X_1,A_1}) \leq 1$, then

$$\|Tx\|_A \leq C\|Tx\|_{G(\mathbf{A})} \leq C\phi_*(\|T\|_{X_0,A_0})\bar{\phi}(\|T\|_{X_1,A_1})\|x\|_{G(\mathbf{X})} \leq C\|x\|_{G(\mathbf{X})},$$

so that

$$\|x\|_{F(\mathbf{X})} \leq C\|x\|_{G(\mathbf{X})},$$

as desired. □

2.3 The Generalized K-Method

Mimicking the usual K-method [98], one can construct here a family of interpolation functors on $\mathscr{N}$; we obtain the real interpolation spaces with a function parameter (see [88, 99] and references therein).

Let us recall that given a couple $\mathbf{A}$ and $t > 0$,

$$K(t, a) = K(t, a; \mathbf{A}) := \inf_{a=a_0+a_1}\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\},$$

for $a \in \Sigma(\mathbf{A})$. The function $t \mapsto K(t, a)$ is positive, increasing and concave. We also have $K(t, a) \leq \max\{1, t/s\}K(s, a)$.

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $K_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^\phi(\mathbf{A})} := (\int_0^\infty (\frac{1}{\phi(t)} K(t, a))^q \frac{dt}{t})^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Remark 2.3.1. We usually define $K_q^\phi(\mathbf{A})$ for $\underline{b}(\phi), \overline{b}(\phi) \in [0,1]$. Indeed, since

$$(\int_0^\infty (\frac{1}{\phi(t)} K(t,a))^q \frac{dt}{t})^{1/q} \geq K(1,a)(\int_0^1 \frac{t^{q-1}}{\phi(t)^q} dt + \int_1^\infty \frac{dt}{t\phi(t)^q})^{1/q},$$

$K_q^\phi(\mathbf{A}) = \{0\}$ if $q < \infty$ and $\underline{b}(\phi) < 0$ or $\overline{b}(\phi) > 1$. Unlike the classical case where $\phi(t) = t^\theta$, the scenarios $\underline{b}(\phi) = 0$ and $\overline{b}(\phi) = 1$ can lead to non-degenerate spaces: see Section 3.4.

Remark 2.3.2. One may work with $q \in (0,1)$ by considering the category of quasi-normed Abelian groups. Most of the results are identical, except that we use only the discrete definitions of the spaces to avoid the need for integration [15, 45].

Theorem 2.3.3. *For $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, K_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$. Moreover, we have*

$$K(t,a) \leq C\phi(t)\|a\|_{K_q^\phi(\mathbf{A})}.$$

Proof. The classical proof (see [88, 99, 15]) can be easily adapted to show that this functor is an exact interpolation functor of exponent ϕ : since

$$\begin{aligned}(\int_0^\infty (\frac{\min\{1,u\}}{\phi(us)})^q \frac{du}{u})^{1/q} K(s,a) &= (\int_0^\infty (\frac{\min\{1,t/s\}}{\phi(t)} K(t,a))^q \frac{dt}{t})^{1/q} \\ &\leq \|a\|_{K_q^\phi(\mathbf{A})},\end{aligned}$$

we have

$$K(s,a) \leq C\phi(s)\|a\|_{K_q^\phi(\mathbf{A})}.$$

This inequality for $s = 1$ gives $K_q^\phi(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A})$. From

$$K(t,a) \leq \min\{1,t\}\|a\|_{\Delta(\mathbf{A})},$$

we also get that $\Delta(\mathbf{A})$ is continuously embedded in $K_q^\phi(\mathbf{A})$. Now, let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathcal{N}_c$; for $T : \mathbf{A} \to \mathbf{B}$, we have

$$K(t, Ta; \mathbf{B}) \leq \|T\|_{A_0,B_0} K(t\frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}, a; \mathbf{A})$$

and thus, for $\alpha := \|T\|_{A_1,B_1}/\|T\|_{A_0,B_0}$,

$$\begin{aligned}\|Ta\|_{K_q^\phi(\mathbf{B})} &\leq \|T\|_{A_0,B_0}(\int_0^\infty (\frac{K(t\alpha, a; \mathbf{A})}{\phi(t)})^q \frac{dt}{t})^{1/q} \\ &\leq \|T\|_{A_0,B_0}\, \bar{\phi}(\alpha)\, \|a\|_{K_q^\phi(\mathbf{A})} \\ &\leq \bar{\phi}_*(\|T\|_{A_0,B_0})\, \bar{\phi}(\|T\|_{A_1,B_1})\, \|a\|_{K_q^\phi(\mathbf{A})},\end{aligned}$$

which shows that K_q^ϕ is exact of exponent ϕ. □

Obviously, $K_q^{1/\phi(1/\cdot)}$ is also exact of exponent ϕ on $\mathscr{N}$.

Remark 2.3.4. From the fact that, for $a \in \Sigma(\mathbf{A})$, $t \mapsto K(t,a)$ is non-decreasing and $t \mapsto K(t,a)/t$ is non-increasing, since ϕ is a Boyd function, a belongs to $K_q^\phi(\mathbf{A})$ if and only if $\big(K(2^j,a)/\phi(2^j)\big)_{j\in\mathbb{Z}} \in \ell^q$.

Remark 2.3.5. One might wonder whether it is possible to generalize method K using two parameter functions, i.e. if $\phi, \gamma \in \mathcal{B}$ with both $\underline{b}(\phi), \underline{b}(\gamma) > 0$, $\overline{b}(\phi), \overline{b}(\gamma) < 1$, define the space of all $a \in \Sigma(\mathbf{A})$ such that

$$(\int_0^\infty \big(\frac{1}{\phi(t)}\, K(\gamma(t),a)\big)^q \,\frac{dt}{t})^{1/q} < \infty.$$

Since $\underline{b}(\gamma) > 0$ and thanks to Proposition 1.1.29, one may employ the change of variables $u = \gamma(t)$ and observe that the space defined is in fact $K_q^{\phi\circ\gamma^{-1}}(\mathbf{A})$.

The usual results can be revised to get the following propositions. They can be easily obtained from the original ones (see [15], Section 3.4).

Proposition 2.3.6. *Given $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$, for $1 \le p \le q \le \infty$, we have the continuous inclusion $K_p^\phi(\mathbf{A}) \hookrightarrow K_q^\phi(\mathbf{A})$.*

Proof. Let us first consider the case where $q = \infty$. For $t_0 > 0$, the change of variable $t \mapsto t_0/t$ and Remark 2.3.4 allow us to write

$$\|a\|^p_{K_p^\phi(\mathbf{A})} \ge C K(t_0,q)^p \int_0^\infty \frac{1}{\phi(t)^p}\frac{dt}{t} \ge C(\frac{K(t_0,q)}{\phi(t_0)})^p,$$

so that we have $K(t_0,a)/\phi(t_0) \le C\|a\|_{K_p^\phi(\mathbf{A})}$ and thus

$$\|a\|_{K_\infty^\phi(\mathbf{A})} \le C\|a\|_{K_p^\phi(\mathbf{A})}.$$

Let us now suppose that $q < \infty$. This time, the Hölder inequality and the previous case give

$$\begin{aligned}\|a\|_{K_q^\phi(\mathbf{A})} = \|(\frac{K(t,a)}{\phi(t)})^p(\frac{K(t,a)}{\phi(t)})^{q-p}\|^{1/q}_{L^1_*} &\le \|(\frac{K(\cdot,a)}{\phi})^p\|^{1/q}_{L^1_*}\ \|(\frac{K(\cdot,a)}{\phi})^{q-p}\|^{1/q}_{L^\infty}\\ &\le C\|a\|^{p/q}_{K_p^\phi(\mathbf{A})}\|a\|^{(q-p)/q}_{K_p^\phi(\mathbf{A})},\end{aligned}$$

which ends the proof. □

The following result is obvious.

Proposition 2.3.7. *For $\phi \in \mathcal{B}$ and $q \in [1,\infty]$, we have*

$$K_q^\phi(A_0,A_1) = K_q^{\phi^*}(A_1,A_0).$$

Remark 2.3.8. We can easily show that, in general, the functions ϕ_* and ϕ^* are not equivalent since $t \mapsto 1/\phi(t)$ and $t \mapsto \phi(1/t)$ are not equivalent in general. But one can remark that $\bar{\phi}_* = \bar{\phi}^*$. Indeed, if $t > 0$,

$$\bar{\phi}_*(t) = \sup_{s>0} t \frac{\phi(s)}{\phi(st)} = \sup_{u>0} \frac{tu\,\phi(1/tu)}{u\,\phi(1/u)} = \bar{\phi}^*(t).$$

In the same way, one can show that $\bar{\phi} = \bar{\check{\phi}}$.

Proposition 2.3.9. *Let $\phi, \phi_0, \phi_1 \in \mathcal{B}$ and $q, q_0, q_1 \in [1, \infty]$; if $\overline{b}(\phi_0) < \underline{b}(\phi)$ and $\overline{b}(\phi) < \underline{b}(\phi_1)$, then*

$$K_{q_0}^{\phi_0}(\mathbf{A}) \cap K_{q_1}^{\phi_1}(\mathbf{A}) \hookrightarrow K_q^{\phi}(\mathbf{A}).$$

Proof. For $a \in K_{q_0}^{\phi_0}(\mathbf{A}) \cap K_{q_1}^{\phi_1}(\mathbf{A})$, we have

$$\begin{aligned} \|a\|_{K_q^{\phi}(\mathbf{A})} &\leq (\int_0^1 (\frac{K(t,a)}{\phi(t)})^q \frac{dt}{t})^{1/q} + (\int_1^\infty (\frac{K(t,a)}{\phi(t)})^q \frac{dt}{t})^{1/q} \\ &\leq \|\frac{K(\cdot,a)}{\phi_1}\|_{L_*^{q_1}(0,1)} \|\frac{\phi_1}{\phi}\|_{L_*^{\alpha}(0,1)} + \|\frac{K(\cdot,a)}{\phi_0}\|_{L_*^{q_0}(1,\infty)} \|\frac{\phi_0}{\phi}\|_{L_*^{\beta}(1,\infty)} \\ &\leq C(\|a\|_{K_{q_1}^{\phi_1}(\mathbf{A})} + \|a\|_{K_{q_0}^{\phi_0}(\mathbf{A})}) \leq C\|a\|_{K_{q_0}^{\phi_0}(\mathbf{A}) \cap K_{q_1}^{\phi_1}(\mathbf{A})}, \end{aligned}$$

hence the conclusion. □

Proposition 2.3.10. *Given $\phi_0, \phi_1 \in \mathcal{B}$ such that $\overline{b}(\phi_0) < \underline{b}(\phi_1)$ and $q \in [1, \infty]$, if $A_1 \hookrightarrow A_0$ then $K_q^{\phi_1}(\mathbf{A}) \hookrightarrow K_q^{\phi_0}(\mathbf{A})$.*

Proof. As there exists a constant $C_0 > 0$ such that $\|a\|_{A_0} \leq C\|a\|_{A_1}$ for all $a \in A_1$, for $t > C_0$, we have $\|a\|_{A_0} = K(t, a)$ for all $a \in A_1$. Therefore, we have

$$\|a\|_{K_q^{\phi_0}(\mathbf{A})} \leq \|\frac{K(\cdot,a)}{\phi_1}\|_{L_*^q(0,C_0)} \|\frac{\phi_1}{\phi_0}\|_{L_*^q(0,C_0)} + C\|a\|_{A_0} \leq C\|a\|_{K_q^{\phi_1}(\mathbf{A})}.$$

□

Proposition 2.3.11. *Let $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1, \infty]$; if A_0 and A_1 are complete, so is $K_q^{\phi}(\mathbf{A})$.*

Proof. Let $(a_j)_{j \in \mathbb{N}}$ be a sequence of $K_q^{\phi}(\mathbf{A})$ such that $\sum_j \|a_j\|_{K_q^{\phi}} < \infty$. We have $K(1, a_j) \leq C\|a_j\|_{K_q^{\phi}(\mathbf{A})}$ for all j and since $\Sigma(\mathbf{A})$ is complete, $\sum_j a_j$ converges to an element a in this space. Consequently, for any $j_0 \in \mathbb{N}$ larger than 2,

$$\|a - \sum_{j<j_0} a_j\|_{K_q^{\phi}(\mathbf{A})} = \|\sum_{j \geq j_0} a_j\|_{K_q^{\phi}(\mathbf{A})} \leq \sum_{j \geq j_0} \|a_j\|_{K_q^{\phi}(\mathbf{A})},$$

which tends to 0 as j_0 tends to infinity. □

2.4 The Generalized J-Method

We can also consider the J-method to get a second family of explicit interpolation functors.

Let us recall that given a couple $\mathbf{A}$ and $t > 0$,

$$J(t, a) = J(t, a; \mathbf{A}) := \max\{(\|a\|_{A_0}, t\|a\|_{A_1}\},$$

for $a \in \Delta(\mathbf{A})$.

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $J_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ which can be represented by $a = \int_0^\infty b(t)\, dt/t$, with convergence in $\Sigma(\mathbf{A})$, where b is measurable, takes its values in $\Delta(\mathbf{A})$ for $t > 0$ and

$$t \mapsto \frac{J(t, b(t))}{\phi(t)} \in L_*^q. \tag{2.4}$$

This space is equipped with the norm

$$\|a\|_{J_q^\phi(\mathbf{A})} := \inf_b \|\frac{J(t, b(t))}{\phi(t)}\|_{L_*^q}.$$

the infimum being taken on all $b : (0, \infty) \to \Delta(\mathbf{A})$ measurable such that $a = \int_0^\infty b(t)\, dt/t$.

Theorem 2.4.1. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1, \infty]$, J_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$. Moreover, for all $a \in \Delta(\mathbf{A})$, we have*

$$\|a\|_{J_q^\phi(\mathbf{A})} \le C \frac{J(t, a)}{\phi(t)}.$$

Proof. Although an explicit proof could be provided, the Equivalence Theorem 2.4.3 allows us to directly conclude the first part. For the inequality, it comes from the fact that any $a \in \Delta(\mathbf{A})$ has the representation

$$a = \frac{1}{\log 2} \int_0^\infty a\, \chi_{(1,2)}\, \frac{dt}{t}.$$

□

It is well known that the equivalence Theorem still holds [42]; however, as we use here slightly different arguments, we sketch a proof.

Lemma 2.4.2. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1, \infty]$, we have*

$$J_q^\phi(\mathbf{A}) \hookrightarrow K_q^\phi(\mathbf{A}).$$

Proof. Let a be an element of $J_q^\phi(\mathbf{A})$, so that $a = \int_0^\infty b(s) ds/s$ with condition (2.4) satisfied. From the trivial decomposition $b = b + 0 = 0 + b$, we get $b \in \Sigma(\bar{A})$ and we have

$$K(t, a) \le \int_0^\infty \min\{\|b\|_{A_0}, t\|b\|_{A_1}\}\, \frac{ds}{s}.$$

We get

$$\frac{K(t,a)}{\phi(t)} \le \int_0^\infty \min\{(\frac{\phi(t)}{\phi(s)})^{-1}, \frac{t}{s}(\frac{\phi(t)}{\phi(s)})^{-1}\} \frac{J(s,b(s))}{\phi(s)} \frac{ds}{s}$$
$$\le \int_0^\infty \min\{\bar{\phi}(\frac{s}{t}), \frac{t}{s}\bar{\phi}(\frac{s}{t})\} \frac{J(s,b(s))}{\phi(s)} \frac{ds}{s}.$$

The last expression is a convolution product (for the multiplicative group $(0,\infty)$ and the Haar measure ds/s) of the function $s \mapsto J(s,b(s))/\phi(s)$ from L^q_* and $s \mapsto \min\{\bar{\phi}(1/s), \bar{\phi}(1/s)s\}$. This last function belongs to L^1_* if $0 < \underline{b}(\phi)$ and $\bar{b}(\phi) < 1$. By Young's inequality, we get

$$\|\frac{K(t,a)}{\phi(t)}\|_{L^q_*} \le C\|\frac{J(t,b(t))}{\phi(t)}\|_{L^q_*},$$

which is sufficient to conclude. □

Theorem 2.4.3. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\bar{b}(\phi) < 1$ and $q \in [1,\infty]$, we have*

$$J^\phi_q(\mathbf{A}) = K^\phi_q(\mathbf{A}).$$

Proof. Let a be an element of $K^\phi_q(\mathbf{A})$; one has $K(t,a) \le C\phi(t)$ for any $t > 0$. For $j \in \mathbb{Z}$, let $a_0^{(j)} \in A_0$ and $a_1^{(j)} \in A_1$ be such that $a = a_0^{(j)} + a_1^{(j)}$ and

$$\|a_0^{(j)}\|_{A_0} + \|a_1^{(j)}\|_{A_1} \le 2K(e^j,a).$$

Since ϕ is a Boyd function, we have

$$0 \le \|a_0^{(j)}\|_{A_0} \le C\phi(e^j) \le C(e^j)^{\underline{b}(\phi)-\underline{b}(\phi)/2},$$

where the right-hand side tends to 0 as j tends to $-\infty$. On the other hand,

$$0 \le \|a_1^{(j)}\|_{A_1} \le C(e^j)^{\bar{b}(\phi)+\varepsilon}e^{-j}$$

tends to 0 as j tends to ∞.

For $j \in \mathbb{Z}$, let us set

$$b_j := a_0^{(j+1)} - a_0^{(j)} = a_1^{(j)} - a_1^{(j+1)} \in \Delta(\mathbf{A}),$$

so that $\sum_{j\in\mathbb{Z}} b_j = a$ with convergence in $\Sigma(\mathbf{A})$. For $t \in (e^j, e^{j+1})$, we get

$$\|b_j\|_{A_0} \le 2K(e^{j+1},a) + 2K(e^j,a)$$

and so $t\|b_j\|_{A_1} \le CK(t,a)$. Finally, by setting $b(t) = b_j$ for $t \in (e^j, e^{j+1})$, we get $a = \int_0^\infty b(t)dt/t$ and thus $J(t,a) \le CK(t,a)$. □

Remark 2.4.4. One can check that a belongs to $J^\phi_q(\mathbf{A})$ if and only if $a = \sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\mathbf{A})$ with $b_j \in \Delta(\mathbf{A})$ for all j and $\big(J(2^j,b_j)/\phi(2^j)\big)_{j\in\mathbb{Z}}$ belongs to ℓ^q.

Considering the classical results, we get the following properties.

Proposition 2.4.5. *For* $\phi \in \mathcal{B}$ *such that* $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ *and* $q \in [1,\infty)$, $\Delta(\mathbf{A})$ *is dense in* $K_q^\phi(\mathbf{A})$*.*

Proof. For $a \in K_q^\phi(\mathbf{A})$, let us write $a = \sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\mathbf{A})$ with $b_j \in \Delta(\mathbf{A})$ and

$$\big(J(2^j, b_j)/\phi(2^j)\big)_{j\in\mathbb{Z}} \in \ell^q.$$

We have

$$\|a - \sum_{|j|<j_0} b_j\|^q_{K_q^\phi(\mathbf{A})} \leq \sum_{|j|\geq j_0} (\frac{J(2^j,a)}{\phi(2^j)})^q,$$

for any $j_0 \in \mathbb{N}$. □

Remark 2.4.6. When $q = \infty$, $\Delta(\mathbf{A})$ is not dense in $K_\infty^\phi(\mathbf{A})$ (see Proposition 3.1.5).

The interpolation functors J_1^ψ and K_∞^ψ are extremal (in the sense of Theorem 2.4.7), using the appropriate function ψ.

Theorem 2.4.7. *If* F *is an interpolation functor of exponent* $\phi \in \mathcal{B}$ *with* $\underline{b}(\phi) > 0$ *and* $\overline{b}(\phi) < 1$*, then, for any compatible Banach couple* $\mathbf{A} = (A_0, A_1)$*, one has*

$$J_{1/\bar{\phi}(1/\cdot),1}(\mathbf{A}) \hookrightarrow F(\mathbf{A}).$$

Moreover, if $\Delta(\mathbf{A})$ *is dense in* A_0 *and* A_1*, then*

$$F(\mathbf{A}) \hookrightarrow K_{\bar{\phi},\infty}(\mathbf{A}).$$

Proof. For $b \in \Delta(\mathbf{A})$, let us define

$$T_b : \mathbb{C} \to F(\mathbf{A}) \,:\, z \mapsto zb,$$

so that

$$\begin{aligned}\|b\|_{F(\mathbf{A})} &= \|T_b\|_{\mathbb{C},F(\mathbf{A})}\\ &\leq C\bar{\phi}_*(\|T_b\|_{\mathbb{C},A_0})\bar{\phi}(\|T_b\|_{\mathbb{C},A_1}) = C\bar{\phi}_*(\|b\|_{A_0})\bar{\phi}(\|b\|_{A_1}).\end{aligned}$$

We can thus write

$$\begin{aligned}\|b\|_{F(\mathbf{A})} &\leq C\bar{\phi}_*(\|b\|_{A_0})\bar{\phi}(\|2^j b/2^j\|_{A_1})\\ &\leq C\bar{\phi}_*(J(2^j,b))\bar{\phi}(J(2^j,b))\bar{\phi}(1/2^j) \leq CJ(2^j,b)\,\bar{\phi}(1/2^j).\end{aligned}$$

Therefore, for $a = \sum_{j\in\mathbb{Z}} b_j$, with $b_j \in \Delta(\mathbf{A})$, we have

$$\|a\|_{F(\mathbf{A})} \leq C\sum_{j\in\mathbb{Z}} J(2^j, b_j)\,\bar{\phi}(1/2^j),$$

so that we can write $J_{1/\bar{\phi}(1/\cdot),1}(\mathbf{A}) \hookrightarrow F(\mathbf{A})$. Now, let us suppose that $\Delta(\mathbf{A})$ is dense in both A_0 and A_1, let a' be an element of $\Delta(\mathbf{A}')$ and set $Ta = \langle a', a\rangle$. We have

$$\begin{aligned}\|T\|_{A,\mathbb{C}} &\leq C\bar{\phi}_*(\|T\|_{A_0,\mathbb{C}})\bar{\phi}(\|T\|_{A_1,\mathbb{C}}) = C\bar{\phi}_*(\|a'\|_{A_0'})\bar{\phi}(\|a'\|_{A_1'}) \\ &\leq C\bar{\phi}(t)J(1/t, a'; \mathbf{A}'),\end{aligned}$$

so that

$$\frac{|\langle a', a\rangle|}{J(1/t, a'; \mathbf{A}')} \leq C\bar{\phi}(t)\|a\|_{F(\mathbf{A})}.$$

Since $\Sigma(\mathbf{A})' = \Delta(\mathbf{A}')$ (see Lemma 2.4.8), we get $K(t, a) \leq C\bar{\phi}(t)\|a\|_{F(\mathbf{A})}$, as desired. □

Let us also recall the duality Theorem, which is also well-known in this context (and can be easily obtained from our equivalence Theorem and from the proofs in [15], Section 3.7). First, we need a lemma from [15].

Lemma 2.4.8. *Suppose that $\Delta(\mathbf{A})$ is dense in both A_0 and A_1. Then $\Delta(\mathbf{A})' = \Sigma(\mathbf{A}')$ and $\Sigma(\mathbf{A})' = \Delta(\mathbf{A}')$, with $\mathbf{A}' = (A_0', A_1')$. More precisely,*

$$\|a'\|_{\Sigma(\mathbf{A}')} = \sup_{a\in\Delta(\mathbf{A})} \frac{|\langle a', a\rangle|}{\|a\|_{\Delta(\mathbf{A})}}$$

and

$$\|a'\|_{\Delta(\mathbf{A}')} = \sup_{a\in\Sigma(\mathbf{A})} \frac{|\langle a', a\rangle|}{\|a\|_{\Sigma(\mathbf{A})}}.$$

From this result, one can remark that if $\Delta(\mathbf{A})$ is dense in both A_0 and A_1, then

$$K^{\mathbf{A}'}(t, a') = \sup_{a\in\Delta(\mathbf{A})} \frac{|\langle a', a\rangle|}{J(1/t, a)}, \tag{2.5}$$

and

$$J^{\mathbf{A}'}(t, a') = \sup_{a\in\Sigma(\mathbf{A})} \frac{|\langle a', a\rangle|}{K(1/t, a)}. \tag{2.6}$$

Theorem 2.4.9. *Let $\mathbf{A} = (A_0, A_1)$ be a couple of Banach spaces such that $\Delta(\mathbf{A})$ is dense in both A_0 and A_1; for $1 \leq q < \infty$, $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, we have*

$$K_q^{\phi}(\mathbf{A})' = K_{q'}^{\check{\phi}}(\mathbf{A}'),$$

where q' is the exponent conjugate to q and $\check{\phi} = 1/\phi(1/t)$.

Proof. If we show that

$$J_q^{\phi}(\mathbf{A})' \hookrightarrow K_{q'}^{\gamma}((A_1', A_0')) \tag{2.7}$$

and

$$K_q^{\phi}(\mathbf{A})' \hookleftarrow J_{q'}^{\gamma}((A_1', A_0')), \tag{2.8}$$

where $\gamma(t) = t/\phi(t)$, then we have the assertion since

$$K_q^\phi(\mathbf{A})' = J_q^\phi(\mathbf{A})' \hookrightarrow K_{q'}^\gamma((A_1', A_0')) = K_{q'}^{\check{\phi}}(\mathbf{A}') = J_{q'}^\gamma((A_1', A_0')) \hookrightarrow K_q^\phi(\mathbf{A})'.$$

• Let $a' \in J_q^\phi(\mathbf{A})'$, by using (2.5), given $\varepsilon > 0$, one can find $b_j \in \Delta(\mathbf{A})\backslash\{0\}$ such that

$$K^{\mathbf{A}'}(2^{-j}, a') - \varepsilon \min\{1, 2^{-j}\} \leq \frac{\langle a', b_j\rangle}{J(2^j, b_j)}.$$

Let $\alpha = (\alpha_j)_j \in l_{1/\phi}^q$ and set

$$a_\alpha = \sum_{j\in\mathbb{Z}} \frac{1}{J(2^j, b_j)} \alpha_j b_j.$$

Since $\|a_\alpha\|_{J_q^\phi(\mathbf{A})} \leq \|\alpha\|_{l_{1/\phi}^q}$, $a_\alpha \in J_q^\phi(\mathbf{A})$,

$$\langle a', a_\alpha\rangle \leq \|\alpha\|_{l_{1/\phi}^q} \|a'\|_{J_q^\phi(\mathbf{A})'},$$

and

$$\langle a', a_\alpha\rangle \geq \sum_{j\in\mathbb{Z}} \left(K^{\mathbf{A}'}(2^{-j}, a') - \varepsilon \min\{1, 2^{-j}\}\right) \alpha_j.$$

Therefore,

$$\sum_{j\in\mathbb{Z}} \frac{1}{2^j} \alpha_j \left(K^{(A_1', A_0')}(2^j, a') - \varepsilon \min\{1, 2^j\}\right) \leq \|\alpha\|_{l_{1/\phi}^q} \|a'\|_{J_q^\phi(\mathbf{A})'}.$$

Since $\ell_{1/\phi}^q$ and $\ell_{1/\gamma}^q$ are dual with the duality $\sum_{j\in\mathbb{Z}} 2^{-j}\alpha_j\beta_j$ and ε is arbitrary, (2.7) follows.

• Let $a' \in J_{q'}^\gamma((A_1', A_0'))$, $a = \sum_{j\in\mathbb{Z}} b_j'$ with convergence in $\Sigma(\mathbf{A}') = \Delta(\mathbf{A})'$. Thus, by using (2.6), one has

$$|\langle a', a\rangle| \leq \sum_{j\in\mathbb{Z}} |\langle b_j', a\rangle| \leq \sum_{j\in\mathbb{Z}} J^{\mathbf{A}'}(2^{-j}, b_j') K(2^j, a),$$

hence (2.8) by Hölder's inequality.

□

Remark 2.4.10. For the case $q = \infty$, see Proposition 3.1.6.

Let us also recall the power Theorem, which is also well-known in this context (and can be easily obtained from the proofs in [15], Section 3.11). Let us remind ourselves that if $\|\cdot\|$ is a quasi-norm on A, then $\|\cdot\|^\alpha$ is also a quasi-norm for $\alpha > 0$. We denote the space equipped with this functional by $A^{(\alpha)}$.

Theorem 2.4.11. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$K_q^\phi(A_0^{(\alpha)}, A_1^{(\alpha)})^{(1/\alpha)} = K_{\alpha q}^{\phi_\alpha}(\mathbf{A}),$$

where $\phi_\alpha(t) = \phi(t^\alpha)^{1/\alpha}$.

Proof. It is proved in a more general context in Subsection 4.1.3. □

2.5 Other real Interpolation Methods

As expected, the "espaces de moyennes" [76] and the trace spaces ("espaces de trace") [75] can be generalized in the context of the Boyd functions. The induced methods are equivalent to the K-method.

Given a compatible Banach couple $\mathbf{A} = (A_0, A_1)$ and $q \in [1, \infty]$, let $X_q^\phi(\mathbf{A})$ be the subspace of $\Sigma(\mathbf{A})$ defined by the norm

$$\|a\|_{X_q^\phi(\mathbf{A})} := \inf_{a=a_0(t)+a_1(t)} \{\| \frac{a_0(t)}{\phi(t)} \|^q_{L^q_*(A_0)} + \|\phi_*(t) a_1(t)\|^q_{L^q_*(A_1)}\}^{1/q}.$$

Theorem 2.5.1. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$X_q^\phi(\mathbf{A}) = K_q^\phi(\mathbf{A}).$$

Proof. One has

$$\begin{aligned} \|a\|^q_{X_q^\phi(\mathbf{A})} &\asymp \int_0^\infty \inf_{a=a_0(t)+a_1(t)} \{ \frac{1}{\phi(t)^q} \|a_0(t)\|^q_{A_0} + \phi_*(t)^q \|a_1(t)\|^q_{A_1} \} \frac{dt}{t} \\ &\asymp \int_0^\infty \inf_{a=\tilde{a}_0(\tau)+\tilde{a}_1(\tau)} \{ \frac{1}{\phi(\tau^{1/q})^q} (\|\tilde{a}_0(\tau)\|^q_{A_0} + \tau \|\tilde{a}_1(\tau)\|^q_{A_1}) \} \frac{d\tau}{\tau}. \end{aligned}$$

Using the power Theorem, we get

$$X_q^\phi(\mathbf{A})^q = K_1^{\phi_{1/p}}(A_0^q, A_1^q) = K_q^\phi(\mathbf{A})^q,$$

as desired. □

If f is an A-valued function on $(0, \infty)$, $f^{(m)}$ will denote the derivative of order m of f in the sense of the distribution theory. The space $X^\phi_{q,m}(\mathbf{A})$ is the space of $\Sigma(\mathbf{A})$-valued functions f on $(0, \infty)$ that are locally A_0-integrable and such that $f^{(m)}$ is locally A_1-integrable with

$$\|f\|_{X^\phi_{q,m}(\mathbf{A})} := \max\{\|\phi f\|_{L^q_*(A_0)}, \| \frac{1}{\phi_*(t)} f^{(m)}(t) \|_{L^q_*(A_1)}\} < \infty.$$

We shall say that f has a trace in $\Sigma(\mathbf{A})$ if $f(t)$ converges in $\Sigma(\mathbf{A})$ as $t \to 0^+$; in this case we set

$$\mathrm{trace}(f) := \lim_{t \to 0^+} f(t).$$

The trace space of functions in $X^\phi_{q,m}(\mathbf{A})$ will be denoted $TX^\phi_{q,m}(\mathbf{A})$; it is the space of all $a \in \Sigma(\mathbf{A})$ such that there exists $f \in X^\phi_{q,m}(\mathbf{A})$ with $\mathrm{trace}(f) = a$. This space is a Banach space for the norm

$$\|a\|_{TX^\phi_{q,m}(\mathbf{A})} := \inf_{\mathrm{trace}(f)=a} \|f\|_{X^\phi_{q,m}(\mathbf{A})}.$$

Theorem 2.5.2. *For $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, we have*

$$TX^\phi_{q,m}(\mathbf{A}) = K^\phi_q(\mathbf{A}).$$

Proof. This result can be shown using the same proof as for the corresponding Theorem of [15], Section 3.12. □

2.6 A Reiteration Theorem

We give here a stability result for the repeated use of the real interpolation method.

Let us recall (see [88]) that given $\psi := \phi_1/\phi_0$ with $\phi_0, \phi_1 \in \mathcal{B}$, $q_0, q_1 \in [1, \infty]$, $E_0 := K_{q_0}^{\phi_0}(\mathbf{A})$ and $E_1 := K_{q_1}^{\phi_1}(\mathbf{A})$, if $\underline{b}(\psi) > 0$, there exists a non-decreasing bijection $\xi \in \mathcal{B}'$ such that $\psi \asymp \xi$ and

$$K(t, a; \mathbf{E}) \asymp \|\frac{K(\cdot, a)}{\phi_0}\|_{L_*^{q_0}(0,\xi^{-1}(t))} + t\|\frac{K(\cdot, a)}{\phi_1}\|_{L_*^{q_1}(\xi^{-1}(t),\infty)},$$

with $\mathbf{E} = (E_0, E_1)$ as soon as both the following conditions are satisfied:

- $\underline{b}(\phi_0) > 0$ if $q_0 < \infty$ or $\sup_{t \le 1} \bar{\phi}_0(t) < \infty$ if $q_0 = \infty$,
- $\bar{b}(\phi_1) < 1$ if $q_1 < \infty$ or $\sup_{t \ge 1} \bar{\phi}_1(t)/t < \infty$ if $q_1 = \infty$.

Let us recall the following notions. Let $\mathbf{A}$ be a couple of normed vector spaces and $\phi \in \mathcal{B}$; if X is an intermediate spaces with respect to $\mathbf{A}$, X is of class $\mathcal{C}_K(\phi; \mathbf{A})$ if

$$K(t, a) \le C\phi(t)\|a\|_X$$

for all $a \in X$. In the same way, X is of class $\mathcal{C}_J(\phi; \mathbf{A})$ if

$$\phi(t)\|a\|_X \le CJ(t, a)$$

for all $a \in \Delta(\mathbf{A})$. Finally, X is of class $\mathcal{C}(\phi; \mathbf{A})$ if it is both of class $\mathcal{C}_K(\phi; \mathbf{A})$ and $\mathcal{C}_J(\phi; \mathbf{A})$. For example, if $0 < \underline{b}(\phi)$ and $\bar{b}(\phi) < 1$, then $K_q^{\phi}(\mathbf{A})$ is of class $\mathcal{C}(\phi; \mathbf{A})$.

Proposition 2.6.1. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$; X is of class $\mathcal{C}_K(\phi; \mathbf{A})$ if and only if $\Delta(\mathbf{A}) \hookrightarrow X \hookrightarrow K_\infty^{\phi}(\mathbf{A})$.*

Proof. This is clear since we have $X \hookrightarrow K_\infty^{\phi}(\mathbf{A})$ if and only if

$$\sup_{t>0} \frac{K(t, a)}{\phi(t)} \le C\|a\|_X,$$

X being an intermediate space with respect to $\mathbf{A}$. □

Proposition 2.6.2. *Let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$; a Banach space X is of class $\mathcal{C}_J(\phi; \mathbf{A})$ if and only if $K_1^{\phi}(\mathbf{A}) \hookrightarrow X \hookrightarrow \Sigma(\mathbf{A})$.*

Proof. Let us suppose that X is Banach space of class $\mathcal{C}_J(\phi; \mathbf{A})$; for $a = \sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\mathbf{A})$, we have

$$\|a\|_X \le \sum_{j\in\mathbb{Z}} \|b_j\|_X \le C \sum_{j\in\mathbb{Z}} \frac{J(2^j, a)}{\phi(2^j)},$$

so that $K_1^{\phi}(\mathbf{A})$ is included in X.

On the other hand, if $K_1^\phi(\mathbf{A})$ is included in X, let m be an integer and set $b_m = a$ and $b_j = 0$ for $j \neq m$. In this case, we have

$$\|a\|_X \leq C\|a\|_{K_1^\phi(\mathbf{A})} = C\frac{J(2^m, a)}{\phi(2^m)},$$

so that X is of class $\mathcal{C}_J(\phi; \mathbf{A})$. □

Let us now give a generalization of the Reiteration Theorem from [15]. This reveals a certain stability when repeatedly applying the generalized method K.

Theorem 2.6.3. *If for $j \in \{0,1\}$, X_j is a Banach space of class $\mathcal{C}(\phi_j; \mathbf{A})$ with $\underline{b}(\phi_j) \geq 0$ and $\overline{b}(\phi_j) \leq 1$, let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$ and set $\theta = \phi_1/\phi_0$, $\psi = (\phi \circ \theta)\phi_0$; if $\underline{b}(\theta) > 0$ or $\overline{b}(\theta) < 0$ then*

$$K_q^\phi(\mathbf{X}) = K_q^\psi(\mathbf{A}).$$

In particular, if for $\underline{b}(\phi_j) > 0$ and $\overline{b}(\phi_j) < 1$, the spaces $K_{q_j}^{\phi_j}(\mathbf{A})$ are complete ($j \in \{0,1\}$), then

$$K_q^\phi(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) = K_q^\psi(\mathbf{A}).$$

Proof. For $a = a_0 + a_1 \in K_q^\phi(\mathbf{X})$, we have

$$K(t, a; \mathbf{A}) \leq C\phi_0(t)K(\theta(t), a; \mathbf{X}).$$

Therefore,

$$\|a\|_{K_q^\psi(\mathbf{A})} \leq C(\int_0^\infty (\frac{K(\phi_1(t)/\phi_0(t), a; \mathbf{X})}{\psi(t)/\phi_0(t)})^q \frac{dt}{t})^{1/q},$$

so that for $s = \theta(t)$, we get

$$\|a\|_{K_q^\psi(\mathbf{A})} \leq C(\int_0^\infty \frac{K(s, a; \mathbf{X})}{\phi(s)})^q \frac{ds}{s})^{1/q}$$

and thus $K_q^\phi(\mathbf{X}) \hookrightarrow K_q^\psi(\mathbf{A})$.

Now, for $a = \int_0^\infty b(t)dt/t \in J_q^\phi(\mathbf{X})$, we have

$$\begin{aligned}
\phi_0(t)\, K(\theta(t), a; \mathbf{X}) &\leq \int_0^\infty \phi_0(t)\, K(\theta(t), b(s); \mathbf{X})\, \frac{ds}{s} \\
&\leq \int_0^\infty \phi_0(t)\, \min\{1, \frac{\theta(t)}{\theta(s)}\}\, J(\theta(s), b(s); \mathbf{X})\, \frac{ds}{s} \\
&\leq C \int_0^\infty \min\{\bar{\phi}_0(t/s), \bar{\phi}_1(t/s)\}\, J(s, b(s); \mathbf{A})\, \frac{ds}{s},
\end{aligned}$$

so that for $u = \theta(t)$ and $s = \sigma t$, we get

$$\begin{aligned}
\|a\|_{K_q^\phi(\mathbf{X})} &\leq C\Big(\int_0^\infty \big(\frac{\phi_0(t)K(\theta(t),a;\mathbf{X})}{\phi_0(t)\phi(\theta(t))}\big)^q \frac{dt}{t}\Big)^{1/q} \\
&\leq C\Big(\int_0^\infty \big(\int_0^\infty \frac{1}{\psi(s/\sigma)} \min\{\bar{\phi}_0(1/\sigma), \bar{\phi}_1(1/\sigma)\}\, J(s,b(s);\mathbf{A})\, \frac{ds}{s}\big)^q \frac{d\sigma}{\sigma}\Big)^{1/q} \\
&\leq C\|\frac{J(s,b(s);\mathbf{A})}{\psi(s)}\|_{L_\star^q}
\end{aligned}$$

and thus, using the equivalence Theorem, $K_q^\psi(\mathbf{A}) \hookrightarrow K_q^\phi(\mathbf{X})$. □

Remark 2.6.4. Reiteration Theorem can be expressed with this equivalent form (see Proposition 3.3.9) :
If for $j \in \{0,1\}$, X_j is of class $\mathcal{C}(\phi_j;\mathbf{A})$ with $\underline{b}(\phi_j) \geq 0$ and $\overline{b}(\phi_j) \leq 1$, let $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$ and set $\theta = \phi_0/\phi_1$, $\psi = \phi_0/(\phi \circ \theta)$; if $\underline{b}(\theta) > 0$ or $\overline{b}(\theta) < 0$ then

$$K_q^\phi(\mathbf{X}) = K_q^\psi(\mathbf{A}).$$

In particular, if for $\underline{b}(\phi_j) > 0$ and $\overline{b}(\phi_j) < 1$, the spaces $K_{q_j}^{\phi_j}(\mathbf{A})$ are complete ($j \in \{0,1\}$), then

$$K_q^\phi(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) = K_q^\psi(\mathbf{A}).$$

2.7 A Compactness Theorem

Using the previous reiteration Theorem, one can show that the classical compactness Theorem [15] still holds in the setting of Boyd functions. A similar result can be obtained with bilinear operators [38].

Theorem 2.7.1. *Let $\mathbf{A}$ be a couple of Banach spaces, B be a Banach space and consider a bounded linear operator T such that $T : A_0 \to B$ is compact and $T : A_1 \to B$ (not necessarily compact); if $E \in \mathcal{C}_K(\phi;\mathbf{A})$ for some $\phi \in \mathcal{B}$ such that $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) < 1$, then $T : E \to B$ is also compact.*

Proof. Let $(a_k)_{k\in\mathbb{N}}$ be a bounded sequence in A_0 such that $\|a\|_E \leq 1$. Given $\varepsilon > 0$, let $t_\varepsilon > 0$ be such that $\phi(t_\varepsilon) < \varepsilon t_\varepsilon$. Now, let us choose $a_k^{(0)} \in A_0$ and $a_k^{(1)} \in A_1$ such that $a_k = a_k^{(0)} + a_k^{(1)}$ and

$$\|a_k^{(0)}\|_{A_0} + t_\varepsilon \|a_k^{(1)}\|_{A_1} \leq 2K(t_\varepsilon, a_k) \leq C\phi(t_\varepsilon)\|a_k\|_E,$$

so that $a_k^{(0)}$ is a bounded sequence in A_0. Since $T : A_0 \to B$ is compact, there exists a subsequence $(a_{l(k)}^{(0)})_{k\in\mathbb{N}}$ of $(a_k^{(0)})_{k\in\mathbb{N}}$ such that

$$\|Ta_{l(k_1)}^{(0)} - Ta_{l(k_2)}^{(0)}\|_B \leq \varepsilon,$$

for k_1 and k_2 sufficiently large. Since we also have

$$\|Ta^{(0)}_{l(k_1)} - Ta^{(0)}_{l(k_2)}\|_B \leq \|T\|_{A_1,B}\, \|a^{(1)}_{l(k_1)} - a^{(1)}_{l(k_2)}\|_{A_1} \leq C\|T\|_{A_1,B} \frac{\phi(t_\varepsilon)}{t_\varepsilon},$$

we get

$$\|Ta_{l(k_1)} - Ta_{l(k_2)}\|_B \leq \varepsilon(1 + C\|T\|_{A_1,B}),$$

which proves that $T : E \to B$ is compact. □

In the same way, we get the following Theorem.

Theorem 2.7.2. *Let* $\mathbf{A}$ *be a couple of Banach spaces,* B *be a Banach space and consider a bounded linear operator* T *such that* $T : B \to A_0$ *is compact and* $T : B \to A_1$ *(not necessarily compact); if* $E \in \mathcal{C}_J(\phi; \mathbf{A})$ *for some* $\phi \in \mathcal{B}$ *such that* $\underline{b}(\phi) > 0$ *and* $\overline{b}(\phi) < 1$*, then* $T : B \to E$ *is also compact.*

Proof. Let $(b_k)_{k\in\mathbb{N}}$ be a sequence in B such that $\|b_k\|_B \leq 1$. Given $\varepsilon > 0$, let $t_\varepsilon > 0$ be such that $t_\varepsilon < \varepsilon\phi(t_\varepsilon)$. There exists a subsequence $(b_{l(k)})_{k\in\mathbb{N}}$ of $(b_k)_{k\in\mathbb{N}}$ such that $\|Tb_{l(k_1)} - Tb_{l(k_2)}\|_{A_0} \leq t_\varepsilon$ for k_1 and k_2 sufficiently large. We then have

$$\begin{aligned}\phi(t_\varepsilon)\|Tb_{l(k_1)} - Tb_{l(k_2)}\|_E &\leq J(t_\varepsilon, Tb_{l(k_1)} - Tb_{l(k_2)}) \\ &\leq Ct_\varepsilon \max\{1, 2\|T\|_{B,A_1}\} \leq C\varepsilon\phi(t_\varepsilon),\end{aligned}$$

so that $T : B \to E$ is compact. □

Corollary 2.7.3. *Let* $\phi_0, \phi_1 \in \mathcal{B}$ *be such that*

$$0 < \underline{b}(\phi_0) \leq \overline{b}(\phi_0) < \underline{b}(\phi_1) \leq \overline{b}(\phi_1) < 1;$$

if A_0 *and* A_1 *are two Banach spaces such that* $A_1 \hookrightarrow A_0$ *compactly, then*

$$K^{\phi_1}_{q_1}(\mathbf{A}) \hookrightarrow K^{\phi_0}_{q_0}(\mathbf{A}),$$

with compact inclusion.

Proof. Since the identity $A_1 \to A_0$ is compact, Theorem 2.7.1 implies $K^{\phi_1}_{q_1}(\mathbf{A}) \hookrightarrow A_0$ with compact inclusion. Now, from Theorem 2.7.2, we also have

$$K^{\phi_1}_{q_1}(\mathbf{A}) \hookrightarrow K^f_{q_0}(A_0, K^{\phi_1}_{q_1}(\mathbf{A})),$$

with compact inclusion. Since Theorem 2.6.3 implies

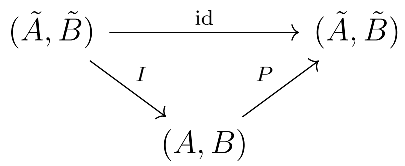
$$K^f_{q_0}(A_0, K^{\phi_1}_{q_1}(\mathbf{A})) = K^{\phi_0}_{q_0}(\mathbf{A}),$$

for f such that $\phi_0 = f \circ \phi_1$, we can conclude. □

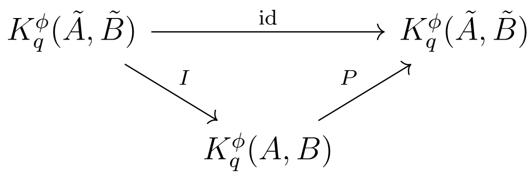
2.8 About Retracts

The proofs of the following results are straightforward. These results will be used to obtain the interpolation of Besov spaces through that of Lebesgue spaces (Theorem 3.2.9).

Lemma 2.8.1. *Let (A, B), $(\tilde{A}, \tilde{B})$ be compatible couples and suppose that $\tilde{A}$ (resp. $\tilde{B}$) is a retract of A (resp. B) with the same P and I, then $(\tilde{A}, \tilde{B})$ is a retract of (A, B).*

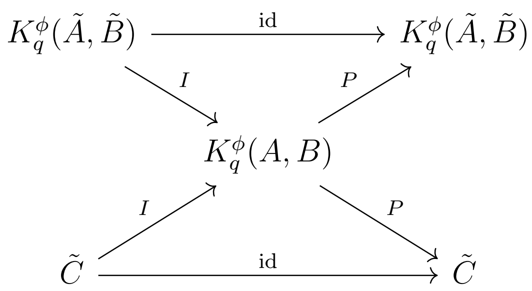


Lemma 2.8.2. *Let $q \in [1, \infty]$, $\phi \in \mathcal{B}$, if $(\tilde{A}, \tilde{B})$ is a retract of (A, B), then $K_q^\phi(\tilde{A}, \tilde{B})$ is a retract of $K_q^\phi(A, B)$.*



Proposition 2.8.3. *Let $q \in [1, \infty]$, $\phi \in \mathcal{B}$, (A, B), $(\tilde{A}, \tilde{B})$ be compatible couples and suppose that $\tilde{A}$ (resp. $\tilde{B}$, resp. $\tilde{C}$) is a retract of A (resp. B, resp. $K_q^\phi(A, B)$) with the same P and I, then*

$$K_q^\phi(\tilde{A}, \tilde{B}) = \tilde{C}.$$



2.9 About Complex Interpolation

The space $\mathfrak{F}(\mathbf{A})$ is the space of all $\Sigma(\mathbf{A})$-valued functions f, which are bounded and continuous on the strip

$$S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$$

and analytic on the open strip

$$S^\circ = \{z \in \mathbb{C} : 0 < \Re z < 1\},$$

and moreover, the function $t \in \mathbb{R} \mapsto f(it) \in A_0$ (resp. $t \in \mathbb{R} \mapsto f(1+it) \in A_1$) is continuous and tends to zero as $|t| \to \infty$. We provide $\mathfrak{F}(\mathbf{A})$ with the norm

$$\|f\|_{\mathfrak{F}(\mathbf{A})} = \max\{\sup \|f(it)\|_{A_0}, \sup \|f(1+it)\|_{A_1}\}.$$

Then, $\mathfrak{F}(\mathbf{A})$ is a Banach space.
The space $C_\theta(\mathbf{A})$ consists of all $a \in \Sigma(\mathbf{A})$ such that $a = f(\theta)$ for some $f \in \mathfrak{F}(\mathbf{A})$. The norm on $C_\theta(\mathbf{A})$ is

$$\|a\|_{C_\theta(\mathbf{A})} := \inf\{\|f\|_{\mathfrak{F}(\mathbf{A})} : f(\theta) = a, f \in \mathfrak{F}(\mathbf{A})\}.$$

The functor C_θ is an exact interpolation functor of exponent θ (see [15]).

Proposition 2.9.1. *Let $\theta \in (0,1)$, then*

$$K_1^\theta(\mathbf{A}) \hookrightarrow C_\theta(\mathbf{A}) \hookrightarrow K_\infty^\theta(\mathbf{A}).$$

Proof. The first continuous embedding comes from Theorem 2.4.7 since C_θ is a functor of exponent ϕ with $\phi(t) = t^\theta$ $(t > 0)$ and $1/\bar{\phi}(1/t) = t^\theta$ for $t > 0$. The second one comes from the Phragmén-Lindelöf extension of the maximum principle (see [15]). □

One may observe that generalizing the complex interpolation method using Boyd's functions appears to be challenging, as only the exponent θ is involved, rather than the function $t \mapsto t^\theta$. Nonetheless, the following connections with the generalized real interpolation method can be established.

Theorem 2.9.2. *Let $q \in [1,\infty]$, $0 < \theta_0 < \theta_1 < 1$, $\gamma \in \mathcal{B}$ such that $\psi \in \mathcal{B}$ is defined by*

$$\psi(t) = \frac{t^{\theta_0}}{\gamma(t^{\theta_0 - \theta_1})}$$

for all $t > 0$. Then,

$$K_q^\gamma(C_{\theta_0}(\mathbf{A}), C_{\theta_1}(\mathbf{A})) = K_q^\psi(\mathbf{A}).$$

Proof. Thanks to Propositions 2.9.1, 2.6.1 and 2.6.2, it is clear that C_{θ_0} (resp. C_{θ_1}) is of class $\mathcal{C}(\theta_0; \mathbf{A})$ (resp. $\mathcal{C}(\theta_1; \mathbf{A})$). Therefore, using Reiteration Theorem 2.6.3, we get the conclusion. □

Theorem 2.9.3. *Let $q, q_0, q_1 \in [1,\infty]$, $\kappa \in (0,1)$, $\phi_0, \phi_1, \gamma \in \mathcal{B}$ with*

$$0 < \underline{b}(\phi_0) \leq \overline{b}(\phi_0) < \underline{b}(\phi_1) \leq \overline{b}(\phi_1) < 1$$

such that $\psi \in \mathcal{B}$ is defined by

$$\psi(t) = \phi_0(t)^{1-\kappa}\phi_1(t)^\kappa$$

for all $t > 0$. Suppose that

$$\frac{1}{q} = \frac{1-\kappa}{q_0} + \frac{\kappa}{q_1}.$$

Then,

$$C_\kappa(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) = K_q^\psi(\mathbf{A}).$$

Proof. • Let $a \in K_q^\psi(\mathbf{A})\backslash\{0\}$. There exists a decomposition $a = \sum_{j\in\mathbb{Z}} b_j$ in $\Sigma(\mathbf{A})$ with $b_j \in \Delta(\mathbf{A})$ and

$$(\sum_{j\in\mathbb{Z}}(\frac{J(2^j,b_j)}{\psi(2^j)})^q)^{1/q} \le C\|a\|_{K_q^\psi(\mathbf{A})}.$$

For $\delta > 0$ and $z \in S$, we set

$$f(z) = \exp(\delta(z-\kappa)^2)\sum_{j\in\mathbb{Z}} f_j(z),$$

where, for $j \in \mathbb{Z}$,

$$f_j(z) = b_j\left(\frac{\phi_1(2^j)}{\phi_0(2^j)}(\frac{J(2^j,b_j)}{\psi(2^j)\|a\|_{K_q^\psi(\mathbf{A})}})^{q(1/q_0-1/q_1)}\right)^{z-\kappa}.$$

Since $-\kappa q(1/q_0 - 1/q_1) = 1 - q/q_0$, one has

$$\begin{aligned}
|\exp(-\delta(it-\kappa)^2)|\|f(it)\|_{K_{q_0}^{\phi_0}(\mathbf{A})} &\le \sum_{j\in\mathbb{Z}}\left\|b_j\left(\frac{\phi_1(2^j)}{\phi_0(2^j)}(\frac{J(2^j,b_j)}{\psi(2^j)\|a\|_{K_q^\psi(\mathbf{A})}})^{q(1/q_0-1/q_1)}\right)^{it-\kappa}\right\|_{K_{q_0}^{\phi_0}(\mathbf{A})} \\
&= \sum_{j\in\mathbb{Z}}\left\|b_j\frac{\phi_1(2^j)^{-\kappa}}{\phi_0(2^j)^{-\kappa}}(\frac{J(2^j,b_j)}{\phi_0(2^j)^{1-\kappa}\phi_1(2^j)^\kappa\|a\|_{K_q^\psi(\mathbf{A})}})^{1-q/q_0}\right\|_{K_{q_0}^{\phi_0}(\mathbf{A})} \\
&\le C(\sum_{j\in\mathbb{Z}}(\frac{J(2^j,f_j(it))}{\phi_0(2^j)})^{q_0})^{1/q_0} \\
&\le C\|a\|_{K_q^\psi(\mathbf{A})}.
\end{aligned}$$

Since $(1-\kappa)q(1/q_0 - 1/q_1) = 1 - q/q_1$, we also get

$$|\exp(-\delta(1+it-\kappa)^2)|\|f(1+it)\|_{K_{q_1}^{\phi_1}(\mathbf{A})} \le C(\sum_{j\in\mathbb{Z}}(\frac{J(2^j,f_j(1+it))}{\phi_1(2^j)})^{q_1})^{1/q_1} \le C\|a\|_{K_q^\psi(\mathbf{A})}.$$

Therefore, $f \in \mathfrak{F}(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A}))$ with $f(\kappa) = a$ so that $C_\kappa(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) \hookleftarrow K_q^\psi(\mathbf{A})$.

• Let $a \in C_\kappa(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A}))$ and let $f \in \mathfrak{F}(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A}))$ with $f(\kappa) = a$. For $j \in \mathbb{Z}$ and $z \in S$, set

$$g_j(z) = \left(\frac{\phi_0(2^j)}{\phi_1(2^j)}\right)^{z-\kappa} 2^{\gamma(z-\kappa)} f(z).$$

Then, for all $j \in \mathbb{Z}$, $g_j \in \mathfrak{F}(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A}))$ and $g_j(\kappa) = a$. Let P_j $(j = 0,1)$ be the Poisson kernels for the strip S: explicitly, for $s, t \in \mathbb{R}$,

$$P_j(s+it,\tau) = \frac{e^{-\pi(\tau-t)}\sin(\pi s)}{\sin^2(\pi s) + (\cos(\pi s) - e^{-ij\pi-\pi(\tau-t)})^2}.$$

Using Cauchy integral formula in $\Sigma(\mathbf{A})$, one has

$$a = \int P_0(\kappa,\tau) g_j(i\tau)d\tau + \int P_1(\kappa,\tau) g_j(1+i\tau)d\tau.$$

Therefore, with an appropriate γ,

$$\begin{aligned}\frac{K(2^j,a)}{\psi(2^j)} &\leq \frac{1}{\psi(2^j)}\frac{\phi_0(2^j)^{-\kappa}}{\phi_1(2^j)^{-\kappa}}\, 2^{-\gamma\kappa}\int P_0(\kappa,\tau)K(2^j,f(i\tau))d\tau \\ &+ \frac{1}{\psi(2^j)}\frac{\phi_0(2^j)^{1-\kappa}}{\phi_1(2^j)^{1-\kappa}}\, 2^{\gamma(1-\kappa)}\int P_1(\kappa,\tau)K(2^j,f(1+i\tau))d\tau \\ &\leq C(\int P_0(\kappa,\tau)\frac{K(2^j,f(i\tau)}{\phi_0(2^j)}d\tau)^{1-\kappa}(\int P_1(\kappa,\tau)\frac{K(2^j,f(1+i\tau)}{\phi_1(2^j)}d\tau)^{\kappa}\end{aligned}$$

Using Hölder inequality, we get

$$\begin{aligned}\|a\|_{K_q^{\psi}(\mathbf{A})} &\leq C(\int P_0(\kappa,\tau)\|f(i\tau)\|^{q_0}_{K_{q_0}^{\phi_0}(\mathbf{A})}d\tau)^{(1-\kappa)/q_0}(\int P_1(\kappa,\tau)\|f(1+i\tau)\|^{q_1}_{K_{q_1}^{\phi_1}(\mathbf{A})}d\tau)^{\kappa/q_1} \\ &\leq C\|f\|_{\mathfrak{F}(K_{q_0}^{\phi_0}(\mathbf{A}),K_{q_1}^{\phi_1}(\mathbf{A}))},\end{aligned}$$

so that $C_\kappa(K_{q_0}^{\phi_0}(\mathbf{A}), K_{q_1}^{\phi_1}(\mathbf{A})) \hookrightarrow K_q^{\psi}(\mathbf{A})$. □

Chapter 3

Continuous Interpolation Spaces, Applications and Limiting cases

In the context of solving boundary value problems involving partial differential equations, the method of traces is frequently employed [40, 74, 94, 41]. This method is often applied to tackle problems with boundary value or non-homogeneous boundary conditions. Within this framework, trace Theorems provide intriguing interpolation methods [55, 75, 81, 107]. Let $K_\infty^{0,\theta}(A_0, A_1)$ denote the space of all $a \in K_\infty^\theta(A_0, A_1)$ such that

$$\lim_{t\to 0} \frac{K(t,a)}{t^\theta} = \lim_{t\to\infty} \frac{K(t,a)}{t^\theta} = 0.$$

The space is called a continuous interpolation space. These spaces are particularly valuable in contexts where asymptotic regularity is pivotal, such as the analysis of operators in weighted functional spaces, the study of solutions to partial differential equations with precise asymptotic behavior, or the theory of traces on manifolds. We explore the properties of those new spaces defined through a function parameter, which we shall refer to as generalized continuous interpolation spaces, and establish their interpretation within a functorial framework. Specifically, we investigate the density problem in the context of L^∞ spaces. We also present equivalent definitions and an adapted J-method in this context.

Additionally, we illustrate the versatility of these spaces by presenting a range of examples. For instance, the generalized continuous interpolation between the space of bounded continuous functions and the space of continuously differentiable functions with bounded derivatives is a weighted form of the so-called little Hölder space [81].

By analyzing the conditions on Boyd indices in the previous examples, we notice that they frequently coincide. Therefore, we identify and characterize the families of generalized spaces that adhere to these conditions.

The last Section is about regarding the limiting cases $\theta = 0$ and $\theta = 1$ in our general setting. Indeed, in the classical case, if $\theta = 0$ or $\theta = 1$, the spaces $K_q^\theta(\mathbf{A})$ are degenerate. It may not be the case when $0 = \underline{b}(\phi)$ and $\overline{b}(\phi) = 1$ (same question with the continuous setting): if

$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1,t\})^q \frac{dt}{t} < \infty,$$

with the usual modification if $q = \infty$, then K_q^ϕ is an exact interpolation functor of exponent ϕ.

One also might wonder what happens with the Equivalence Theorem 2.4.3 when $0 = \underline{b}(\phi)$ and $\overline{b}(\phi) = 1$. We obtain conditions on ϕ in this Section in order to answer those questions.

The results established in this Section were published in [69].

3.1 Continuous Interpolation Spaces

For $\phi \in \mathcal{B}$, let $K_\infty^{0,\phi}(\mathbf{A})$ denote the space comprising all $a \in \Sigma(\mathbf{A})$ such that

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,a) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,a) = 0.$$

Given that $t \mapsto K(t,a)$ is continuous on $(0,\infty)$, $K_\infty^{0,\phi}(\mathbf{A})$ is in $K_\infty^\phi(\mathbf{A})$. We naturally equip $K_\infty^{0,\phi}(\mathbf{A})$ with the norm induced by $K_\infty^\phi(\mathbf{A})$. It is evident that $K_\infty^{0,\phi}(\mathbf{A})$ constitutes a closed subspace of $K_\infty^\phi(\mathbf{A})$.

Now, let us address some straightforward observations.

Remark 3.1.1. For analogous reasons to Proposition 2.3.7, we observe that

$$K_\infty^{0,\phi}(A_0, A_1) = K_\infty^{0,\phi^*}(A_1, A_0).$$

Remark 3.1.2. For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q < \infty$, it holds that $K_q^\phi(\mathbf{A})$ is continuously embedded in $K_\infty^{0,\phi}(\mathbf{A})$. Indeed, for $x \in K_q^\phi(\mathbf{A})$, we have

$$\begin{aligned} \|1/\bar{\phi}\|_{L_*^q(1,\infty)} \frac{1}{\phi(t)} K(t,x) &\le C(\int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s})^{1/q} K(t,x) \\ &\le C(\int_t^\infty (\frac{1}{\phi(s)} K(s,x))^q \frac{ds}{s})^{1/q}. \end{aligned}$$

Consequently, $K(t,a)/\phi(t) \to 0$ as $t \to \infty$. Utilizing the fact that

$$K_q^\phi(A_0, A_1) = K_q^{\phi^*}(A_1, A_0),$$

we also deduce

$$\begin{aligned} 0 = \lim_{t\to\infty} \frac{1}{\phi^*(t)} K(t,a;(A_1,A_0)) &= \lim_{t\to\infty} \frac{1}{\phi(1/t)} K(1/t,a) \\ = \lim_{t\to 0} \frac{1}{\phi(t)} K(t,a). \end{aligned}$$

Theorem 3.1.3. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$, $K_\infty^{0,\phi}$ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.*

Moreover, the inequality

$$K(t,a) \leq C\phi(t)\|a\|_{K_\infty^{0,\phi}(\mathbf{A})}$$

holds.

Proof. We have the following embeddings:

$$\Delta(\mathbf{A}) \hookrightarrow K_1^\phi(\mathbf{A}) \hookrightarrow K_\infty^{0,\phi}(\mathbf{A}) \hookrightarrow K_\infty^\phi(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A}).$$

Let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathscr{N}_c$; we know that for $T : \mathbf{A} \to \mathbf{B}$, we have

$$K(t, Ta; \mathbf{B}) \leq \|T\|_{A_0,B_0} K(t\frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}, a; \mathbf{A}).$$

Therefore,

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,a) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,a) = 0$$

implies

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,Ta) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,Ta) = 0.$$

Since K_∞^ϕ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$, we conclude the proof. □

Just as in the standard case, we observe the following inclusion.

Proposition 3.1.4. *Given $\phi_0, \phi_1 \in \mathcal{B}$ such that $\overline{b}(\phi_0) < \underline{b}(\phi_1)$, if $A_1 \hookrightarrow A_0$, then $K_\infty^{0,\phi_1}(\mathbf{A}) \hookrightarrow K_\infty^{0,\phi_0}(\mathbf{A})$.*

The space $K_\infty^\phi(\mathbf{A})$ can be characterized as follows.

Proposition 3.1.5. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$ is $K_\infty^{0,\phi}(\mathbf{A})$.*

Proof. Let a be such that $K(t,a)/\phi(t)$ tends to 0 as t or $1/t$ tends to infinity. We can assume that $a = \sum_j b_j$ in $\Sigma(\mathbf{A})$, with $b_j \in \Delta(\mathbf{A})$ and $J(2^j, b_j) \leq CK(2^j, a)$ for all $j \in \mathbb{Z}$ (see the proof of Theorem 2.4.3). Consequently, we have

$$\|a - \sum_{|j|<j_0} b_j\|_{J_{\phi,\infty}(\mathbf{A})} \leq C \sup_{|j|\geq j_0} \frac{K(2^j, a)}{\phi(2^j)},$$

for $j_0 \in \mathbb{N}$, so a is in the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$.

Now, if a is in the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$, for $\varepsilon > 0$, there exists $b \in \Delta(\mathbf{A})$ such that $\|a - b\|_{K_\infty^\phi(\mathbf{A})} < \varepsilon$. We obtain

$$K(t,a) \leq K(t, a-b) + K(t,b) \leq \phi(t)\|a-b\|_{K_\infty^\phi(\mathbf{A})} + \min\{1,t\}J(1,b).$$

Thus

$$\frac{K(t,a)}{\phi(t)} \le \varepsilon + \min\{1,t\}\frac{J(1,b)}{\phi(t)},$$

which suffices to conclude. □

Let us also recall the duality Theorem in this context, well-known and easily derived from Section 3.7 of [15].

Proposition 3.1.6. *Assuming $\Delta(\mathbf{A})$ is dense in A_0 and A_1, then*

$$K_\infty^{0,\phi}(\mathbf{A})' = K_1^{\check{\phi}}(\mathbf{A}').$$

3.1.1 Equivalent definitions

It is straightforward to verify that for $q \in [1,\infty]$, $X_q^\phi(\mathbf{A})$ coincides with the space comprising all $a \in \Sigma(\mathbf{A})$ for which there exists a representation $a = a_0(t) + a_1(t)$ for almost all $t > 0$ with $t \mapsto 1/\phi(t)a_0(t) \in L_*^q(A_0)$ and $t \mapsto t/\phi(t)a_1(t) \in L_*^q(A_1)$, equipped with the norm

$$\|a\|_{X_q^\phi(\mathbf{A})} = \inf_{a=a_0(t)+a_1(t)} \|\frac{1}{\phi(t)}a_0(t)\|_{L_*^q(A_0)} + \|\frac{t}{\phi(t)}a_1(t)\|_{L_*^q(A_1)}.$$

Proposition 3.1.7. *For $\phi \in \mathcal{B}$, $K_\infty^{0,\phi}(\mathbf{A})$ coincide with the subspace $X_\infty^{0,\phi}(\mathbf{A})$ of $X_\infty^\phi(\mathbf{A})$ (equipped with the induced norm) such that there exists a representation $a = a_0(t) + a_1(t)$ satisfying*

$$\lim_{t\to 0}\frac{1}{\phi(t)}\|a_0(t)\|_{A_0} = \lim_{t\to\infty}\frac{1}{\phi(t)}\|a_0(t)\|_{A_0} = 0$$

and

$$\lim_{t\to 0}\frac{t}{\phi(t)}\|a_1(t)\|_{A_1} = \lim_{t\to\infty}\frac{t}{\phi(t)}\|a_1(t)\|_{A_1} = 0.$$

Proof. If $a \in K_\infty^{0,\phi}(\mathbf{A})$, then for every $t > 0$, there exist $a_0(t) \in A_0$ and $a_1(t) \in A_1$ such that

$$\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \le 2K(t,a),$$

which provides a valid representation for a.

Conversely, if $a \in X_\infty^{0,\phi}(\mathbf{A})$, utilizing a valid representation, we have

$$K(t,a) \le \|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1},$$

and the conclusion follows. □

Now, we provide an equivalence Theorem in this scenario. Let $J_\infty^{0,\phi}(\mathbf{A})$ denote the subspace of $J_\infty^\phi(\mathbf{A})$, equipped with induced norm, such that there exists a representation $a = \int_0^\infty b(t)\,dt/t$ satisfying

$$\lim_{t\to 0}\frac{1}{\phi(t)}\|b(t)\|_{A_0} = \lim_{t\to\infty}\frac{1}{\phi(t)}\|b(t)\|_{A_0} = 0$$

and

$$\lim_{t\to 0}\frac{t}{\phi(t)}\|b(t)\|_{A_1}=\lim_{t\to\infty}\frac{t}{\phi(t)}\|b(t)\|_{A_1}=0.$$

Theorem 3.1.8. *For $\phi\in\mathcal{B}$ satisfying $0<\underline{b}(\phi)$ and $\overline{b}(\phi)<1$, we have*

$$J^{0,\phi}_\infty(\mathbf{A})=K^{0,\phi}_\infty(\mathbf{A}).$$

Proof. Thanks to the preceding proposition, it suffices to demonstrate that $J^{0,\phi}_\infty(\mathbf{A})=X^{0,\phi}_\infty(\mathbf{A})$. Suppose that $a=\int_0^\infty b(t)\,dt/t$ belongs to $J^{0,\phi}_\infty(\mathbf{A})$. Then, for $t>0$,

$$a_0(t)=\int_0^1 b(t\tau)\,\frac{d\tau}{\tau}\quad\text{and}\quad a_1(t)=\int_1^\infty b(t\tau)\,\frac{d\tau}{\tau}.$$

These equalities provide a representation for a in $X^{\phi}_\infty(\mathbf{A})$. Moreover, since both

$$\int_0^1\bar\phi(\tau)\,\frac{d\tau}{\tau}\quad\text{and}\quad\int_1^\infty\frac{\bar\phi(\tau)}{\tau}\,\frac{d\tau}{\tau}$$

are finite, the dominated convergence Theorem implies that a belongs to $X^{0,\phi}_\infty(\mathbf{A})$.

For $a\in X^{0,\phi}_\infty(\mathbf{A})$, suppose a_0 and a_1 constitute a valid representation of a, and let φ be a test function such that $\int\varphi(\tau)d\tau/\tau=1$. Define

$$\tilde a_0(t)=\int_0^\infty\varphi(\frac{t}{\tau})a_0(\tau)\,\frac{d\tau}{\tau}\quad\text{and}\quad\tilde a_1(t)=\int_0^\infty\varphi(\frac{t}{\tau})a_1(\tau)\,\frac{d\tau}{\tau},$$

so that $a=\tilde a_0(t)+\tilde a_1(t)$ for all $t>0$. By defining $b(t)=tD_t\tilde a_0(t)=-tD_t\tilde a_1(t)$, we obtain

$$\int_0^\infty b(t)\frac{dt}{t}=\int_0^1 D_t\tilde a_0(t)dt-\int_1^\infty D_t\tilde a_1(t)dt=\tilde a_0(1)+\tilde a_1(1)=a.$$

Utilizing the fact that the support of $D\varphi$ is contained in a compact set, the dominated convergence Theorem enables us to conclude. □

3.2 Applications

3.2.1 Interpolation of Hölder spaces

Theorem 3.2.1. *Let $\phi\in\mathcal{B}$ with $0<\underline{b}(\phi)$, $\overline{b}(\phi)<1$,*

$$K^\phi_\infty(C_b(\mathbb{R}^d),C^1_b(\mathbb{R}^d))=C^\phi_b(\mathbb{R}^d).$$

Proof. Let $a=a_0+a_1\in K^\phi_\infty(C_b(\mathbb{R}^d),C^1_b(\mathbb{R}^d))$, then

$$\begin{aligned}\|a\|_{C^\phi_b(\mathbb{R}^d)}&=\|a\|_\infty+|a|_{C^\phi}\\&\le\|a_0\|_\infty+\|a_1\|_\infty+\sup_{x\neq y}\frac{|a_0(x)-a_0(y)|}{\phi(|x-y|)}+\frac{|a_1(x)-a_1(y)|}{\phi(|x-y|)}\\&\le K(1,a)+\sup_{x\neq y}\frac{2}{\phi(|x-y|)}K(|x-y|,a)\\&\le C\|a\|_{K^\phi_\infty(C_b(\mathbb{R}^d),C^1_b(\mathbb{R}^d))}.\end{aligned}$$

Now, if $a \in C_b^\phi(\mathbb{R}^d)$, let $\varphi \in \mathcal{D}$ be a non-negative function with support in the unit ball satisfying $\int \varphi(x)dx = 1$. For each $t > 0$, we set

$$a_0^{(t)}(x) = \frac{1}{t^d}\int (a(x) - a(x-y))\varphi(\frac{y}{t})dy \quad \text{and} \quad a_1^{(t)}(x) = \frac{1}{t^d}\int a(y)\varphi(\frac{x-y}{t})dy.$$

One can remark that $a = a_0^{(t)} + a_1^{(t)}$ for all $t > 0$. On the one hand,

$$\|a_0^{(t)}\|_\infty \leq \phi(t)|a|_{C^\phi}\int \bar{\phi}(|u|)\varphi(u)du$$

and on the other hand, $\|a_1^{(t)}\|_\infty \leq \|a\|_\infty$ and since for $k \in \{1, ..., d\}$,

$$D_k a_1^{(t)}(x) = \frac{1}{t^{d+1}}\int (a(x-y) - a(x))D_k\varphi(\frac{y}{t})dy,$$

we get

$$\|D_k a_1^{(t)}\|_\infty \leq \frac{\phi(t)}{t}|a|_{C^\phi}\int \bar{\phi}(|u|)|D_k\varphi(u)|du.$$

Therefore, we have

$$\frac{1}{\phi(t)}K(t,a) \leq \frac{1}{\phi(t)}(\|a_0^{(t)}\|_\infty + t\|a_1^{(t)}\|_{C_b^1(\mathbb{R}^d)}) \leq C\|a\|_{C_b^\phi(\mathbb{R}^d)} \quad \text{for } t \in (0,1].$$

If $t > 1$, one can take $a_0^{(t)} = a$ and $a_1^{(t)} = 0$. □

Using Reiteration Theorem 2.6.3, we obtain the following Corollary.

Corollary 3.2.2. *Let* $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ *to set* $f = \phi_0/\phi_1$ *and* $\psi = \phi_0/(\gamma \circ f)$. *If* $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$ *and if* $\underline{b}(f) > 0$ *or* $\overline{b}(f) < 0$, *then*

$$K_\infty^\gamma(C_b^{\phi_0}(\mathbb{R}^d), C_b^{\phi_1}(\mathbb{R}^d)) = C_b^\psi(\mathbb{R}^d).$$

In the context of $K_\infty^{0,\phi}$, we obtain a weighted form of the so-called little Hölder space [81].

Theorem 3.2.3. *For* $\phi \in \mathcal{B}$ *such that* $0 < \underline{b}(\phi)$ *and* $\overline{b}(\phi) < 1$, *we have*

$$K_\infty^{0,\phi}(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) = h^\phi(\mathbb{R}^d),$$

where $h^\phi(\mathbb{R}^d)$ *is the space consisting of bounded functions* a *such that*

$$\lim_{h\to 0}\sup_{x\in\mathbb{R}^d}\frac{|a(x+h) - a(x)|}{\phi(|h|)} = 0.$$

Proof. Let $a = a_0 + a_1$ be in $K_\infty^{0,\phi}(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) \hookrightarrow K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))$. We have

$$\begin{aligned}\frac{|a(x+h)-a(x)|}{\phi(|h|)} &\leq \frac{|a_0(x+h)-a_0(x)|}{\phi(|h|)} + \frac{|a_1(x+h)-a_1(x)|}{\phi(|h|)} \\ &\leq \frac{2\|a_0\|_\infty}{\phi(|h|)} + \frac{\|a_1\|_{C_b^1}|h|}{\phi(|h|)}.\end{aligned}$$

Therefore, we obtain

$$0 \leq \sup_{x\in\mathbb{R}^d} \frac{|a(x+h)-a(x)|}{\phi(|h|)} \leq C\frac{1}{\phi(|h|)}K(|h|, a),$$

implying that a belongs to $h^\phi(\mathbb{R}^d)$.

Now, for a in $h^\phi(\mathbb{R}^d) \hookrightarrow C_b^\phi(\mathbb{R}^d)$, as in Theorem 3.2.1, define

$$a_0^{(t)}(x) = \frac{1}{t^d}\int (a(x) - a(x-y))\varphi(\frac{y}{t})\,dy \quad \text{and} \quad a_1^{(t)}(x) = \frac{1}{t^d}\int a(y)\varphi(\frac{x-y}{t})\,dy.$$

We directly get

$$\frac{1}{\phi(t)}K(t,a) \leq \frac{1}{\phi(t)}(\|a_0^{(t)}\|_\infty + t\|a_1^{(t)}\|_{C_b^1(\mathbb{R}^d)}).$$

That being said, we have

$$\frac{1}{\phi(t)}\|a_0^{(t)}\|_\infty \leq \int \sup_{x\in\mathbb{R}^d} \frac{|a(x)-a(x-tu)|}{\phi(t)}\varphi(u)\,du,$$

$$\frac{t}{\phi(t)}\|a_1^{(t)}\|_\infty \leq \frac{t}{\phi(t)}\|a\|_\infty,$$

and, for $k \in \{1, \dots, d\}$,

$$\frac{t}{\phi(t)}\|D_k a_1^{(t)}\|_\infty \leq \int \sup_{x\in\mathbb{R}^d} \frac{|a(x-tu)-a(x)|}{\phi(t)}|D_k\varphi(u)|\,du,$$

which implies that $K(t,a)/\phi(t)$ tends to 0 as t tends to 0^+.

To check the other limit, we can simply use the decomposition $a = a + 0$ to directly obtain

$$\frac{1}{\phi(t)}K(t,a) \leq \frac{1}{\phi(t)}\|a\|_\infty,$$

which tends to 0 as t tends to infinity. □

Using Proposition 3.1.5, the previous result can be rewritten as follows.

Corollary 3.2.4. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $C_b^1(\mathbb{R}^d)$ in $K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))$ is $h^\phi(\mathbb{R}^d)$.*

3.2.2 Interpolation of weighted l^q spaces and generalized Besov spaces using retracts

We present here a first way to interpolate generalized Besov spaces, following the path of [27].

Lemma 3.2.5. *Let X a Banach space, $\phi \in \mathcal{B}$ and $q_0, q_1, q \in [1, \infty]$. If $s_0, s_1 \in \mathbb{R}$ are such that $s_1 \leq \underline{b}(\phi)$ and $\overline{b}(\phi) \leq s_0$, if $\gamma \in \mathcal{B}$ is defined by*

$$\gamma(t) = \frac{t^{s_0/(s_0-s_1)}}{\phi(t^{1/(s_0-s_1)})}$$

for $t > 0$, then we have $K_q^\gamma(\ell_{s_0}^{q_0}(X), \ell_{s_1}^{q_1}(X)) = \ell_\phi^q(X)$.

Proof. Let us first show that

$$K_q^\gamma(\ell_{s_0}^\infty(X), \ell_{s_1}^\infty(X)) \hookrightarrow \ell_\phi^q(X). \tag{3.1}$$

For $\alpha \in \ell_{s_1}^\infty(X)$, we easily get

$$\begin{aligned} K(t, \alpha; \ell_{s_0}^\infty(X), \ell_{s_1}^\infty(X)) &= \inf\{\|\alpha^{(0)}\|_{\ell_{s_0}^\infty(X)} + t\|\alpha^{(1)}\|_{\ell_{s_1}^\infty(X)} : \alpha = \alpha^{(0)} + \alpha^{(1)}, \alpha^{(k)} \in \ell_{s_k}^\infty(X)\} \\ &\geq C \sup_j \min(2^{js_0}, t2^{js_1})\|\alpha_j\|_X. \end{aligned}$$

Suppose that $q < \infty$ and set $\alpha_j = 0$ if $j \in -\mathbb{N}$; one has

$$\begin{aligned} \|\alpha\|^q_{K_q^\gamma(\ell_{s_0}^\infty(X), \ell_{s_1}^\infty(X))} &= \int_0^\infty \frac{K^{\ell_{s_0}^\infty(X), \ell_{s_1}^\infty(X)}(t, \alpha)^q}{\gamma(t)^q} \frac{dt}{t} \\ &= \sum_{k \in \mathbb{Z}} \int_{2^{(k-1)(s_0-s1)}}^{2^{k(s_0-s1)}} \frac{\phi(t^{1/(s_0-s_1)})^q K^{\ell_{s_0}^\infty(X), \ell_{s_1}^\infty(X)}(t, \alpha)^q}{t^{qs_0/(s_0-s_1)}} \frac{dt}{t} \\ &\geq C \sum_{k \in \mathbb{Z}} \frac{1}{2^{qks_0}} \phi(2^k)^q (\sup_j \min(2^{js_0}, 2^{k(s_0-s_1)}2^{js_1})\|\alpha_j\|_X)^q \\ &\geq C \sum_{k \in \mathbb{Z}} \frac{\phi(2^k)^q 2^{qks_0} (\sup_j \|\alpha_j\|_X)^q}{2^{qks_0}} \\ &\geq C\|\alpha\|^q_{\ell_\phi^q(X)}. \end{aligned}$$

The usual modification leads to the same inclusion if $q = \infty$. Now, since

$$0 < \frac{s_0}{s_0 - s_1} - \frac{\overline{b}(\phi)}{s_0 - s_1} = \underline{b}(\gamma) \leq \overline{b}(\gamma) = \frac{s_0}{s_0 - s_1} - \frac{\underline{b}(\phi)}{s_0 - s_1} < 1,$$

we have

$$\int_0^\infty \frac{\overline{\gamma}(t)}{\max(1, t)} \frac{dt}{t} < \infty,$$

so that $\sum_{j\in\mathbb{Z}} \min(1,2^{-js})\overline{\gamma}(2^{js}) < \infty$ for all $s \in \mathbb{R}$. For $\alpha \in \ell^q_\phi(X)$, one has

$$K(t,\alpha;\ell^1_{s_0}(X),\ell^1_{s_1}(X)) \leq C \sum_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\alpha_j\|_X.$$

If $q<\infty$, set $\alpha_j = 0$ if $j \in -\mathbb{N}$ to get

$$\begin{aligned}
\|\alpha\|^q_{K^\gamma_q(\ell^1_{s_0}(X),\ell^1_{s_1}(X))} &= \int_0^\infty \frac{K(t,\alpha;\ell^1_{s_0}(X),\ell^1_{s_1}(X))^q}{\gamma(t)^q}\frac{dt}{t} \\
&= \sum_{k\in\mathbb{Z}} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s1)}} \frac{\phi(t^{1/(s_0-s_1)})^q K(t,\alpha;\ell^1_{s_0}(X),\ell^1_{s_1}(X))^q}{t^{qs_0/(s_0-s_1)}}\frac{dt}{t} \\
&\leq C\sum_{k\in\mathbb{Z}} \frac{\phi(2^k)^q}{2^{qks_0}} (\sum_{j\in\mathbb{N}_0} \min(2^{js_0}, 2^{k(s_0-s_1)}2^{js_1})\|\alpha_j\|_X)^q,
\end{aligned}$$

which gives the converse inclusion (with the usual modification for $q=\infty$). We conclude using Reiteration Theorem 2.6.3. □

Theorem 3.2.6. *Let $q_0, q_1, q \in [1,\infty]$ and $\gamma, \phi_0, \phi_1 \in \mathcal{B}$. Set $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma\circ f)$. If $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then*

$$K^\gamma_q(\ell^{q_0}_{\phi_0}(X),\ell^{q_1}_{\phi_1}(X)) = \ell^q_\psi(X).$$

Proof. It is clear that $\psi \in \mathcal{B}$. Let $s_0, s_1 \in \mathbb{R}$ be such that

$$s_1 < \min(\underline{b}(\phi_0),\underline{b}(\phi_1),\underline{b}(\psi)) \leq \max(\overline{b}(\phi_0),\overline{b}(\phi_1),\overline{b}(\psi)) < s_0.$$

By Lemma 3.2.5, given $j \in \{0,1\}$, for

$$\gamma_j(t) := \frac{t^{s_0/(s_0-s_1)}}{\phi_j(t^{1/(s_0-s_1)})},$$

one has

$$K^{\gamma_j}_{q_j}(\ell^1_{s_0}(X),\ell^1_{s_1}(X)) = \ell^{q_j}_{\phi_j}(X).$$

We thus have

$$\begin{aligned}
K^\gamma_q(\ell^{q_0}_{\phi_0}(X),\ell^{q_1}_{\phi_1}(X)) &= K^\gamma_q(K^{\gamma_0}_{q_0}(\ell^1_{s_0}(X),\ell^1_{s_1}(X)),K^{\gamma_1}_{q_1}(\ell^1_{s_0}(X),\ell^1_{s_1}(X))) \\
&= K^\Psi_q(l^1_{s_0}(X),l^1_{s_1}(X)),
\end{aligned}$$

for $\Psi(t) := \gamma_0(t)\,\gamma(\frac{\gamma_1(t)}{\gamma_0(t)})$. By lemma 3.2.5, we thus get the desired result. □

In the context of $K^{0,\phi}_\infty$, the preceding result is reformulated as follows:

Theorem 3.2.7. *Let $q_0, q_1 \in [1,\infty]$, $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ and define $f = \phi_0/\phi_1$. If $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then*

$$K_\infty^{0,\gamma}(\ell_{\phi_0}^{q_0}(X), \ell_{\phi_1}^{q_1}(X)) = c_{0,\psi}(X),$$

with $\psi = \phi_0/(\gamma \circ f)$, where $c_{0,\psi}(X)$ is the subspace of $\ell_\psi^\infty(X)$ such that

$$\lim_j \psi(2^j)\|a_j\|_X = 0.$$

Proof. We know that

$$K_\infty^{\gamma}(\ell_{\phi_0}^{\infty}(X), \ell_{\phi_1}^{\infty}(X)) = \ell_\psi^\infty(X).$$

Assuming $\underline{b}(f) > 0$, we can write

$$\ell_{\phi_0}^{\infty}(X) \hookrightarrow \ell_{\phi_1}^{\infty}(X).$$

Let $a = a^{(0,j)} + a^{(1,j)}$ be a discrete representation of a in $K_\infty^{0,\gamma}(\ell_{\phi_0}^{\infty}(X), \ell_{\phi_1}^{\infty}(X))$. We have

$$\psi(2^j)\|a_j\|_X \leq \frac{\|a^{(0,j)}\|_{l_{\phi_0}^\infty(X)}}{\gamma(f(2^j))} + f(2^j)\frac{\|a^{(1,j)}\|_{l_{\phi_1}^\infty(X)}}{\gamma(f(2^j))},$$

the RHS tending to 0 as $j \to \infty$.

Now, let $a \in c_{0,\psi}(X) \hookrightarrow \ell_\psi^\infty(X)$. Set $a^{(0,k)} = 0$ and $a^{(1,k)} = a_k e_k$, providing a discrete representation of a in $K_\infty^{\gamma}(\ell_{\phi_0}^{\infty}(X), \ell_{\phi_1}^{\infty}(X))$ such that

$$\frac{f(2^j)}{\gamma(f(2^j))}\|a^{(1,j)}\|_{l_{\phi_1}^\infty(X)} \leq \psi(2^j)\|a_j\|_X.$$

Hence, a belongs to $K_\infty^{0,\gamma}(\ell_{\phi_0}^{\infty}(X), \ell_{\phi_1}^{\infty}(X))$. □

Remark 3.2.8. If $\phi_0/(\gamma \circ f) = \mathrm{id}$, then we can express the usual space $c_0(X)$ as

$$c_0(X) = K_\infty^{0,\gamma}(\ell_{\phi_0}^{\infty}(X), \ell_{\phi_1}^{\infty}(X)).$$

Theorem 3.2.9. *Given $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, define $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma \circ f)$. If $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$ and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then*

$$K_q^\gamma(B_{p,q_0}^{\phi_0}, B_{p,q_1}^{\phi_1}) = B_{p,q}^\psi \quad \text{for} \quad p, q, q_0, q_1 \in [1,\infty]$$

and

$$K_q^\gamma(F_{p,q_0}^{\phi_0}, F_{p,q_1}^{\phi_1}) = B_{p,q}^\psi, \quad \text{for} \quad p, q_0, q_1 \in (1,\infty), q \in [1,\infty].$$

Proof. This is a direct consequence of Theorems 1.2.2, 2.8.3 and 3.2.6. □

3.2.3 Interpolation of generalized Besov spaces using Sobolev spaces

Let us present another way to interpolate generalized Besov spaces involving Sobolev spaces and Reiteration. We reconsider certain proofs from [88], as this context will be necessary for the continuous interpolation results.

Theorem 3.2.10. *Let $\gamma \in \mathcal{B}$, $\phi_0, \phi_1 \in \mathcal{B}''$, $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$ and $p, q \in [1, \infty]$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1}) = B_{p,q}^\psi.$$

Proof. If $\underline{b}(f) > 0$, $\psi \in \mathcal{B}$ and we can suppose that $f \in \mathcal{B}'$. Let $a = a_0 + a_1$ be an element of $K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1})$, with $a_k \in H_p^{\phi_k}$ $(k \in \{0, 1\})$. By Proposition 1.2.5, one has

$$\begin{aligned}
\|a\|_{B_{p,q}^\psi} &\leq \sum_{k=0}^{1} \|(\psi(2^j)\|\mathcal{J}^{1/\phi_k}\mathcal{J}^{\phi_k}\varphi_j * a_k\|_{L^p})_j\|_{\ell^q} \\
&\leq C\sum_{k=0}^{1} \|(\psi(2^j)\frac{1}{\phi_k(2^j)}\|\mathcal{J}^{\phi_k}\varphi_j * a_k\|_{L^p})_j\|_{\ell^q} \\
&\leq C\sum_{k=0}^{1} \|(\psi(2^j)\frac{1}{\phi_k(2^j)}\|\mathcal{J}^{\phi_k} a_k\|_{L^p})_j\|_{\ell^q}
\end{aligned}$$

and thus, by taking the infimum,

$$\|a\|_{B_{p,q}^\psi} \leq C\|\frac{1}{\gamma(f(t))}K(f(t), a; H_p^{\phi_0}, H_p^{\phi_1})\|_{L_*^q}.$$

The change of variables $s = f(t)$ leads to

$$K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1}) \hookrightarrow B_{p,q}^\psi. \tag{3.2}$$

Now, let a be an element of $B_{p,q}^\psi$. Again, by Proposition 1.2.5, for $k \in \{0, 1\}$, one has $\|\varphi_j * a\|_{H_p^{\phi_k}} \leq C\phi_k(2^j)\,\|\varphi_j * a\|_{L^p}$ for all $j \in \mathbb{N}_0$. For $a_j = \varphi_j * a$ $(j \in \mathbb{N}_0)$, we get

$$\|(\frac{1}{\gamma(f(2^j))}J(f(2^j), a_j; H_p^{\phi_0}, H_p^{\phi_1}))_j\|_{\ell^q} \leq C\|a\|_{B_{p,q}^\psi}. \tag{3.3}$$

Since $\underline{b}(f) > 0$, one has $H_p^{\phi_1} = \Sigma(H_p^{\phi_0}, H_p^{\phi_1})$, by Proposition 1.2.7. Let $q' \in [1, \infty]$ be the exponent conjugate to q; by Holder's inequality, one has

$$\begin{aligned}
\sum_{j\in\mathbb{N}_0} \|a_j\|_{H_p^{\phi_1}} &\leq C\sum_{j\in\mathbb{N}_0} \phi_1(2^j)\|a_j\|_{L^p} \\
&= C\sum_{j\in\mathbb{N}_0} \frac{\phi_0(2^j)(\gamma\circ f)(2^j)}{(\gamma\circ f)(2^j)f(2^j)}\|a_j\|_{L^p} \\
&\leq C\|(\psi(2^j)\|a_j\|_{L^p})_j\|_{l^q}\|(\frac{(\gamma\circ f)(2^j)}{f(2^j)})_j\|_{\ell^{q'}}.
\end{aligned}$$

The change of variables $s = f(t)$ leads to $\|(\frac{(\gamma\circ f)(2^j)}{f(2^j)})_j\|_{\ell^{q'}} \leq C\|\frac{\gamma(s)}{s}\|_{L_*^{q'}}$, which is finite since $\overline{b}(\gamma) < 1$. As a consequence, $\sum_{j\in\mathbb{N}_0} a_j$ converges to a in $\Sigma(H_p^{\phi_0}, H_p^{\phi_1})$. Finally, for $t \in [f(2^l), f(2^{l+1}))$ and $l \in \mathbb{Z}$, let us define

$$b(t) := \frac{f(2^l)}{\log f(2^l)} a_l,$$

where $a_l = 0$ if $l \in -\mathbb{N}$. We directly get $a = \int_0^\infty b(t)dt/t$ in $\Sigma(H_p^{\phi_0}, H_p^{\phi_1})$. For $t \in [f(2^l), f(2^{l+1}))$, one has

$$\frac{J(t, b(t); H_p^{\phi_0}, H_p^{\phi_1})}{\gamma(t)} \leq C_j \frac{J(f(2^j), a_j; H_p^{\phi_0}, H_p^{\phi_1})}{\gamma(f(2^j))},$$

with $\sup_{j\in\mathbb{N}} C_j \leq C$. By the Equivalence Theorem 2.4.3 and (3.3), we get

$$K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1}) \hookleftarrow B_{p,q}^\psi.$$

The case $\overline{b}(f) < 0$ is obtained by interchanging the spaces $H_p^{\phi_0}$ and $H_p^{\phi_1}$. □

With the previous result and using the reiteration Theorem, we can prove again the first part of Theorem 3.2.9.

Corollary 3.2.11. *Let $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma\circ f)$ and $p, q, q_0, q_1 \in [1,\infty]$; If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K_q^\gamma(B_{p,q_0}^{\phi_0}, B_{p,q_1}^{\phi_1}) = B_{p,q}^\psi.$$

Proof. Let $\psi_0, \psi_1 \in \mathcal{B}''$ be such that $\underline{b}(\psi_0/\psi_1) > 0$ or $\overline{b}(\psi_0/\psi_1) < 0$. There exist $\gamma_0, \gamma_1 \in \mathcal{B}$ such that, for $k \in \{0,1\}$, $0 < \underline{b}(\gamma_k) \leq \overline{b}(\gamma_k) < 1$ and

$$\phi_k = \frac{\psi_0}{\gamma_k \circ \frac{\psi_0}{\psi_1}}.$$

Let us define $\tilde{\psi} := (\gamma\circ\frac{\gamma_1}{\gamma_0})\gamma_0$; by Theorem 3.2.10, one has

$$K_q^\gamma(B_{p,q_0}^{\phi_0}, B_{p,q_1}^{\phi_1}) = K_q^{\tilde{\psi}}(H_p^{\psi_0}, H_p^{\psi_1}) = B_{p,q}^\psi,$$

with $\psi := \frac{\psi_0}{\tilde{\psi}\circ\frac{\psi_0}{\psi_1}} = \frac{\phi_0}{\gamma\circ f}$. □

In the context of $K_\infty^{0,\phi}$, we get:

Theorem 3.2.12. *Let $\gamma \in \mathcal{B}$, $\phi_0, \phi_1 \in \mathcal{B}''$ and define $f = \phi_0/\phi_1$. For $p \in [1,\infty]$, if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K_\infty^{0,\gamma}(H_p^{\phi_0}, H_p^{\phi_1}) = b_{p,\infty}^\psi,$$

for $\psi = \phi_0/(\gamma\circ f)$, where $b_{p,\infty}^\psi$ is the subspace of the elements a of $B_{p,\infty}^\psi$ (equipped with the induced norm) such that

$$\lim_j \psi(2^j)\|\varphi_j * a\|_{L^p} = 0.$$

Proof. For $a = a_0 + a_1$ in $K_\infty^{0;\gamma}(H_p^{\phi_0}, H_p^{\phi_1})$, we have

$$\psi(2^j)\|\varphi_j * a\|_{L^p} \leq C\psi(2^j)(\frac{1}{\phi_0(2^j)}\|\mathcal{J}^{\phi_0}a_0\|_{L^p} + \frac{1}{\phi_1(2^j)}\|\mathcal{J}^{\phi_1}a_1\|_{L^p}),$$

so that

$$\psi(2^j)\|\varphi_j * a\|_{L^p} \leq C\frac{1}{\gamma(f(2^j))}K(f(2^j), a),$$

the RHS tending to 0 as j tends to infinity, since $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$.

Now, let a belong to $b^{\psi}_{p,\infty}$. If $\underline{b}(f) > 0$, then $\Sigma(H_p^{\phi_0}, H_p^{\phi_1}) = H_p^{\phi_1}$. For $a_j = \varphi_j * a$ ($j \in \mathbb{N}_0$), using Hölder inequality, we get that $\sum_j \|a_j\|_{H_p^{\phi_1}}$ is finite and thus $\sum_{j\in\mathbb{N}} a_j$ converges to a in $\Sigma(H_p^{\phi_0}, H_p^{\phi_1})$. For $l \in \mathbb{Z}$ and $t \in [f(2^l), f(2^{l+1}))$, let us define

$$b(t) = \frac{f(2^l)}{\log f(2^l)}a_l,$$

where $a_l = 0$ if $l \in -\mathbb{N}$. Since for $k \in \{0, 1\}$, we have

$$\|\varphi_j * a\|_{H_p^{\phi_k}} \leq C\phi_k(2^j)\|\varphi_j * a\|_{L^p},$$

for all $j \in \mathbb{N}$, we get

$$\frac{1}{\gamma(t)}J(t, b(t)) \leq C\frac{J(f(2^j), a_j)}{\gamma(f(2^j))} \leq \frac{\phi_0(2^j)}{\gamma(f(2^j))}\|\varphi_j * a\|_{L^p},$$

hence the conclusion. □

3.2.4 Interpolation of Lorentz spaces

We follow the definition of [88].

Definition 3.2.13. Let (Ω, μ) be a space with totally σ-finite measure μ. If $\phi \in \mathcal{B}$ and $p \in (0, \infty]$, the Lorentz space Λ^p_ϕ is the class of functions μ-measurable from Ω such that

$$\|a\|_{\Lambda^p_\phi} := \|\phi\, a^*\|_{L^p_*}$$

is finite, with a^* being the non-increasing rearrangement of $|a|$ in $(0, \infty)$. This space is quasi-normed w.r.t $\|\cdot\|_{\Lambda^p_\phi}$.

Example 3.2.14. If $\phi(t) = t^{1/q}$, one recovers the usual Lorentz-Zygmund space (see, for example, [15])

$$\Lambda^p_\phi = L^{p,q}.$$

In [88], they proved that interpolation with a function parameter with quasi-normed Abelian groups is perfectly suited to identify interpolation spaces between two generalized Lorentz spaces. We did not explore this direction further in this work.

Theorem 3.2.15. *Let $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$ and $p, p_0, p_1, q \in [1, \infty]$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K_q^\gamma(\Lambda_{\phi_0}^{p_0}, \Lambda_{\phi_1}^{p_1}) = \Lambda_\psi^q.$$

Using Theorem 2.4 of [45], one can show that there exists $\alpha > 0$ such that

$$K(t,a) \asymp (\int_0^{t^\alpha} (\phi_0(s)a^*(s))^{p_0} \frac{ds}{s})^{1/p_0} + t(\int_{t^\alpha}^\infty (\phi_1(s)a^*(s))^{p_1} \frac{ds}{s})^{1/p_1}.$$

3.2.5 Interpolation of Pointwise spaces

The real interpolation of pointwise spaces has not been extensively studied, and even for the simplest pointwise spaces, such as Hölder spaces, it remains unclear whether a result similar to Corollary 3.2.2 can be obtained. In general, the embedding of the interpolation space in the space is easy to establish:

Theorem 3.2.16. *Let $x_0 \in \mathbb{R}^d$, $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_1/\phi_0$, $\psi = (\gamma \circ f)\phi_0$ and $p, q \in [1, \infty]$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K_q^\gamma(T_{\phi_0}^p(x_0), T_{\phi_1}^p(x_0)) \hookrightarrow T_\psi^{p,q}(x_0).$$

Remark 3.2.17. If $p = q = \infty$, the last result can be written as

$$K_q^\gamma(\Lambda^{\phi_0}(x_0), \Lambda^{\phi_1}(x_0)) \hookrightarrow \Lambda^\psi(x_0).$$

For the reverse inclusion, if one wishes to adapt the arguments of the proof of Theorem 3.2.1, a neighborhood of x_0 is needed. Therefore, we can only obtain the following result.

Theorem 3.2.18. *Let Ω be an open set that contains $x_0 \in \mathbb{R}^d$, $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_1/\phi_0$, $\psi = (\gamma \circ f)\phi_0$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and if $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$C^\psi(\Omega) \hookrightarrow K_q^\gamma(\Lambda^{\phi_0}(x_0), \Lambda^{\phi_1}(x_0)) \hookrightarrow \Lambda^\psi(x_0).$$

3.3 Interpolated Spaces of a certain type

By examining the conditions on Boyd indices in the last examples, we observe that these conditions are often the same. The idea here is to characterize the families of generalized spaces that satisfy these conditions.

Definition 3.3.1. A family of space $\{X_\phi^p, \phi \in \mathcal{B}, p \in [1, \infty]\}$ is of type D if one has the equality

$$K_q^\gamma(X_{\phi_0}^p, X_{\phi_1}^p) = X_\psi^q$$

whenever $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_0/\phi_1$,

$$\psi = \phi_0/(\gamma \circ f)$$

and $p, q \in [1, \infty]$ with $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and with $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$.

Example 3.3.2. Weighted Lebesgue spaces l^q_ϕ form a family of type D.

Example 3.3.3. Let $p \in [1, \infty]$, the family $\{B^\phi_{p,q}, \phi \in \mathcal{B}, q \in [1, \infty]\}$ is of type D.

Example 3.3.4. Lorentz spaces Λ^p_ϕ form a family of type D.

Remark 3.3.5. If the family is stable by real interpolation, then it is of type D using Reiteration Theorem 2.6.3.

Remark 3.3.6. The condition for D-type families is the generalized version of the classical relation

$$\theta = \alpha_0(1 - \alpha) + \alpha_1 \alpha,$$

using $\psi(t) = t^\theta$, $\phi_0(t) = t^{\alpha_0}$, $\phi_1(t) = t^{\alpha_1}$ and $\gamma(t) = t^\alpha$.

Definition 3.3.7. A family of space $\{X_\phi, \phi \in \mathcal{B}\}$ is of type D_∞ if one has the equality

$$K^\gamma_\infty(X_{\phi_0}, X_{\phi_1}) = X_\psi$$

whenever $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_0/\phi_1$ and

$$\psi = \phi_0/(\gamma \circ f)$$

with $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and with $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$.

Example 3.3.8. Hölder spaces C^ϕ_b form a family of type D_∞.

Proposition 3.3.9. *A family*

$$\{X^p_\phi, \phi \in \mathcal{B}, p \in [1, \infty]\}$$

is of type D if and only if one has the equality

$$K^\gamma_q(X^p_{\phi_0}, X^p_{\phi_1}) = X^q_\psi$$

whenever $\gamma, \phi_0, \phi_1 \in \mathcal{B}$, $f = \phi_1/\phi_0$,

$$\psi = \phi_0(\gamma \circ f)$$

and $p, q \in [1, \infty]$ *with* $\underline{b}(f) > 0$ *or* $\overline{b}(f) < 0$ *and with* $0 < \underline{b}(\gamma), \overline{b}(\gamma) < 1$.

Proof. Given $\phi_0, \phi_1, \psi \in \mathcal{B}$, just set $\widetilde{\phi}_0 = 1/\phi_0$, $\widetilde{\phi}_1 = 1/\phi_1$ and $\widetilde{\psi} = 1/\psi$. □

3.4 Limiting cases

One might wonder what happens with the Equivalence Theorem 2.4.3 when $0 = \underline{b}(\phi)$ and $\overline{b}(\phi) = 1$. In [96], they have conditions on weights defined by

$$t \in (0,\infty) \mapsto t^{-\theta}\phi(t),$$

where ϕ is a slowly varying function on $(0,\infty)$: it means that for each $\varepsilon > 0$, $t \mapsto t^{\varepsilon}\phi(t)$ is equivalent to a non decreasing function and $t \mapsto t^{-\varepsilon}\phi(t)$ is equivalent to a non increasing function. In our setting, this corresponds to Boyd functions with Boyd indices equal to 0. Following the idea of Example 4.3 of [67], we thus have more general weights and therefore, we can look at how the conditions are adapted.

Proposition 3.4.1. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$, $\overline{b}(\phi) \leq 1$ and $q \in [1,\infty]$.*

(i) If

$$\int_0^1 (\frac{t}{\phi(t)})^q \frac{dt}{t} = \infty \quad \text{or} \quad \int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t} = \infty \tag{3.4}$$

with the usual modification if $q = \infty$, then $K_q^\phi(\mathbf{A}) = \{0\}$.

(ii) If

$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1,t\})^q \frac{dt}{t} < \infty, \tag{3.5}$$

with the usual modification if $q = \infty$, then K_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$.

Proof. (i) Let $a \in \Sigma(\mathbf{A})\backslash\{0\}$. Since $K(\cdot,a)$ is non decreasing, we get

$$\|a\|_{K_q^\phi(\mathbf{A})} \geq K(a,1)(\int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t})^{1/q}$$

and since $K(\cdot,a)/\cdot$ is non increasing,

$$\|a\|_{K_q^\phi(\mathbf{A})} \geq K(a,1)(\int_0^1 (\frac{t}{\phi(t)})^q \frac{dt}{t})^{1/q},$$

hence the conclusion.
(ii) Since, for $a \in \Delta(\mathbf{A})$,

$$K(t,a) \leq \min\{1,t\}\|a\|_{\Delta(\mathbf{A})},$$

and for $a \in \Sigma(\mathbf{A})$,

$$K(t,a) \geq \min\{1,t\}\|a\|_{\Sigma(\mathbf{A})},$$

$K_q^\phi(\mathbf{A})$ is an intermediate space between A_0 and A_1, we get the conclusion since the others points to check use the same proofs as for the usual case. □

Remark 3.4.2. Condition (3.4) is satisfied if and only if condition (3.5) is not satisfied.

Remark 3.4.3. In case (i), it implies that $K_q^\phi(\mathbf{A})$ is not an intermediate space between A_0 and A_1 if $\Delta(\mathbf{A}) \neq \{0\}$ and K_q^ϕ is thus not an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Remark 3.4.4. If $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, then

$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1,t\})^q \frac{dt}{t} < \infty$$

and we find the usual case. If $q = \infty$ and $\phi = 1$ or $\phi = \mathrm{id}$, then (3.4) is also verified since, in this case,

$$\sup_{t\in(0,\infty)} \frac{\min\{1,t\}}{\phi(t)} = 1.$$

Example 3.4.5. As remarked in [35], if

$$\phi(t) = \begin{cases} t^\theta(1+|\log t|)^{\alpha_0} & \text{if } t \in (0,1], \\ t^\theta(1+|\log t|)^{\alpha_\infty} & \text{if } t \in (1,\infty), \end{cases}$$

then condition (3.5) is equivalent to the validity of one of those conditions :

(i) $0 < \theta < 1$,

(ii) $\theta = 0$ and $\alpha_\infty + \frac{1}{q} < 0$,

(iii) $\theta = 0$, $q = \infty$ and $\alpha_\infty = 0$,

(iv) $\theta = 1$ and $\alpha_0 + \frac{1}{q} < 0$,

(v) $\theta = 1$, $q = \infty$ and $\alpha_0 = 0$;

We get the adapted conditions for the continuous interpolation spaces.

Proposition 3.4.6. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$, $\overline{b}(\phi) \leq 1$ and $q \in [1,\infty]$.*

(i) If

$$\lim_{t\to 0^+} \frac{t}{\phi(t)} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{1}{\phi(t)} > 0, \tag{3.6}$$

then $K_\infty^{0;\phi}(\mathbf{A}) = \{0\}$.

(ii) If

$$\lim_{t\to 0^+} \frac{t}{\phi(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{\phi(t)} = 0, \tag{3.7}$$

then $K_\infty^{0;\phi}$ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Proof. (i) Let $a \in \Sigma(\mathbf{A})\backslash\{0\}$. Since $K(\cdot, a)$ is non decreasing, we get

$$\frac{K(t,a)}{\phi(t)} \geq K(a,1)\frac{1}{\phi(t)} \qquad \text{for all } t \geq 1$$

and since $K(\cdot, a)/\cdot$ is non increasing,

$$\frac{K(t,a)}{\phi(t)} \geq K(a,1)\frac{t}{\phi(t)} \qquad \text{for all } t \leq 1,$$

hence the conclusion.
(ii) Since, for $a \in \Delta(\mathbf{A})$,

$$K(t,a) \leq \min\{1,t\}\|a\|_{\Delta(\mathbf{A})},$$

and for $a \in \Sigma(\mathbf{A})$,

$$K(t,a) \geq \min\{1,t\}\|a\|_{\Sigma(\mathbf{A})},$$

□

Remark 3.4.7. If $\phi = 1$ or $\phi = \mathrm{id}$, then (3.6) is verified and $K_\infty^{0,\phi}(\mathbf{A}) = \{0\}$.

We get similar conditions for the J_q^ϕ method.

Proposition 3.4.8. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$, $\overline{b}(\phi) \leq 1$, $q \in [1,\infty]$ and q' its conjugate exponent.*

(i) If

$$\int_0^1 (\frac{1}{\phi(t)})^{q'}\frac{dt}{t} = \infty \quad \text{or} \quad \int_1^\infty (\frac{1}{t\phi(t)})^{q'}\frac{dt}{t} = \infty \tag{3.8}$$

with the usual modification if $q' = \infty$, then $\|\cdot\|_{J_q^\phi(\mathbf{A})}$ equals zero on $\Delta(\mathbf{A})$.

(ii) If

$$\int_0^\infty (\frac{1}{\phi(t)}\min\{1,1/t\})^{q'}\frac{dt}{t} < \infty, \tag{3.9}$$

with the usual modification if $q' = \infty$, then J_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathcal{N}$.

Proof. One can easily adapt arguments of the appendix of [96]. □

Remark 3.4.9. Condition (3.8) is satisfied if and only if condition (3.9) is not satisfied.

3.4.1 J and K

The following famous lemma is proved in [15] for example.

Lemma 3.4.10. (Usual Fundamental Lemma)
Let $a \in \Sigma(\mathbf{A})$ and assume that

$$\lim_{t\to 0^+} \min\{1, 1/t\}K(t,a) = \lim_{t\to\infty} \min\{1, 1/t\}K(t,a) = 0. \tag{3.10}$$

Then, for any $\varepsilon > 0$, there exists a representation $a = \int b(t)dt$ with convergence in $\Sigma(\mathbf{A})$ such that

$$J(t, b(t)) \leq (C_0 + \varepsilon)K(t,a)$$

for all $t > 0$, where $C_0 \leq 3$ is a universal constant.

Remark 3.4.11. If $\phi \in \mathcal{B}$ is such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) > 1$, then condition (3.10) is satisfied for $a \in K_q^\phi(\mathbf{A})$ since for those a and for all $t > 0$,

$$K(t,a) \leq C\phi(t)\|a\|_{K_q^\phi(\mathbf{A})}.$$

This is how we obtain the equivalence Theorem.

If we look at the limiting cases, one cannot necessarily check condition (3.10) and thus cannot prove the equivalence Theorem this way. We need to adapt the Fundamental Lemma.

Lemma 3.4.12. *(Modified Fundamental Lemma)*
Let $a \in \Sigma(\mathbf{A})$ and $q \in [1, \infty)$. Assume that condition (3.10) is satisfied. Let $\phi \in \mathcal{B}$ such that

$$\int_1^\infty \left(\frac{1}{\phi(t)}\right)^q \frac{dt}{t} < \infty$$

and

$$\int_0^1 \left(\frac{1}{\phi(t)}\right)^q \frac{dt}{t} = \infty.$$

Then, ψ defined for all $s > 0$ by

$$\psi(s) = \phi(s)^{q/q'} \int_s^\infty \left(\frac{1}{\phi(t)}\right)^q \frac{dt}{t}$$

is a Boyd function and there is a representation $a = \int_0^\infty b(s)\frac{ds}{s}$ with convergence in $\Sigma(\mathbf{A})$ such that

$$\psi(t)J(t, b(t)) \leq C\frac{1}{\phi(t)}K(\xi(\eta(t)/4), a)$$

for all $t > 0$, where $\eta(t) = \int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s}$ and $\xi = \eta^{-1}$.

Proof. It is clear that $\eta \in \mathcal{B}$ since $\bar{\eta} \leq \overline{\frac{1}{\phi}}^q$, and therefore $\psi \in \mathcal{B}$. For the remainder of the proof, we can use the calculations from Lemma 4.3 of [96]. □

Remark 3.4.13. If $\underline{b}(\phi) > 0$, then $1/\phi \in L^q_*(1,\infty)$ since $\overline{b}(1/\phi) = -\underline{b}(\phi) < 0$.

Proposition 3.4.14. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, $q \in [1,\infty)$ and q' its conjugate exponent. Suppose that*

$$\int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t} < \infty$$

and

$$\int_0^1 (\frac{1}{\phi(t)})^q \frac{dt}{t} = \infty.$$

Then, ψ defined for all $s > 0$ by

$$\psi(s) = \frac{1}{\phi(t)^{q/q'}} \int_s^\infty \phi(t)^q \frac{dt}{t}$$

is a Boyd function such that

$$K_q^\phi(\mathbf{A}) = J_q^{1/\psi}(\mathbf{A}).$$

Proof. Let $a = \int_0^\infty b(s)\frac{ds}{s} \in J_q^{1/\psi}(\mathbf{A})$, one has

$$K(t,a) \leq \int_0^\infty \min\{1, \frac{t}{s}\} J(s, b(s))\frac{ds}{s}.$$

We get

$$\|a\|_{K_q^\phi(\mathbf{A})} \leq I_1 + I_2,$$

where

$$I_1 = (\int_0^\infty (\int_0^t J(s,b(s))\frac{ds}{s}\frac{1}{\phi(t)})^q \frac{dt}{t})^{1/q}$$

and

$$I_2 = (\int_0^\infty (\int_t^\infty \frac{J(s,b(s))}{s}\frac{ds}{s}\frac{t}{\phi(t)})^q \frac{dt}{t})^{1/q}.$$

Using Hardy-type inequalities (see [95]), one has

$$I_1 \leq (\int_0^\infty (\frac{t^{1/q'} J(t,b(t))\psi(t)}{t})^q dt)^{1/q} = \|a\|_{J_q^{1/\psi}(\mathbf{A})}$$

and

$$I_2 \leq (\int_0^\infty (\frac{t^{1+1/q'} J(t,b(t))\psi(t)}{t^2})^q dt)^{1/q} = \|a\|_{J_q^{1/\psi}(\mathbf{A})}.$$

Therefore, $J_q^{1/\psi}(\mathbf{A}) \hookrightarrow K_q^\phi(\mathbf{A})$. Now, if $a \in K_q^\phi(\mathbf{A})$, for all $t > 0$,

$$K(t,a)(\int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s})^{1/q} \leq \|a\|_{K_q^\phi(\mathbf{A})} < \infty,$$

so that, using our hypothesis, we get that

$$\lim_{t\to 0^+} K(t,a) = 0.$$

Moreover, for all $t > 0$, we also have

$$\frac{K(t,a)}{t}\frac{t}{\phi(t)} \le C\,\frac{K(t,a)}{t}\Big(\int_0^t \big(\frac{s}{\phi(s)}\big)^q \frac{ds}{s}\Big)^{1/q} \le \|a\|_{K_q^{\phi}(\mathbf{A})} < \infty,$$

and thus

$$\lim_{t\to\infty} \frac{K(t,a)}{t} = 0,$$

since $\lim_{t\to\infty} \frac{t}{\phi(t)} = \infty$. Therefore, condition (3.10) is satisfied and using the modified fundamental Lemma, we get $K_q^{\phi}(\mathbf{A}) \hookrightarrow J_q^{1/\psi}(\mathbf{A})$, applying $s = \eta(t)/4$ and then $u = \eta(s)$. □

For the case $q = \infty$, we also need a modified fundamental Lemma.

Lemma 3.4.15. *(Second Modified Fundamental Lemma)*
Let $a \in \Sigma(\mathbf{A})$. Assume that condition (3.10) is satisfied. Let $\phi \in \mathcal{B}$ be a non-decreasing bijection and define $\psi(t)$ for all $t > 0$ by

$$\psi(t) = -\frac{\eta^2(t)}{t\eta'(t)},$$

where $\eta(t) = \frac{1}{t}\int_0^t \frac{1}{\phi(s)}ds$. Then, ψ is a Boyd function and there is a representation $a = \int_0^\infty b(s)\frac{ds}{s}$ with convergence in $\Sigma(\mathbf{A})$ such that

$$\psi(t)J(t,b(t)) \le C\frac{1}{\phi(t)}K(\xi(1/4\phi(t)),a)$$

for all $t > 0$, where $\xi = (1/\phi)^{-1}$.

Proof. It is clear that $\eta \in \mathcal{B}$ since $\bar{\eta} \le \bar{\phi}$, and thus $\psi \in \mathcal{B}$. For the remainder of the proof, we can use the calculations from Lemma 9.5 of [96]. □

Proposition 3.4.16. *Let $\phi \in \mathcal{B}$ such that $0 \le \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$. Suppose that ϕ is a non-decreasing bijection. Then,*

$$K_\infty^{\phi}(\mathbf{A}) = J_\infty^{1/\psi}(\mathbf{A})$$

and

$$K_c^{\phi}(\mathbf{A}) = J_c^{1/\psi}(\mathbf{A}),$$

where, for all $t > 0$, $\psi(t) = -\frac{\eta^2(t)}{t\eta'(t)}$, with $\eta(t) = \frac{1}{t}\int_0^t \frac{1}{\phi(s)}ds$.

Remark 3.4.17. One can easily adapt the previous Theorems for the case $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) \le 1$, i.e. the other limit case.

Chapter 4

Interpolation of several Spaces with function parameters

Another extension of real interpolation surfaces when the scope expands from interpolating between two spaces to encompassing multiple spaces. The initial instance of interpolating across multiple spaces is documented in [39]. Subsequent exploration into this topic can be found in [6, 56, 108, 118], for example. This Section aims to extend the theory of interpolation beyond the traditional framework involving two Banach spaces with a function parameter, providing a functorial interpretation. Our work primarily follows the path set by [108]. This framework establishes the foundation for generalizing the Stein-Weiss interpolation Theorem, drawing inspiration from the approach outlined in [6], which requires a reiteration formula for triples. Moreover, families of smooth functions are not generally stable under real interpolation of couples; however, the scenario can differ significantly when interpolating between triples of smooth spaces [6, 99, 98].
We commence by introducing the notions of interpolation functors and the interpolation of multiple spaces with function parameters. Then, we explore the equivalence between the K-method and the J-method in this context, shedding light on some essential properties of the associated spaces. Subsection 4.1.3 is dedicated to presenting the power Theorem, followed by a discussion on the stability of the K- and J- spaces. Moving on to Section 4.2, we demonstrate that, in a rather general setting, these methods exclusively involve 1-functors. Lastly, we apply the developed theory to illustrate that the interpolation of several generalized Sobolev spaces still results in a generalized Besov space.

The results established in this Section were published in [70].

4.1 Notion of Interpolation with several Spaces

In the theory of real interpolation, the general correspondence from two to several Banach spaces is not always feasible. However, in a multitude of specific instances, comparable outcomes can be ascertained.

Definition 4.1.1. We denote $\mathbf{A}$ (resp. $\mathbf{B}$) as a $(n+1)$-tuples of Banach spaces $A_0, \ldots, A_n$

(resp. $B_0, \dots, B_n$) such that each A_j(resp. each B_j) can be linearly and continuously imbedded in a Hausdorff topological vector space $\mathcal{A}$ (resp. $\mathcal{B}$): $\mathbf{A} = (A_0, \dots, A_n)$. Given two such sets $\mathbf{A}$ and $\mathbf{B}$, morphisms $T : \mathbf{A} \to \mathbf{B}$ are morphisms $T : \mathcal{A} \to \mathcal{B}$ such that $T : A_j \to B_j$ are also morphisms.

Definition 4.1.2. The standard norm on $\Sigma(\mathbf{A}) = A_0 + \cdots + A_n$ is defined by

$$a \mapsto \|a\|_{\Sigma(\mathbf{A})} = \inf\{\|a_0\|_{A_0} + \cdots + \|a_n\|_{A_n}\},$$

where the infimum is taken over all decompositions $a = a_0 + \cdots + a_n$, with $a_j \in A_j$.

Definition 4.1.3. The standard norm on $\Delta(\mathbf{A}) = A_0 \cap \cdots \cap A_n$ is defined by

$$a \mapsto \|a\|_{\Delta(\mathbf{A})} = \max\{\|a\|_{A_0}, \dots, \|a\|_{A_n}\}.$$

Let $\mathscr{B}_n$ be the category of the elements $\mathbf{A}$, it is evident that Σ and Δ are functors from $\mathscr{B}_n$ to $\mathscr{B}$.

Definition 4.1.4. An interpolation functor of order n is a functor $F : \mathscr{B}_n \to \mathscr{B}$ such that

(i) $\Delta(\mathbf{A}) \hookrightarrow F(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A})$ for any $\mathbf{A}$ in $\mathscr{B}_n$,

(ii) $F(T) = T|_{F(\mathbf{A})}$ for any morphism $T : \mathbf{A} \to \mathbf{B}$.

Definition 4.1.5. Let f be a function from $(0,\infty)^{n+1}$ to $(0,\infty)$; an interpolation functor F is of type f if there exists a constant $C \geq 1$ such that

$$\|T\|_{F(\mathbf{A}),F(\mathbf{B})} \leq C\, f\left(\|T\|_{A_0,B_0}, \dots, \|T\|_{A_n,B_n}\right),$$

for any morphism $T : \mathbf{A} \to \mathbf{B}$. If, moreover,

- $f(t_0, \dots, t_n) = \max\{t_0, \dots, t_n\}$, we say that F is bounded,
- $C = 1$, we say that F is exact.

Let $\sim$ be the equivalence relation defined by $(t_0, ..., t_n) \sim (s_0, .., s_n)$ if and only if there exists $\lambda > 0$ such that $t_j = \lambda s_j$ for all $j \in \{0, ..., n\}$.

Definition 4.1.6. Let $q \in [1,\infty]$, $\phi_1, \dots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$ and f be a function from $(0,\infty)^{n+1}$ to $(0,\infty)$; one defines

$$\Phi_q^{\phi_1,\dots,\phi_n}(f) = \Big(\int_{\mathbb{P}^n_+} \big(\frac{\phi_1(t_0)\cdots\phi_n(t_0)}{t_0\phi_1(t_1)\cdots\phi_n(t_n)} f(t_0, t_1, \dots, t_n)\big)^q d\omega(t)\Big)^{1/q},$$

with the usual modification in the case $q = \infty$ and where ω is the Haar measure on $\mathbb{P}^n_+ = (0,\infty)^{n+1}/\sim$.

Remark 4.1.7. For the Haar measure ω in the previous definition, we have

$$d\omega(t) = \sum_{j=0}^{n} (-1)^j \frac{dt_0}{t_0} \wedge \cdots \wedge \frac{\check{dt_j}}{t_j} \wedge \cdots \wedge \frac{dt_n}{t_n},$$

so that

$$\Phi_q^{\phi_1,\dots,\phi_n}(f) = \Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(t_1)} \cdots \frac{1}{\phi_n(t_n)} f(1, t_1, \dots, t_n) \big)^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big)^{1/q},$$

with the usual modification in the case $q = \infty$.

Definition 4.1.8. Given $a \in \mathbf{A}$ and $t \in (0, \infty)^{n+1}$, let

$$K(t, a) = \inf_a \{ t_0 \|a_0\|_{A_0} + t_1 \|a_1\|_{A_1} + \cdots + t_n \|a_n\|_{A_n} \},$$

where the infimum is taken over all the decompositions $a = a_0 + \cdots + a_n$, $a_j \in A_j$. In the same way, for $a \in \Delta(\mathbf{A})$, we set

$$J(t, a) = \max\{ t_0 \|a\|_{A_0}, t_1 \|a\|_{A_1}, \dots, t_n \|a\|_{A_n} \}.$$

It is noteworthy that

$$K(t, a) \le \max \left\{ \frac{t_0}{s_0}, \dots, \frac{t_n}{s_n} \right\} K(s, a),$$

$$J(t, a) \le \max \left\{ \frac{t_0}{s_0}, \dots, \frac{t_n}{s_n} \right\} J(s, a)$$

and

$$K(t, a) \le \max \left\{ \frac{t_0}{s_0}, \dots, \frac{t_n}{s_n} \right\} J(s, a).$$

Definition 4.1.9. Let $q \in [1, \infty]$, $\phi_1, \dots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; we define $K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})$ as the set of $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})} = \Phi_q^{\phi_1,\dots,\phi_n}\big(K(t, a)\big) < \infty.$$

Proposition 4.1.10. *The functor $K_q^{\phi_1,\dots,\phi_n}$ is an exact interpolation functor of type f where $f(t_0, \dots, t_n) = t_0 \overline{\phi_1}(t_1/t_0) \cdots \overline{\phi_n}(t_n/t_0)$.*

Moreover,

$$K(1, t_1, \dots, t_n, a) \le C\, \phi_1(t_1) \cdots \phi_n(t_n) \|a\|_{K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})}.$$

Proof. One has

$$\begin{aligned}
&\Big(\int_{(0,\infty)^n}\big(\frac{1}{\phi_1(u_1s_1)\cdots\phi_n(u_ns_n)}\min\{1,u_1,\ldots,u_n\}\big)^q\frac{du_1}{u_1}\cdots\frac{du_n}{u_n}\Big)^{1/q} \\
&\quad K(1,s_1,\ldots,s_n,a) \\
&\leq \Big(\int_{(0,\infty)^n}\big(\frac{1}{\phi_1(t_1)\cdots\phi_n(t_n)}\min\{1,t_1/s_1,\ldots,t_n/s_n\} \\
&\quad K(1,s_1,\ldots,s_n)\big)^q\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}\Big)^{1/q} \\
&\leq \|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})},
\end{aligned}$$

so that

$$K(1,s_1,\ldots,s_n,a)\leq C\,\phi_1(s_1)\cdots\phi_n(s_n)\|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})}. \tag{4.1}$$

By using (4.1) with $s_j=1$, one has

$$\|a\|_{\Sigma(\mathbf{A})}=K(1,a)\leq C\,\|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})},$$

so that $K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})\hookrightarrow\Sigma(\mathbf{A})$.

Now, since

$$K(1,t_1,\ldots,t_n,a)\leq\min\{1,t_1,\ldots,t_n\}\|a\|_{\Delta(\mathbf{A})},$$

one has

$$\|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})}\leq C\,\|a\|_{\Delta(\mathbf{A})},$$

so that $\Delta(\mathbf{A})\hookrightarrow K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$.

Now, let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathscr{B}_n$ and let $T:\mathbf{A}\to\mathbf{B}$; one has

$$K^{\mathbf{B}}(1,t_1,\ldots,t_n,Ta)\leq\|T\|_{A_0,B_0}\,K^{\mathbf{A}}(1,t_1\frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}},\ldots,t_n\frac{\|T\|_{A_n,B_n}}{\|T\|_{A_0,B_0}},a),$$

so that, for $\alpha_j=\frac{\|T\|_{A_j,B_j}}{\|T\|_{A_0,B_0}}$, we have

$$\begin{aligned}
&\|Ta\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{B})} \\
&\leq\|T\|_{A_0,B_0}\Big(\int_{(0,\infty)^n}\big(\frac{1}{\phi_1(u_1/\alpha_1)\cdots\phi_n(u_n/\alpha_n)}K^{\mathbf{A}}(1,u_1,\ldots,u_n,a)\big)^q \\
&\quad\frac{du_1}{u_1}\cdots\frac{du_n}{u_n}\Big)^{1/q} \\
&\leq\|T\|_{A_0,B_0}\overline{\phi_1}(\alpha_1)\cdots\overline{\phi_n}(\alpha_n)\|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})},
\end{aligned}$$

thus completing the demonstration. □

Remark 4.1.11. In practice, we deal with $K(1,t_1,...,t_n)$. For $t>0$, we know that $t\mapsto K(t,a)$ has a geometrical interpretation in the Gagliardo diagram (see [15]): let

$$\Gamma(a)=\{(x_0,x_1)\in\mathbb{R}^2:\exists a_0\in A_0,a_1\in A_1:a=a_0+a_1 \text{ and } \|a_0\|_{A_0}\le x_0,\|a_1\|_{A_1}\le x_1\}.$$

It is clear that $\Gamma(a)$ is a convex subset of $\mathbb{R}^2$ with

$$K(t,a)=\inf_{x\in\Gamma(a)}\{x_0+tx_1\}=\inf_{x\in\partial\Gamma(a)}\{x_0+tx_1\},$$

i.e. $K(t,a)$ is the x_0-intercept of the tangent to $\partial\Gamma(a)$, with slope $\frac{-1}{t}$.
In three dimensions, there is the adapted interpretation :

$$\Gamma(a)=\{x\in\mathbb{R}^3: \text{ for } j\in\{0,1,2\},\exists a_j\in A_j:a=a_0+a_1+a_2 \text{ and } \|a_j\|_{A_j}\le x_j\}$$

and

$$K(1,t_1,t_2,a)=\inf_{x\in\Gamma(a)}\{x_0+t_1x_1+t_2x_2\}=\inf_{x\in\partial\Gamma(a)}\{x_0+t_1x_1+t_2x_2\},$$

i.e. $K(1,t_1,t_2,a)$ is the intersection of $\partial\Gamma(a)$ and its tangent plane with normal vector $(1,t_1,t_2)$.

Definition 4.1.12. Let $q\in[1,\infty]$, $\phi_1,\dots,\phi_n\in\mathcal{B}$ be such that $0<\underline{b}(\phi_1)+\cdots+\underline{b}(\phi_n)$ and $\overline{b}(\phi_1)+\cdots+\overline{b}(\phi_n)<1$; $J_q^{\phi_1,\dots,\phi_n}(\mathbf{A})$ is the set of $a\in\Sigma(\mathbf{A})$ such that there exists a function $u:\mathbb{P}_+^n\to\Delta(\mathbf{A})$ for which we have

$$a=\int u(t)d\omega(t)=\int_{(0,\infty)^n}u(1,t_1,\dots,t_n)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}\quad \text{in } \Sigma(\mathbf{A})$$

and

$$\Phi_q^{\phi_1,\dots,\phi_n}\Big(J\big(t,u(t)\big)\Big)<\infty.$$

This space is equipped with the norm

$$\|a\|_{J_q^{\phi_1,\dots,\phi_n}(\mathbf{A})}=\inf_u\Phi_q^{\phi_1,\dots,\phi_n}\Big(J\big(t,u(t)\big)\Big),$$

the infimum being taken on all $u:\mathbb{P}_+^n\to\Delta(\mathbf{A})$ such that $a=\int u(t)d\omega(t)$.

Proposition 4.1.13. *The functor $J_q^{\phi_1,\dots,\phi_n}$ is an exact interpolation functor of type f where $f(t_0,\dots,t_n)=t_0\overline{\phi_1}(t_1/t_0)\cdots\overline{\phi_n}(t_n/t_0)$.*

Moreover,

$$\|a\|_{J_q^{\phi_1,\dots,\phi_n}(nA)}\le C\,\frac{1}{\phi_1(t_1)\cdots\phi_n(t_n)}J(1,t_1,\dots,t_n,a).$$

Proof. For $a\in\Delta(\mathbf{A})$, we have

$$a=\frac{1}{(\log 2)^n}\int_{(0,\infty)^n}a\chi_{(1,2)^n}(t)\frac{dt}{t},$$

and

$$\begin{aligned}
&\|a\|_{J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})}\\
&\leq \Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(t_1)\cdots\phi_n(t_n)} J(1,t_1,\ldots,t_n, a\frac{\chi_{(1,2)^n}(t)}{(\log 2)^n})\big)^q \frac{dt}{t}\Big)^{1/q}\\
&\leq \Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(t_1)\cdots\phi_n(t_n)(\log 2)^n} \max\{1, t_1/s_1,\ldots,t_n/s_n\}\\
&\quad J(1,s_1,\ldots,s_n,a)\big)^q \frac{dt}{t}\Big)^{1/q}\\
&\leq C\,\frac{1}{\phi_1(s_1)\cdots\phi_n(s_n)} J(1,s_1,\ldots,s_n,a).
\end{aligned}$$

In particular, by taking $s_j = 1$ in the previous inequality, one gets

$$\|a\|_{J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})} \leq C\|a\|_{\Delta(\mathbf{A})},$$

so that $\Delta(\mathbf{A}) \hookrightarrow J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$. Moreover, with the inclusion of the equivalence Theorem, one has

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) \hookrightarrow K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A}).$$

Now, let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathscr{B}_n$ and let $T:\mathbf{A}\to\mathbf{B}$. If $a = \int u(s)d\omega(s) \in J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$, then $Ta = \int Tu(s)d\omega(s) \in J_q^{\phi_1,\ldots,\phi_n}(\mathbf{B})$ and

$$\begin{aligned}
&J^{\mathbf{B}}(1,s_1,\ldots,s_n,Tu(1,s_1,\ldots,s_n))\\
&\leq \|T\|_{A_0,B_0} J^{\mathbf{A}}\big(1, s_1\frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}},\ldots, s_n\frac{\|T\|_{A_n,B_n}}{\|T\|_{A_0,B_0}}, u(1,s_1,\ldots,s_n)\big),
\end{aligned}$$

so that we have, for $\alpha_j = \frac{\|T\|_{A_j,B_j}}{\|T\|_{A_0,B_0}}$,

$$\begin{aligned}
&\|Ta\|_{J_q^{\phi_1,\ldots,\phi_n}(\mathbf{B})}\\
&\leq \|T\|_{A_0,B_0}\Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(t_1)\cdots\phi_n(t_n)} J^{\mathbf{A}}(1,t_1\alpha_1,\ldots,t_n\alpha_n,u(t))\big)^q \frac{dt}{t}\Big)^{1/q}\\
&= \|T\|_{A_0,B_0}\Big(\int_{(0,\infty)^n} \big(\frac{1}{\phi_1(s_1/\alpha_1)\cdots\phi_n(s_n/\alpha_n)} J^{\mathbf{A}}(1,s_1,\ldots,s_n,u(t(s)))\big)^q \frac{ds}{s}\Big)^{1/q}\\
&\leq \|T\|_{A_0,B_0}\overline{\phi_1}(\alpha_1)\cdots\overline{\phi_n}(\alpha_n)\|a\|_{J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})},
\end{aligned}$$

which ends the proof. □

By following the methodology outlined in [108] and mimicking tools used in Section 2.5, one can readily verify the following equivalent definitions.

Proposition 4.1.14. *Let $q \in [1,\infty]$, $\phi_1, \ldots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; an equivalent norm on $K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$ is given by*

$$\inf\Big\{\Big(\int \|a_0(t)\|_{A_0}^q d\omega(t)\Big)^{1/q} + \sum_{j=1}^n \Big(\int \big(\frac{t_j\phi_1(t_0)\cdots\phi_n(t_0)}{t_0\phi_1(t_1)\cdots\phi_n(t_n)}\|a_j(t)\|_{A_j}\big)^q d\omega(t)\Big\}^{1/q},$$

where the infimum is taken over all decompositions $a = a_0(t) + \cdots + a_n(t)$, $a_j(t) \in A_j$.

Proposition 4.1.15. *Let $q \in [1,\infty]$, $\phi_1, \ldots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; an equivalent norm on $J_q^{\phi_1,\cdots,\phi_n}(\mathbf{A})$ is given by*

$$\inf_u \max\Big\{\Big(\int \|u(t)\|_{A_0}^q d\omega(t)\Big)^{1/q},$$
$$\max_{1\le j\le n}\Big\{\int \big(\frac{t_j\phi_1(t_0)\cdots\phi_n(t_0)}{t_0\phi_1(t_1)\cdots\phi_n(t_n)}\|u(t)\|_{A_j}\big)^q d\omega(t)\Big\}^{1/q}\Big\},$$

where the infimum is taken over all $u : \mathbb{P}_+^n \to \Delta(\mathbf{A})$ such that $a = \int u(t)d\omega(t)$ in $\Sigma(\mathbf{A})$.

4.1.1 On the Equivalence Theorem

For $n > 1$, the K- and the J- methods are not equivalent, with the K-method being more general.

Proposition 4.1.16. *Let $q \in [1,\infty]$, $\phi_1, \ldots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; then*

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) \hookrightarrow K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}).$$

Proof. Let $a \in J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$, so there exists $u : \mathbb{P}_+^n \to \Delta(\mathbf{A})$ such that

$$a = \int_{(0,\infty)^n} u(1, t_1, \ldots, t_n)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}.$$

We have

$$\begin{aligned}
&K(1, t_1, \ldots, t_n, a)\\
&\le \int_{\mathbb{R}_+^n} K\big(1, t_1, \ldots, t_n, u(1, s_1, \ldots, s_n)\big)\frac{ds_1}{s_1}\cdots\frac{ds_n}{s_n}\\
&\le \int_{\mathbb{R}_+^n} \min\{1, t_1/s_1, \ldots, t_n/s_n\}J\big(1, s_1, \ldots, s_n, u(1, s_1, \ldots, s_n)\big)\frac{ds_1}{s_1}\cdots\frac{ds_n}{s_n}\\
&\le \int_{\mathbb{R}_+^n} \min\{1, 1/s_1, \ldots, 1/s_n\}J\big(1, s_1t_1, \ldots, s_nt_n, u(1, s_1t_1, \ldots, s_nt_n)\big)\frac{ds_1}{s_1}\cdots\frac{ds_n}{s_n}.
\end{aligned}$$

Using properties of the Boyd functions and Young's inequality, we can conclude. □

Now, we provide a counterexample for the reverse inclusion:

Example 4.1.17. Let A_0 and A_1 be two Banach spaces and $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi), \overline{b}(\phi) < 1$, such that

$$A_1 \subsetneq K^{\phi}_{\infty}(A_0, A_1) \hookrightarrow A_0.$$

Let $a \in K^{\phi}_{\infty}(A_0, A_1) \backslash A_1$ and set $A_2 = \rangle a \langle$ with norm $\| \cdot \|_{A_0}$. We then consider $\mathbf{A} = (A_0, A_1, A_2)$. Clearly, $\Delta(\mathbf{A}) = \{0\}$, so that $J^{\phi_1,\phi_2}_q(\mathbf{A}) = \{0\}$ for all p, ϕ_1, ϕ_2.

Let us demonstrate the existence of $\phi_1, \phi_2 \in \mathcal{B}$ such that $a \in K^{\phi_1,\phi_2}_q(\mathbf{A})$. Without loss of generality, one can assume that

$$\max\{\|a\|_{A_0}, \|a\|_{K^{\phi}_{\infty}(A_0,A_1)}\} \leq 1.$$

By the given assumptions, we obtain

$$\begin{aligned} K(t,a) &\leq \min\{t_0\|a\|_{A_0}, K^{(A_0,A_1)}\big((t_0,t_1),a\big), t_2\|a\|_{A_2}\} \\ &\leq \min\{t_0, \frac{t_0\phi(t_1)}{\phi(t_0)}, t_2\}. \end{aligned}$$

Therefore, using $(s_1, s_2) = (\phi(t_1), t_2)$, one has

$$\begin{aligned} \|a\|_{K^{\phi_1,\phi_2}_q(\mathbf{A})} &\leq \Big(\int_{\mathbb{P}^2_+} \big(\frac{\phi_1(t_0)\phi_2(t_0)}{t_0\phi_1(t_1)\phi_2(t_2)} \min\{t_0, \frac{t_0\phi(t_1)}{\phi(t_0)}, t_2\}\big)^q dw(t)\Big)^{1/q} \\ &= \Big(\int_{(0,\infty)^2} \big(\frac{1}{\phi_1(t_1)\phi_2(t_2)} \min\{1, \phi(t_1), t_2\}\big)^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\Big)^{1/q} \\ &\leq C\Big(\int_{(0,\infty)^2} \big(\frac{1}{\phi_1(\xi(s_1))\phi_2(s_2)} \min\{1, s_1, s_2\}\big)^q \frac{ds_1}{s_1}\frac{ds_2}{s_2}\Big)^{1/q}, \end{aligned}$$

where $\xi = \phi^{-1}$. Thus, for $\overline{b}(\phi_1)/\underline{b}(\phi) + \overline{b}(\phi_2) < 1$, we have $a \in K^{\phi_1,\phi_2}_q(\mathbf{A})$.

Let us introduce a sufficient condition for the equivalence Theorem to hold.

Definition 4.1.18. We denote by $\sigma(\mathbf{A})$ the subspace of all $a \in \Sigma(\mathbf{A})$ for which

$$\int \frac{K(t,a)}{\max t} d\omega(t) < \infty.$$

Definition 4.1.19. The condition $\mathcal{F}(\mathbf{A})$ is satisfied if, for every $a \in \sigma(\mathbf{A})$, there exists a function $u : \mathbb{P}^n_+ \to \Delta(\mathbf{A})$ such that

$$a = \int u(t) d\omega(t) \quad \text{in} \quad \Sigma(\mathbf{A})$$

and

$$J\big(t, u(t)\big) \leq C(\mathbf{A}) K(t,a).$$

In the case $n = 1$, $\mathcal{F}(\mathbf{A})$ is satisfied for any Banach couple (Equivalence Theorem 2.4.3); however, for $n > 1$, it is not necessarily the case.

Theorem 4.1.20. *Let $q \in [1, \infty]$, $\phi_1, \ldots, \phi_n \in \mathcal{B}$ such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; if $\mathcal{F}(\mathbf{A})$ is satisfied, then*

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}). \tag{4.2}$$

Proof. If $a \in K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$, then $a \in \sigma(\mathbf{A})$. Therefore, we get

$$\begin{aligned} \|a\|_{J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})} \leq \Phi_q^{\phi_1,\ldots,\phi_n}\Big(J\big(t, u(t)\big)\Big) &\leq C\Phi_q^{\phi_1,\ldots,\phi_n}\big(K(t,a)\big) \\ &= C\|a\|_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})}, \end{aligned}$$

which is sufficient to conclude. □

If the condition is satisfied, we denote the space (4.2) by $\mathbf{A}_q^{\phi_1,\ldots,\phi_n}$.

Proposition 4.1.21. *If $\mathcal{F}(\mathbf{A})$ is satisfied and $\mathbf{B}$ is a retract of $\mathbf{A}$, then $\mathcal{F}(\mathbf{B})$ is also satisfied.*

Proof. Let $b \in \sigma(\mathbf{B})$, so that $Ib \in \sigma(\mathbf{A})$. Consequently, there exists $u : \mathbb{P}_+^n \to \Delta(\mathbf{A})$ such that

$$Ib = \int u(t) d\omega(t) \quad \text{in} \quad \Sigma(\mathbf{A})$$

and

$$J\big(t, u(t)\big) \leq C(\mathbf{A}) K(t, Ib).$$

We have

$$b = PIb = \int Pu(t) d\omega(t) \quad \text{in} \quad \Sigma(\mathbf{B})$$

and

$$J^{\mathbf{B}}\big(t, Pu(t)\big) \leq CJ^{\mathbf{A}}\big(t, u(t)\big) \leq CK^{\mathbf{A}}(t, Ib) \leq CK^{\mathbf{B}}(t, b),$$

bringing the proof to a close. □

By interpolation, we have the following result:

Proposition 4.1.22. *If $\mathbf{B}$ is a retract of $\mathbf{A}$, then $K_q^{\phi_1,\ldots,\phi_n}(\mathbf{B})$ is a retract of $K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$ and $J_q^{\phi_1,\ldots,\phi_n}(\mathbf{B})$ is a retract of $J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$, with the same P and I.*

4.1.2 Some properties

The following property is a direct consequence of Hölder's inequality.

Proposition 4.1.23. *For $1 \le p \le q \le \infty$, we have*

$$K_p^{\phi_1,\ldots,\phi_n}(\mathbf{A}) \hookrightarrow K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$$

and

$$J_p^{\phi_1,\ldots,\phi_n}(\mathbf{A}) \hookrightarrow J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}).$$

We also directly have

Proposition 4.1.24. (Permutation result)
Let π be a permutation of $\{0, 1, \ldots, n\}$ and set

$$\mathbf{A}^{(\pi)} = (A_{\pi(0)}, A_{\pi(1)}, \ldots, A_{\pi(n)}),$$

$\phi_0(t) = t/(\phi_1(t) \cdots \phi_n(t))$ $(t > 0)$ *and*

$$(\psi_0, \psi_1, \ldots, \psi_n) = \big((\phi_{\pi(0)})_p, (\phi_{\pi(1)})_p, \ldots, (\phi_{\pi(n)})_p\big).$$

We have

$$K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = K_q^{\psi_1,\ldots,\psi_n}(\mathbf{A}^{(\pi)})$$

and

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = J_q^{\psi_1,\ldots,\psi_n}(\mathbf{A}^{(\pi)}).$$

Proposition 4.1.25. (Reduction result)
If $A_{n-1} = A_n$ $(n > 1)$, then

$$K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = K_q^{\phi_1,\ldots,\phi_{n-1}\phi_n}(A_0, \ldots, A_{n-1})$$

and

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = J_q^{\phi_1,\ldots,\phi_{n-1}\phi_n}(A_0, \ldots, A_{n-1}).$$

Proof. Let us show the result for the case K. For $\mathbf{A}' = (A_0, \ldots, A_{n-1})$, we have

$$K^{\mathbf{A}}(1, t_1, \ldots, t_n, a) = K^{\mathbf{A}'}(1, t_1, \ldots, \min\{t_{n-1}, t_n\}, a).$$

This identity allows to conclude, since

$$\begin{aligned}&\int_{(0,\infty)^2} \big(\frac{1}{\phi_{n-1}(t_{n-1})\phi_n(t_n)} K^{\mathbf{A}}(1, t_1, \ldots, t_n, a)\big)^q \frac{dt_{n-1}}{t_{n-1}} \frac{dt_n}{t_n} \\ &= C \int_0^\infty \big(\frac{1}{\phi_{n-1}(s)\phi_n(s)} K^{\mathbf{A}'}(1, t_1, \ldots, s, a)\big)^q \frac{ds}{s}.\end{aligned}$$

□

Let us highlight the following outcome from [108].

Proposition 4.1.26. *If $A_{n-1} = A_n$ and $\mathcal{F}(A_0, \ldots, A_{n-1})$ is satisfied, then $\mathcal{F}(\mathbf{A})$ is also satisfied.*

4.1.3 Power Theorem

If $\mathbf{A} = (A_0, ..., A_n)$ is in $\mathscr{B}_n$, $\alpha > 0$, we naturally define

$$\mathbf{A}^{(\alpha)} = (A_0^{(\alpha)}, \dots, A_n^{(\alpha)}).$$

Theorem 4.1.27. *Let $q \in [1, \infty]$, $\alpha > 0$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1) + \cdots + \underline{b}(\phi_n)$ and $\overline{b}(\phi_1) + \cdots + \overline{b}(\phi_n) < 1$; we have*

$$K_q^{\phi_1,\dots,\phi_n}(\mathbf{A}^{(\alpha)}) = K_{\alpha q}^{(\phi_1)_\alpha,\dots,(\phi_n)_\alpha}(\mathbf{A})^{(\alpha)}.$$

Proof. Let us define

$$K_\infty(t, a) = \inf_a \max\{t_0\|a_0\|_{A_0}, t_1\|a_1\|_{A_1}, \dots, t_n\|a_n\|_{A_n}\},$$

where the infimum is taken over all the decompositions $a = a_0 + \cdots + a_n$, with $a_j \in A_j$. It can be observed that

$$K_\infty(t, a) \le K(t, a) \le nK_\infty(t, a).$$

If $q = \infty$, we have

$$\begin{aligned} \|a\|_{K_\infty^{\phi_1,\dots,\phi_n}(\mathbf{A}^{(\alpha)})} &\asymp \sup_s \frac{1}{\phi_1(s_1)} \cdots \frac{1}{\phi_n(s_n)} K_\infty(1, s_1, \dots, s_n, a; \mathbf{A}^{(\alpha)}) \\ &= \sup_t \frac{1}{(\phi_1)_\alpha(t_1)^\alpha} \cdots \frac{1}{(\phi_n)_\alpha(t_n)^\alpha} K_\infty(1, t_1, \dots, t_n, a; \mathbf{A})^\alpha. \end{aligned}$$

If $q < \infty$, we get

$$\begin{aligned} &\|a\|^q_{K_q^{\phi_1,\dots,\phi_n}(\mathbf{A}^{(\alpha)})} \\ &\asymp \int_{(0,\infty)^n} \Big(\frac{1}{\phi_1(s_1)} \cdots \frac{1}{\phi_n(s_n)} K_\infty(1, s_1, \dots, s_n)\Big)^q \frac{ds_1}{s_1} \cdots \frac{ds_n}{s_n} \\ &= \int_{(0,\infty)^n} \Big(\frac{1}{(\phi_1)_\alpha(t_1)} \cdots \frac{1}{(\phi_n)_\alpha(t_n)} K_\infty(1, t_1, \dots, t_n)\Big)^{\alpha q} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \end{aligned}$$

thereby concluding the proof. □

4.1.4 Stability

In this Section, we use X to denote an intermediate space with respect to $\mathbf{A}$.

Definition 4.1.28. We define the following classes for X:

1. X is of class $\mathscr{C}_K(\phi_1, \dots, \phi_n; \mathbf{A})$ if

$$K(1, t_1, \dots, t_n, a) \le C\, \phi_1(t_1) \cdots \phi_n(t_n)\|a\|_X,$$

   for all $a \in X$;

2. X is of class $\mathscr{C}_J(\phi_1, \ldots, \phi_n; \mathbf{A})$ if

$$\|a\|_X \leq C \frac{1}{\phi_1(t_1) \cdots \phi_n(t_n)} J(1, t_1, \ldots, t_n, a),$$

for all $a \in \Delta(\mathbf{A})$.

We say that X is of class $\mathscr{C}(\phi_1, \ldots, \phi_n; \mathbf{A})$ if X is of class $\mathscr{C}_K(\phi_1, \ldots, \phi_n; \mathbf{A})$ and of class $\mathscr{C}_J(\phi_1, \ldots, \phi_n; \mathbf{A})$.

The two following propositions are straightforward from these definitions.

Proposition 4.1.29. *X is of class $\mathscr{C}_K(\phi_1, \ldots, \phi_n; \mathbf{A})$ if and only if*

$$X \hookrightarrow K_\infty^{\phi_1, \ldots, \phi_n}(\mathbf{A}).$$

Proposition 4.1.30. *X is of class $\mathscr{C}_J(\phi_1, \ldots, \phi_n; \mathbf{A})$ if and only if*

$$K_1^{\phi_1, \ldots, \phi_n}(\mathbf{A}) \hookrightarrow X.$$

We have the following result.

Proposition 4.1.31. *Let $q \in [1, \infty]$, $m \geq n$, $0 < \lambda_1 + \cdots + \lambda_m < 1$, $\phi_1^{(j)}, \ldots, \phi_n^{(j)} \in \mathcal{B}$ such that $0 < \underline{b}(\phi_1^{(j)}) + \cdots + \underline{b}(\phi_n^{(j)})$ and $\overline{b}(\phi_1^{(j)}) + \cdots + \overline{b}(\phi_n^{(j)}) < 1$ for all $j \in \{1, \ldots, m\}$. Set*

$$(\phi_1, \ldots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \cdots (\phi_1^{(m)})^{\lambda_m}, \ldots, (\phi_n^{(1)})^{\lambda_1} \cdots (\phi_n^{(m)})^{\lambda_m})$$

and suppose that the vectors

$$\begin{pmatrix} E\phi_1^{(j)}(t_1) \\ \vdots \\ E\phi_n^{(j)}(t_n) \end{pmatrix} \quad (j \in \{1, \ldots, m\})$$

generate $\mathbb{R}^n$. Finally, let $\mathbf{X} = (X_0, \ldots, X_m)$.

(i) If X_j is of class $\mathscr{C}_K(\phi_1^{(j)}, \ldots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \ldots, m\}$, then, with obvious notations,

$$K_q^{\lambda_1, \ldots, \lambda_m}(\mathbf{X}) \hookrightarrow K_q^{\phi_1, \ldots, \phi_n}(\mathbf{A}).$$

(ii) If X_j is of class $\mathscr{C}_J(\phi_1^{(j)}, \ldots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \ldots, m\}$, then

$$J_q^{\phi_1, \ldots, \phi_n}(\mathbf{A}) \hookrightarrow J_q^{\lambda_1, \ldots, \lambda_m}(\mathbf{X}).$$

Proof. First suppose that $m = n$ and remark that

$$\begin{cases} s_1 = \phi_1^{(1)}(t_1)\cdots\phi_n^{(1)}(t_n), \\ \vdots \\ s_n = \phi_1^{(n)}(t_1)\cdots\phi_n^{(n)}(t_n), \end{cases}$$

is an appropriate change of variable if and only if the vectors

$$\begin{pmatrix} \frac{t_1 D\phi_1^{(j)}(t_1)}{\phi_1^{(j)}(t_1)} \\ \vdots \\ \frac{t_n D\phi_n^{(j)}(t_n)}{\phi_n^{(j)}(t_n)} \end{pmatrix} \quad (j \in \{1,\dots,n\})$$

are linearly independent.

For $a = a_0 + \cdots + a_n \in \Sigma(\mathbf{X})$, we get

$$\begin{aligned} K(1,t_1,\dots,t_n,a;\mathbf{A}) &\le \|a_0\|_{A_0} + K(t,a_1;\mathbf{A}) + \cdots + K(t,a_n;\mathbf{A}) \\ &\le C(\|a_0\|_{X_0} + \phi_1^{(1)}(t_1)\cdots\phi_n^{(1)}(t_n)\|a_1\|_{X_1} \\ &\quad + \cdots + \phi_1^{(n)}(t_1)\cdots\phi_n^{(n)}(t_n)\|a_n\|_{X_n}). \end{aligned}$$

Therefore, using

$$\begin{cases} s_1 = \phi_1^{(1)}(t_1)\cdots\phi_n^{(1)}(t_n), \\ \qquad\qquad \vdots \\ s_n = \phi_1^{(n)}(t_1)\cdots\phi_n^{(n)}(t_n), \end{cases}$$

one gets (i), since

$$\begin{aligned} \|a\|_{K_q^{\phi_1,\dots,\phi_n}(\mathbf{A})} &= \Phi_q^{\phi_1,\dots,\phi_n}\big(K(t,a)\big) \\ &\le C\Phi_q^{\lambda_1,\dots,\lambda_m}\big(K(s,a;\mathbf{X})\big) \\ &= C\|a\|_{K_q^{\lambda_1,\dots,\lambda_m}(\mathbf{X})}. \end{aligned}$$

Let us prove (ii). Set $a = \int u(t)d\omega(t) \in J_q^{\phi_1,\dots,\phi_n}(\mathbf{A})$ and define v so that

$$C(s)v\big(s(t)\big) = u(t).$$

We have

$$\begin{aligned} \|a\|_{J_q^{\lambda_1,\dots,\lambda_m}(\mathbf{X})} &= \Phi_q^{\lambda_1,\dots,\lambda_m}\big(J(s,v(s);\mathbf{X})\big) \\ &= C\Phi_q^{\phi_1,\dots,\phi_n}\Big(J\big(s(t),v\big(s(t)\big);\mathbf{X}\big)\Big) \\ &\le C\Phi_q^{\phi_1,\dots,\phi_n}\big(J(t,u(t);\mathbf{A})\big). \end{aligned}$$

Now suppose $m > n$ and consider the $(m+1)$-tuple

$$\mathbf{A}' = (A_0, \ldots, A_n, A_n, \ldots, A_n).$$

Using Proposition 4.1.25, we get

$$K_q^{\psi_1,\ldots,\psi_m}(\mathbf{A}') = K_q^{\psi_1,\ldots,\psi_{n-1},\psi_n\cdots\psi_m}(\mathbf{A}),$$

bringing the proof to a close. □

By adjusting the reasoning from the preceding demonstration, we derive the following results.

Theorem 4.1.32. *Let $q \in [1,\infty]$, $m \geq n$, $0 < \lambda_1 + \cdots + \lambda_m < 1$, $\phi_1^{(j)}, \ldots, \phi_n^{(j)} \in \mathcal{B}$ such that $0 < \underline{b}(\phi_1^{(j)}) + \cdots + \underline{b}(\phi_n^{(j)})$ and $\overline{b}(\phi_1^{(j)}) + \cdots + \overline{b}(\phi_n^{(j)}) < 1$ for all $j \in \{1, \cdots, m\}$. Set*

$$\big(\phi_1, \ldots, \phi_n\big) = \big((\phi_1^{(1)})^{\lambda_1} \cdots (\phi_1^{(m)})^{\lambda_m}), \ldots, (\phi_n^{(1)})^{\lambda_1} \cdots (\phi_n^{(m)})^{\lambda_m})\big)$$

and suppose that the vectors

$$\begin{pmatrix} E\phi_1^{(j)}(t_1) \\ \vdots \\ E\phi_n^{(j)}(t_n) \end{pmatrix} \quad (j \in \{1, \ldots, m\})$$

generate $\mathbb{R}^n$. Finally, let $\mathbf{X} = (X_0, \ldots, X_m)$. If X_j is of class $\mathscr{C}(\phi_1^{(j)}, \ldots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \ldots, m\}$ and if

$$K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$\mathbf{X}_q^{\lambda_1,\ldots,\lambda_m} = \mathbf{A}_q^{\phi_1,\ldots,\phi_n}.$$

Corollary 4.1.33. *Let $m \geq n$, $q, q_0, \ldots, q_m \in [1,\infty]$, $0 < \lambda_1 + \cdots + \lambda_m < 1$, $\phi_1^{(j)}, \ldots, \phi_n^{(j)} \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1^{(j)}) + \cdots + \underline{b}(\phi_n^{(j)})$ and $\overline{b}(\phi_1^{(j)}) + \cdots + \overline{b}(\phi_n^{(j)}) < 1$ for all $j \in \{1, \ldots, m\}$. Set*

$$(\phi_1, \ldots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \cdots (\phi_1^{(m)})^{\lambda_m}), \ldots, (\phi_n^{(1)})^{\lambda_1} \cdots (\phi_n^{(m)})^{\lambda_m}))$$

and suppose that the vectors

$$\begin{pmatrix} E\phi_1^{(j)}(t_1) \\ \vdots \\ E\phi_n^{(j)}(t_n) \end{pmatrix} \quad (j \in \{1, \ldots, m\})$$

generate $\mathbb{R}^n$. If $K_{q_j}^{\phi_1^{(j)},\ldots,\phi_n^{(j)}}(\mathbf{A}) = J_{q_j}^{\phi_1^{(j)},\ldots,\phi_n^{(j)}}(\mathbf{A})$ for $j \in \{0, \ldots, m\}$ and if

$$K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$(\mathbf{A}_{q_0}^{\phi_1^{(0)},\ldots,\phi_n^{(0)}}, \ldots, \mathbf{A}_{q_m}^{\phi_1^{(m)},\ldots,\phi_n^{(m)}})_q^{\lambda_1,\ldots,\lambda_m} = \mathbf{A}_q^{\phi_1,\ldots,\phi_n}.$$

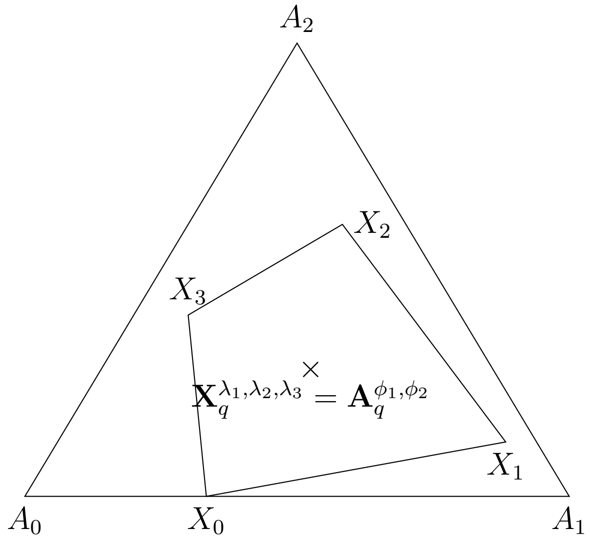


Figure 4.1: Illustration of Theorem 4.1.32 with $m = 3$ and $n = 2$.

Proposition 4.1.34. *Let $m \geq n$, $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1^{(j)}) + \cdots + \underline{b}(\phi_n^{(j)})$ and $\overline{b}(\phi_1^{(j)}) + \cdots + \overline{b}(\phi_n^{(j)}) < 1$ for all $j \in \{1, \dots, m\}$. Suppose that the vectors*

$$\begin{pmatrix} E\phi_1^{(j)}(t_1) \\ \vdots \\ E\phi_n^{(j)}(t_n) \end{pmatrix} \quad (j \in \{1, \dots, m\})$$

generate $\mathbb{R}^n$. Finally, let $\mathbf{X} = (X_0, \dots, X_m)$ such that X_j is of class $\mathscr{C}(\phi_1^{(j)}, \dots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \dots, m\}$. If $\mathcal{F}(\mathbf{A})$ is satisfied, then $\mathcal{F}(\mathbf{X})$ is also satisfied.

Additional stability results can be achieved by imposing restrictions on $\lambda_1, \dots, \lambda_m$, q, $q_0, \dots, q_m$ instead of $\phi_1^{(j)}, \dots, \phi_n^{(j)}$. The ensuing result can be inferred from [108]:

Theorem 4.1.35. *Let $q, q_0, \dots, q_m \in [1, \infty]$, $0 < \lambda_1 + \cdots + \lambda_m < 1$, $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ be such that $0 < \underline{b}(\phi_1^{(j)}) + \cdots + \underline{b}(\phi_n^{(j)})$ and $\overline{b}(\phi_1^{(j)}) + \cdots + \overline{b}(\phi_n^{(j)}) < 1$ for all $j \in \{1, \dots, m\}$. Set*

$$(\phi_1, \dots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \cdots (\phi_1^{(m)})^{\lambda_m}), \dots, (\phi_n^{(1)})^{\lambda_1} \cdots (\phi_n^{(m)})^{\lambda_m}))$$

and suppose that

$$\frac{1}{p} = \sum_{j=0}^{m} \frac{\lambda_j}{p_j}.$$

If $K_{q_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A}) = J_{q_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A})$ for $j \in \{0, \dots, m\}$ and if

$$K_q^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_q^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

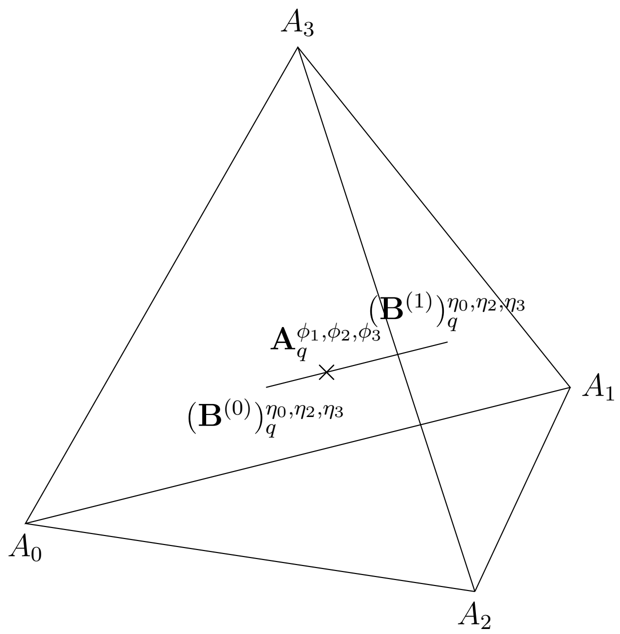


Figure 4.2: Illustration of Theorem 4.2.1 with $m = 1$ and $n = 3$.

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$(\mathbf{A}_{q_0}^{\phi_1^{(0)},\ldots,\phi_n^{(0)}},\ldots,\mathbf{A}_{q_m}^{\phi_1^{(m)},\ldots,\phi_n^{(m)}})_q^{\lambda_1,\ldots,\lambda_m} = \mathbf{A}_q^{\phi_1,\ldots,\phi_n}.$$

4.2 Reduction to 1-functors

It is evidently feasible to define n-functors through the composition of 1-functors. Conversely, on "well-behaved" tuples, every n-functor can be expressed in terms of 1-functors.

Theorem 4.2.1. *Let $1 \leq m \leq n-1$, $0 < \lambda_1 + \cdots + \lambda_m < 1$, $\eta_0, \eta_{m+1}, \ldots, \eta_n \in \mathcal{B}$ be such that $0 < \underline{b}(\eta_0) + \underline{b}(\eta_{m+1}) + \cdots + \underline{b}(\eta_n)$ and $\overline{b}(\eta_0) + \overline{b}(\eta_{m+1}) + \cdots + \overline{b}(\eta_n) < 1$. Set*

$$(\phi_1, \ldots, \phi_m, \phi_{m+1}, \ldots, \phi_n) = (\eta_0^{\lambda_1}, \ldots, \eta_0^{\lambda_m}, \eta_{m+1}, \ldots, \eta_m)$$

and let $\mathbf{B}^{(j)} = (A_j, A_{m+1}, \ldots, A_n)$ for $j \in \{0, \ldots, m\}$. If

$$K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(j)}) = J_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(j)})$$

(this is the case if $\mathcal{F}(\mathbf{B}^{(j)})$ is satisfied) for $j \in \{0, \ldots, m\}$ and if

$$K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}) = J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$\big((\mathbf{B}^{(0)})_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}, \ldots, (\mathbf{B}^{(m)})_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}\big)_q^{\lambda_1,\ldots,\lambda_m} = \mathbf{A}_q^{\phi_1,\ldots,\phi_n}.$$

Proof. First, let us establish the inclusion:

$$K_q^{\lambda_1,\ldots,\lambda_m}(K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(0)}),\ldots,K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(m)}))\hookrightarrow K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A}). \tag{4.3}$$

Define $\mathbf{A}'=(A_{m+1},\ldots,A_n)$,

$$\mathbf{B}_K=(K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(0)}),\ldots,K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(m)})).$$

and

$$\mathbf{B}_J=(J_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(0)}),\ldots,J_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(m)})).$$

Note that

$$K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\mathbf{B}^{(j)})=K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\eta_0^{-1}(s_j)/s_j\,A_j,s_j^{-1}\mathbf{A}').$$

Instead of K, we consider the equivalent functional

$$K_q(t,a;\mathbf{A})=\inf(\|a_0\|_{A_0}^q+(t_1\|a_1\|_{A_1})^q+\ldots+(t_n\|a_n\|_{A_n})^q)^{1/q}.$$

For $a\in K_q^{\lambda_1,\ldots,\lambda_m}(\mathbf{B}_K)$ and $s_0=1$, we have

$$\begin{aligned}K_q(s,a;\mathbf{B}_K)^q&=\inf\sum_{k=0}^m s_k^q\|a_k\|^q_{K_q^{\eta_0,\eta_{m+1},\ldots,\eta_n}(\eta_0^{-1}(s_j)/s_j\,A_j,s_j^{-1}\mathbf{A}')}\\&\ge\int_{(0,\infty)^{n-m}}\Big(\frac{K_q(\eta_0^{-1}(s_1),\ldots,\eta_0^{-1}(s_m),t_{m+1},\ldots,t_n,a;\mathbf{A})}{\eta_{m+1}(t_{m+1})\ldots\eta_n(t_n)}\Big)^q\frac{dt}{t}.\end{aligned}$$

Therefore, we obtain

$$\|a\|^q_{K_q^{\lambda_1,\ldots,\lambda_m}(\mathbf{B}_K)}\ge C\|a\|^q_{K_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})}.$$

One can establish that

$$J_q^{\phi_1,\ldots,\phi_n}(\mathbf{A})\hookrightarrow J_q^{\lambda_1,\ldots,\lambda_m}(\mathbf{B}_J),$$

using arguments similar to those employed in [108]. □

Remark 4.2.2. The continuous interpolation method can also be easily adapted to several spaces.

4.3 Interpolation of Generalized Sobolev Spaces

Let us generalize the first part of Theorem 3.2.9 to several spaces.

Theorem 4.3.1. *Let $p,q\in[1,\infty]$, $\gamma_1,\ldots,\gamma_n\in\mathcal{B}$, $\phi_0,\phi_1,\ldots,\phi_n\in\mathcal{B}''$ and for $l_1,l_2\in\{1,\ldots,n\}$, set $f_{l_1,l_2}=\phi_{l_1}/\phi_{l_2}$. If $\underline{b}(f_{l_1,l_2})>0$ or $\overline{b}(f_{l_1,l_2})<0$ for $l_1<l_2$ and if $0<\underline{b}(\gamma_k)$, $\sum_k\overline{b}(\gamma_k)<1$, then*

$$K_q^{\gamma_1,\ldots,\gamma_n}(H_p^{\phi_0},\ldots,H_p^{\phi_n})=B^{\psi}_{p,q},$$

with $\psi=\phi_0/(f_{0,1}\circ\gamma_1\cdots f_{0,n}\circ\gamma_n)$.

Proof. Let us assume that $\underline{b}(f_{l_1,l_2}) > 0$ for $l_1 < l_2$; it is evident that $\psi \in \mathcal{B}$. Moreover, we can suppose that $f_{l_1,l_2} \in \mathcal{B}'$.

Let us first show that

$$K_q^{\gamma_1,\dots,\gamma_n}(H_p^{\phi_0},\dots,H_p^{\phi_n}) = K_q^{\eta_2,\dots,\eta_n}(H_p^{\phi_1},\dots,H_p^{\phi_n}), \tag{4.4}$$

with

$$\eta_2(t) = \frac{\gamma_1(f_{0,1}(f_{1,2}^{-1}(t)))\gamma_2(f_{0,2}(f_{1,2}^{-1}(t)))}{f_{0,1}(f_{1,2}^{-1}(t))}$$

and

$$\eta_l(t) = \gamma_l(f_{0,l}(f_{1,l}^{-1}(t))),$$

for $l \in \{3,\dots,n\}$. We already know that

$$\Sigma(H_p^{\phi_0},\dots,H_p^{\phi_n}) = \Sigma(H_p^{\phi_1},\dots,H_p^{\phi_n}).$$

From there, we can get (4.4) using $s_l = f_{1,l}^{-1}(t_l)$ $(l = 2,...,n)$, Proposition 4.1.25 and

$$(u_1,\dots,u_n) = (f_{0,1}(s_1),\dots,f_{0,n}(s_n)).$$

Applying the same procedure, we obtain

$$K_q^{\gamma_1,\dots,\gamma_n}(H_p^{\phi_0},\dots,H_p^{\phi_n}) = K_q^{\eta}(H_p^{\phi_{n-1}},H_p^{\phi_n}), \tag{4.5}$$

with η such that $\psi = \phi_{n-1}/(\eta \circ f_{(n-1),n})$. Given that the right-hand side of (4.5) is equal to $B_{p,q}^{\psi}$, we can draw the conclusion. □

In the case where $\phi_j(t) = t^{s_j}$ $(s_j \in \mathbb{R})$ for $j \in \{0,\dots,n\}$, we recover a result of [108]:

$$K_q^{\theta_1,\dots,\theta_n}(H_p^{s_0},\dots,H_p^{s_n}) = B_{p,q}^{s},$$

with $s = (1-\sum_{j=1}^n \theta_j)s_0 + \sum_{j=1}^n \theta_j s_j$, where $\theta_1,\dots,\theta_n > 0$ are such that $\sum_{j=1}^n \theta_j < 1$.

Part II

Pointwise Regularity through Continued Fractions

Chapter 5

Metrical theory of α-continued fractions

In this part, we focus on the pointwise regularity of several functions, all of which are linked to the Diophantine approximation of the point under consideration. First, we examine the Brjuno function, denoted as B, which is an arithmetic $\mathbb{Z}$-periodic function defined on irrational numbers as follows:

$$B : \mathbb{R} \setminus \mathbb{Q} \to \bar{\mathbb{R}} \quad x \mapsto \sum_{n=0}^{\infty} y_0 \cdots y_{n-1} \log \frac{1}{y_n},$$

where y_0 is the fractional part of x, and y_{n+1} is the fractional part of $1/y_n$. The Brjuno function plays a crucial role in the study of dynamical systems generated by iterations of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. Notably, Yoccoz's Theorem [117] states that if $Df(0) = e^{2i\pi x}$ for $x \in (0,1) \setminus \mathbb{Q}$, then f is locally linearizable in a neighborhood of the origin if and only if the series defining $B(x)$ converges – such a x is referred to as a Brjuno number. The global regularity of the Brjuno function and some of its variants is explored in [85]; specifically, it is demonstrated that these functions are of Bounded Mean Oscillation (BMO) [51, 52]. Local properties of B have been investigated in [9], where it is shown that Brjuno numbers are Lebesgue points of B. Finally, a multifractal study of this function is proposed in [48]; this study can be viewed as a refinement of the result from [9] concerning Lebesgue points.

It is interesting to note that originally, the function considered in [117] is not the function B but a variant, introduced to simplify calculations. Instead of considering the integer part (leading to the fractional part in the definition of B), Yoccoz takes the nearest integer. Thus, while the function B is clearly connected to the continued fraction expansion, the function considered in [117] is related to the nearest integer continued fraction expansion. This type of expansion was originally considered by Minnigerode [89]. Naturally, we will refer to this function as the Brjuno-Yoccoz function, denoted as $\mathfrak{B}$. In [85], they consider a more general setting : given $\alpha \in [1/2, 1]$, define

$$[x]_\alpha = \min\{p \in \mathbb{Z} : x < p + \alpha\}$$

for a real x. The (generalized) Gauss map is then given by

$$A_\alpha(x) = \left|\frac{1}{x} - [\frac{1}{x}]_\alpha\right|.$$

Considering iterations of A_α leads to new continued fraction expansions: set $x_0 = |x - [x]_\alpha|$ and $a_0 = a_0(x) = [x]_\alpha$. Consequently, $x_0 = a_0 + \varepsilon_0 x_0$, where

$$\varepsilon_0 = \begin{cases} 1 & \text{if } x \geq a_0, \\ -1 & \text{otherwise.} \end{cases}$$

This initialization defines $x_{n+1} = A_\alpha(x_n)$ and

$$a_{n+1} = a_{n+1}(x) = [\frac{1}{x_n}]_\alpha \geq 1,$$

for $n \in \mathbb{N}_0$ if it is meaningful. Subsequently, $x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1}$, where

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

If $x_n = 0$ for some $n \in \mathbb{N}_0$, the process concludes. Otherwise, this algorithm establishes three sequences $(x_n)_{n\in\mathbb{N}_0}$, $(a_n)_{n\in\mathbb{N}_0}$ and $(\varepsilon_n)_{n\in\mathbb{N}_0}$.

The goal of this Chapter is to explore the metric properties of these continued fractions, such as the notion of cells in this context, with the aim of providing a solid foundation for studying the pointwise regularity of generalized versions of the Brjuno function using those A_α, thus addressing certain questions raised at the end of [48]. Since the regularity of B at x is linked to the irrationality measure of x, one can naturally define an irrationality exponent linked to the convergents (p_n/q_n) obtained by iterations of A_α :

$$\tau^{(\alpha)}(x) = \limsup_n \frac{\log |x - \frac{p_n}{q_n}|}{\log \frac{1}{q_n}}.$$

We proved that those exponents are, in fact, the same. We also consider α-cells: fixing $a_0, ..., a_n$, $\varepsilon_0, ..., \varepsilon_n$, it is the sets of points x such that

$$\forall j \in \{1, \ldots, n\}, a_j(x) = a_j, \varepsilon_j(x) = \varepsilon_j.$$

When $\alpha = 1$, the structure of the cells plays a crucial role in [9] and [48]. In particular, it seems crucial to know the endpoints of a given cell. In the general situation of the α-cell, this only occurs for certain values of α, which we call advantageous: α is advantageous if and only if

$$\alpha \in \Big\{1/2, \frac{\sqrt{5}-1}{2}, 1\Big\} \cup \Big\{1 - \frac{1}{k}, k \geq 3\Big\} \cup \Big\{\frac{-k + \sqrt{k^2+4k}}{2}, k \geq 2\Big\}.$$

As in the next Chapter, we will only focus on the regularity of $\mathfrak{B}$, we explicit our results to $\alpha = 1/2$ in the last Section. In this case, cells can be easily described as intervals with known endpoints.

This Chapter was written in collaboration with Bruno Martin and Samuel Nicolay. However, our collaboration is still ongoing, and we have yet to discuss collectively how to structure our respective contributions. As such, this text represents my personal interpretation of our discussions. Some of the results presented here will not be pursued in forthcoming articles, while others remain, of course, essential. My intention was to capture the original spirit of our collaboration, which has since evolved, shaped by unforeseen results. Certain research directions are yet to be definitively established with my collaborators.

5.1 α-continued fractions

Let us provide a brief introduction to the concept of α-continued fractions [93].

Given $\alpha \in [1/2, 1]$, define

$$[x]_\alpha = \min\{p \in \mathbb{Z} : x < p + \alpha\},$$

for $x \in \mathbb{R}$. Notably, $[x]_1$ represents the integer part of x, while $[x]_{1/2}$ denotes the nearest integer of x. Next, introduce the (generalized) Gauss map:

$$A_\alpha : (0, \alpha) \to [0, \alpha] : x \mapsto |\frac{1}{x} - [\frac{1}{x}]_\alpha|.$$

It can be observed in Figure 5.1 that A_α is composed of decreasing and increasing branches (only decreasing when $\alpha = 1$). In fact, one has

$$A_\alpha(x) = \begin{cases} \dfrac{1}{x} - k & \text{if } \frac{1}{k+\alpha} < x \leq \frac{1}{k}, \\ k - \dfrac{1}{x} & \text{if } \frac{1}{k} < x \leq \frac{1}{k+\alpha-1}. \end{cases}$$

We observe that A_α is continuous on $(0, \alpha) \setminus \{\frac{1}{k+\alpha}, k \in \mathbb{N}\}$ and is right continuous on $(0, \alpha)$. Following notations in [85], we write

$$G = \frac{\sqrt{5}+1}{2} \quad \text{and} \quad g = G^{-1} = \frac{\sqrt{5}-1}{2}.$$

As pointed by Marmi *et al.*([85, 86, 7]), the cases $1/2 \leq \alpha \leq g$ and $g < \alpha \leq 1$ behave differently and we will also observe some bifurcations when addressing the metric theory attached to the iterations of A_α.

Remark 5.1.1. One can also consider $\alpha \in [0, 1]$, as it is done in [82]. The map A_0 is then the so-called by-excess continued fraction map. As the primary focus is on the regularity of the generalized Brjuno functions from [85], we limit our study to $\alpha \in [1/2, 1]$ in this Chapter.

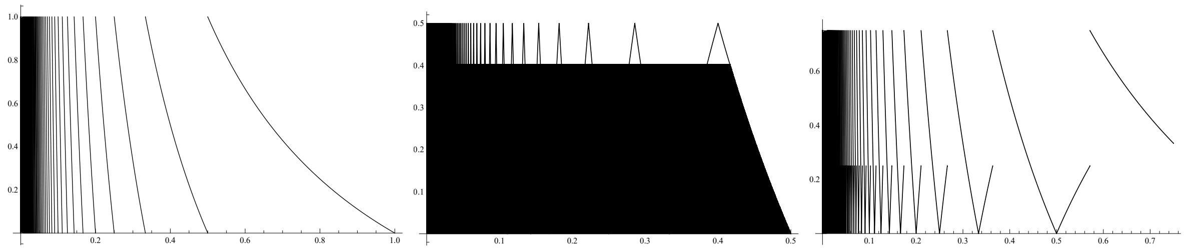


Figure 5.1: $A_\alpha : (0, \alpha) \to [0, \alpha]$ with resp. $\alpha = 1$, $\alpha = 1/2$ and $\alpha = 3/4$.

In [85], it is demonstrated that the dynamical system defined by the iteration of A_α preserves an absolutely continuous probability measure m_α (with respect to the Lebesgue measure). Furthermore, for any $\alpha \in [1/2, 1]$, we have

$$c_0^{-1} \leq \rho_\alpha \leq c_0, \tag{5.1}$$

with $c_0 := 2/\log(G) = 4.156...$ and ρ_α such that $m_\alpha(dx) = \rho_\alpha(x)dx$.

As is customary, one can construct a continued fraction expansion for x by iterating A_α. To elaborate, set $x_0 = |x - [x]_\alpha|$ and $a_0 = a_0(x) = [x]_\alpha$. Consequently, $x = a_0 + \varepsilon_0 x_0$, where

$$\varepsilon_0 = \begin{cases} 1 & \text{if } x \geq a_0, \\ -1 & \text{otherwise.} \end{cases}$$

This initialization defines $x_{n+1} = A_\alpha(x_n)$ and

$$a_{n+1} = a_{n+1}(x) = [\frac{1}{x_n}]_\alpha \geq 1,$$

for $n \in \mathbb{N}_0$ if it is meaningful. Subsequently, $x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1}$, where

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

Remark 5.1.2. If $x \in (0, \alpha)$, we thus obtain $A_\alpha(x) = \varepsilon_1(x)(\frac{1}{x} - a_1(x))$, which establishes a connection between a_1, ε_1, and the increasing and decreasing branches of A_α (see Figure 5.2).

If $x_n = 0$ for some $n \in \mathbb{N}_0$, the process concludes. Otherwise, this algorithm establishes three sequences $(x_n)_{n\in\mathbb{N}_0}$, $(a_n)_{n\in\mathbb{N}_0}$ and $(\varepsilon_n)_{n\in\mathbb{N}_0}$.

If $x_{n-1} \neq 0$, we have

$$x = a_0 + \varepsilon_0 x_0 = \cdots = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n x_n}}}}. \tag{5.2}$$

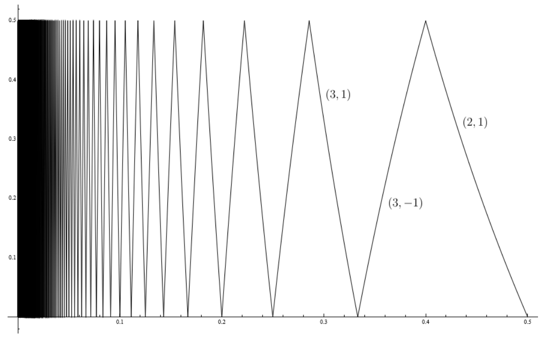


Figure 5.2: Connection between a_1, ε_1, and the increasing and decreasing branches of $A_{1/2}$.

In instances where the process continues indefinitely, we can represent x as:

$$x = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_{n-1}}{a_n + \cdots}}}},$$

interpreted in the sense of convergence in $\mathbb{R}$. This expression is denoted by

$$[(a_0, \varepsilon_0), \ldots, (a_n, \varepsilon_n), \ldots].$$

We define

$$\begin{aligned}[(a_0, \varepsilon_0), \ldots, (a_n, \varepsilon_n), t] &= [(a_0, \varepsilon_0), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n + \frac{\varepsilon_n}{t}] \\ &= a_0 + \cfrac{a_0}{a_1 + \cfrac{\varepsilon_1}{\ddots + a_n + \cfrac{\varepsilon_n}{t}}}.\end{aligned}$$

The n-th α-convergent of x is given by

$$\frac{p_n}{q_n} = [(a_0, \varepsilon_0), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n].$$

Setting $p_{-1} = q_{-2} = 1$ and $p_{-2} = q_{-1} = 0$, the relationships between the numerators p_n and the denominators q_n are expressed as:

$$\begin{cases} p_n = a_n p_{n-1} + \varepsilon_{n-1} p_{n-2}, \\ q_n = a_n q_{n-1} + \varepsilon_{n-1} q_{n-2}. \end{cases} \tag{5.3}$$

Verification reveals that

$$[(a_0,\varepsilon_0),\dots,(a_n,\varepsilon_n),t] = \frac{tp_n+\varepsilon_n p_{n-1}}{tq_n+\varepsilon_n q_{n-1}},$$

hence, from (5.2), we derive

$$x = \frac{p_n + p_{n-1}\varepsilon_n x_n}{q_n + q_{n-1}\varepsilon_n x_n}. \tag{5.4}$$

Utilizing the definition of x_n, we obtain

$$x_n = -\varepsilon_n \frac{q_n x - p_n}{q_{n-1}x - p_{n-1}}.$$

Let us set

$$s_n = (-1)^n \varepsilon_0 \cdots \varepsilon_n$$

One can verify the following relations:

$$q_n p_{n-1} - p_n q_{n-1} = s_n \varepsilon_n \tag{5.5}$$

and

$$p_{n+1}q_{n-1} - p_{n-1}q_{n+1} = -s_n\varepsilon_n a_{n+1}.$$

Additionally, the sequence $(q_n)_{n\in\mathbb{N}}$ is monotonically increasing. For $x>0$, we observe that $p_n>0$ for all the defined indices n, while for $x<0$, it follows that $p_n<0$ for all n. In the case of $\alpha=1$, we retrieve the classical theory of continued fractions [57], whereas $\alpha=1/2$ corresponds to the nearest integer continued fraction expansion [89]. Note that when $\alpha=1/2$, $a_n\geq 2$ for all n.

Remark 5.1.3. In accordance with standard tradition, we refrain from explicitly indicating the dependence of p_n, q_n, and other sequences on x and α, unless explicitly required.

The α-convergents form a subsequence of the 1-convergents. Consider an irrational number x and let P_n/Q_n denote the n-th 1-convergent of x. Set $k^{(\alpha)}(-1)=-1$ and define the arithmetic function $k^{(\alpha)}$ on $\mathbb{N}_0$ as follows:

$$k^{(\alpha)}(n+1) = \begin{cases} k^{(\alpha)}(n)+1 & \text{if } \varepsilon_{n+1}=1, \\ k^{(\alpha)}(n)+2 & \text{if } \varepsilon_{n+1}=-1. \end{cases}$$

As established in [85], the sequence $(k^{(\alpha)}(n))_{n\in\mathbb{N}_0}$ is strictly increasing, and for all $n\in\mathbb{N}$, we have

$$\frac{p_n}{q_n} = \frac{P_{k^{(\alpha)}(n)}}{Q_{k^{(\alpha)}(n)}}.$$

It can be easily obtained using the identity

$$A - \frac{1}{B+x} = A - 1 + \cfrac{1}{1+\cfrac{1}{B-1+x}}.$$

Remark that

$$q_n = Q_{k^{(\alpha)}(n)} \geq Q_{k^{(\alpha)}(n)-1} + Q_{k^{(\alpha)}(n)-2} \geq F_{k^{(\alpha)}(n)+1} \geq F_{n+1}, \tag{5.6}$$

where $(F_k)_{k\in\mathbb{N}}$ denote the Fibonacci sequence.
We will utilize the sequence $(\beta_n)_{n\in\mathbb{N}_0}$. Define $\beta_{-1} = 1$ and

$$\beta_n = x_0 \cdots x_n = s_n(q_n x - p_n) = |q_n x - p_n|$$

for $n \in \mathbb{N}_0$, so that

$$x_n = \frac{\beta_n}{\beta_{n-1}}$$

and

$$\beta_{n-2} = a_n \beta_{n-1} + \varepsilon_n \beta_n.$$

For $n \geq 1$, we have

$$\beta_n = \frac{1}{q_{n+1} + \varepsilon_{n+1} q_n x_{n+1}},$$

resulting in

$$\frac{1}{1+\alpha} < \beta_n q_{n+1} < \frac{1}{\alpha}, \tag{5.7}$$

so that $(\beta_n)_n \asymp (\frac{1}{q_{n+1}})_n$.

If $\alpha > g$, then

$$\beta_n \leq \alpha g^n. \tag{5.8}$$

On the other hand, if $\alpha \leq g$, then

$$\beta_n \leq \alpha(\sqrt{2} - 1)^n. \tag{5.9}$$

Equations (5.8) and (5.9) will play a key role in what follows. A proof may be found in [85], as it is rather involved. Finally, we point out that

$$-\varepsilon_{n+1}(q_{n+1} x_n - p_{n+1})(q_n x - p_n) \geq 0.$$

Hence, contrary to what happens when $\alpha = 1$, $q_n x - p_n$ and $q_{n+1} x_n - p_{n+1}$ may not have opposite sign.

In the perspective of performing a local analysis of α-Brjuno functions, it seems relevant to introduce the notion of α-irrationality exponent.

Definition 5.1.4. For $x \in \mathbb{R} \setminus \mathbb{Q}$, by defining $\tau_n^{(\alpha)}(x)$ as

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n^{(\alpha)}(x)}},$$

we introduce the α-irrationality exponent of x as

$$\tau^{(\alpha)}(x) = \limsup_{n\to\infty} \tau_n^{(\alpha)}(x).$$

We will denote the usual irrationality exponent by τ:

$$\begin{aligned}\tau^{(1)}(x) &= \tau(x)\\ &= \sup\left\{u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z}\times\mathbb{N} : \left|x - \frac{p}{q}\right| < \frac{1}{q^u}\right\}.\end{aligned}$$

Since the sequence of α-convergents is a subsequence of the sequence of convergents, we have $\tau^{(\alpha)}(x) \leq \tau(x)$. In fact, for all irrational x, $\tau^{(\alpha)}(x) = \tau(x)$.

Proposition 5.1.5. *For all $\alpha \in [1/2, 1]$, the α-irrationality exponent is the irrationality exponent: we have $\tau^{(\alpha)}(x) = \tau(x)$ for all $x \in \mathbb{R}\setminus\mathbb{Q}$.*

Proof. Let $(p_n/q_n)_{n\in\mathbb{N}}$ be the α-convergents of x. It follows from (5.7) that

$$\frac{1}{2q_nq_{n+1}} \leq \left|x - \frac{p_n}{q_n}\right| \leq \frac{2}{q_nq_{n+1}}. \tag{5.10}$$

This enables us to apply Theorem 4 of [34]: the irrationality exponent is given by

$$\tau(x) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}. \tag{5.11}$$

Now, (5.10) leads to

$$\frac{1}{2}q_n^{\tau_n^{(\alpha)}(x)-1} \leq q_{n+1} \leq 2q_n^{\tau_n^{(\alpha)}(x)-1},$$

which implies

$$1 + \frac{\log q_{n+1}}{\log q_n} - \frac{\log 2}{\log q_n} \leq \tau_n^{(\alpha)}(x) \leq 1 + \frac{\log q_{n+1}}{\log q_n} + \frac{\log 2}{\log q_n}.$$

This leads to the stated conclusion by taking the limit superior when n tends to infinity. □

We say that x is diophantine if $\tau(x) < \infty$; it is a Liouville number otherwise.

Example 5.1.6. For any irrational number x, we have $\tau(x) \geq 2$ and the irrationality exponent of almost every number x (with respect to the Lebesgue measure) is equal to 2 [43]. Roth's Theorem asserts that the irrationality exponent of any irrational algebraic number is exactly 2 [101]. Notable examples include square roots, the golden ratio G.

Let us delve into the metrical theory of the introduced α-continued fractions. From now on, we only work on the interval $(0,\alpha)$. In this interval, $x_0 = x$, $a_0 = 0$, $\varepsilon_0 = 1$, $q_0 = 1$ and $p_0 = 0$. As a consequence, we omit the rank $k = 0$ in every notation that will follow. Also, we observe that if $x \in (0,\alpha)$, we cannot have $a_1 = 1$ and $\varepsilon_1 = -1$ since this would imply $x > 1$.

We begin with two important observations. First, the functions a_1 and ε_1 are left-continuous on $(0,\alpha)$, and if we introduce the sets

$$\mathscr{D}_1(\alpha) = \left\{\frac{1}{k} \,:\, k \in \mathbb{N}\right\} \cap (0,\alpha),\ \mathscr{D}_2(\alpha) = \left\{\frac{1}{k+\alpha} \,:\, k \in \mathbb{N}\right\} \cap (0,\alpha),$$

and

$$\mathscr{D}(\alpha) = \mathscr{D}_1(\alpha) \cup \mathscr{D}_2(\alpha),$$

then the function a_1 is continuous on $(0,\alpha)\backslash\mathscr{D}_2(\alpha)$ whereas ε_1 is continuous on $(0,\alpha)\backslash\mathscr{D}(\alpha)$. We have

$$\lim_{x\to 1/k} a_1(x) = k, \lim_{x\to(1/(k+\alpha))^+} a_1(x) = k, \lim_{x\to(1/(k+\alpha))^-} a_1(x) = k+1,$$

and

$$\lim_{x\to(1/k)^\pm} \varepsilon_1(x) = \mp 1, \lim_{x\to(1/(k+\alpha))^\pm} \varepsilon_1(x) = \pm 1.$$

Second, we point out to the reader that, with an adapted set X, the couple (X, A_α) can be seen as a numeration system and even as *a fibred numeration system*: we recall those notions here (see [12] for more details).

Definition 5.1.7. A fibred system is a set X and a transformation $T : X \to X$ for which there exist a finite or countable set $\mathfrak{I}$ and a partition $X = \bigcup_{j\in\mathfrak{I}} X_j$ of X such that the restriction T_j of T on X_j is injective, for any $j \in \mathfrak{I}$. This yields a well-defined map $\rho : X \to \mathfrak{I}$ that associates the index j with $x \in X$ such that $x \in X_j$.

Let $\varphi : X \to \mathfrak{I}^{\mathbb{N}}$ be defined by $\varphi(x) = (\rho(T^{n-1}x))_{n\geq 1}$ and write $\rho_n = \rho \circ T^{n-1}$ for short.

Definition 5.1.8. If the function φ is injective, we call the quadruple $\mathcal{N} = (X, T, \mathfrak{I}, \varphi)$ a fibred numeration system. Then $\mathfrak{I}$ is the set of digits of the numeration system; the map φ is the representation map and $\varphi(x)$ is the $\mathcal{N}$-representation of x. In general, the representation map is not surjective. The set of prefixes of $\mathcal{N}$-representations is called the language $\mathcal{L} = \mathcal{L}(\mathcal{N})$ of the fibred numeration system, and its elements are said to be admissible. The admissible sequences are defined as the elements $y \in \mathbb{N}^*$ for which $y = \varphi(x)$ for some $x \in X$.

To fit with Definition 5.1.8, we can restrict A_α to a set X such that $A_\alpha(X) \subseteq X$. Therefore, we introduce

$$Z(\alpha) = \bigcup_{j\geq 0} A_\alpha^{-j}\big(\mathscr{D}(\alpha)\big)\big) = \Big(\bigcup_{j\geq 0} A_\alpha^{-j}\big(\mathscr{D}_2(\alpha)\big)\Big) \cup \big(\mathbb{Q} \cap (0,\alpha)\big),$$

and we set

$$X = X(\alpha) = (0,\alpha) \setminus Z(\alpha).$$

Proposition 5.1.9. $A_\alpha(X) \subseteq X$.

Proof. Let $x \in X$ and suppose by contradiction that $A_\alpha(x) \notin X$. Then, $A_\alpha(x) \in \mathbb{Q}$ or $A_\alpha(x) \in Z(\alpha)$. Since $x \in X$, we cannot have $A_\alpha(x) \in Z(\alpha)$. Hence, $A_\alpha(x) \in \mathbb{Q}$ which implies that $x = 1/(a_1 + \varepsilon_1 A_\alpha(x))$ is rational: a contradiction with $x \in X$. □

Let us set $\mathfrak{I} = \{(a, \varepsilon) : a \in \mathbb{N}, \varepsilon \in \{\pm 1\}\}$, which is *the set of digits*. We introduce the function

$$\rho : X \to \mathfrak{I} : x \mapsto (a_1(x), \varepsilon_1(x))$$

and write $\rho_n = \rho \circ A_\alpha^{n-1}$ for short. Finally, we define $\varphi : X \to \mathfrak{I}^{\mathbb{N}}$ by $\varphi(x) = (\rho_n(x))_{n \geq 1}$, which is injective: this is *the representation map*.Then, $(X, A_\alpha, \mathfrak{I}, \varphi)$ is a fibred numeration system.

The theory of fibered numeration system aims at understanding the ergodic aspects of the system (X, A_α). Even if the existence of an invariant measure and the structure of cylinders (or cells) play a crucial role in our work, we will not directly make use of this theory.

5.2 The Generalized Brjuno functions

We define the (generalized) Brjuno functions as presented in [85].

Definition 5.2.1. Let $\alpha \in [1/2, 1]$; the generalized Brjuno function B_α is defined as

$$B_\alpha : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} \beta_{n-1}(x) \log x_n.$$

The domain of B_α can be extended to $\mathbb{R}$ by setting $B_\alpha(x) = \infty$ for $x \in \mathbb{Q}$. For simplicity, we will denote the usual Brjuno function as $B = B_1$ and the Bruno-Yoccoz function as $\mathfrak{B} = B_{1/2}$. We set $\gamma_n(x) = \beta_{n-1}(x) \log 1/x_n$ for all $n \in \mathbb{N}$.

Definition 5.2.2. An irrational number x is classified as a Brjuno number if $B(x) < \infty$; otherwise, x is termed a Cremer number. The set of Brjuno numbers is denoted by BN .

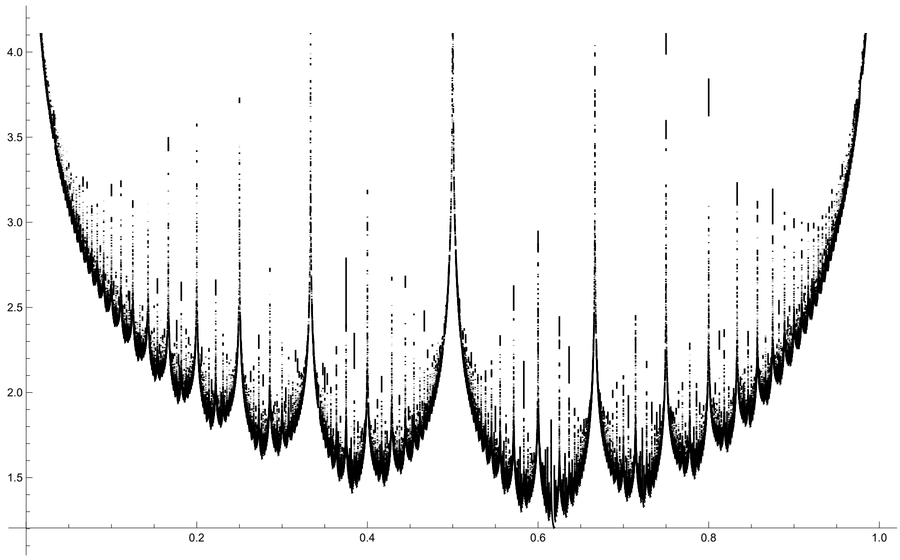


Figure 5.3: Brjuno function B_1.

Let us review some properties of the generalized Brjuno functions B_α derived in [85]:

1. $B_\alpha(x) = B_\alpha(x+1)$ for all $x \in \mathbb{R}$,
2. For all $x \in (0, \alpha]$,
$$B_\alpha(x) = -\log x + xB_\alpha(\frac{1}{x}) = -\log x + xB_\alpha(A_\alpha(x)), \tag{5.12}$$
3. $x \in [\alpha - 1, 0)$ implies $B_\alpha(-x) = B_\alpha(x)$,
4. there exists a constant $C_1 > 0$ (independent of α) such that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, we have
$$|B_\alpha(x) - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}| \leq C_1,$$
5. there exists a constant $C_2 > 0$ such that for all irrational numbers x, we have
$$|B_\alpha(x) - \sum_{n=0}^{\infty} \frac{\log Q_{n+1}}{Q_n}| \leq C_2.$$

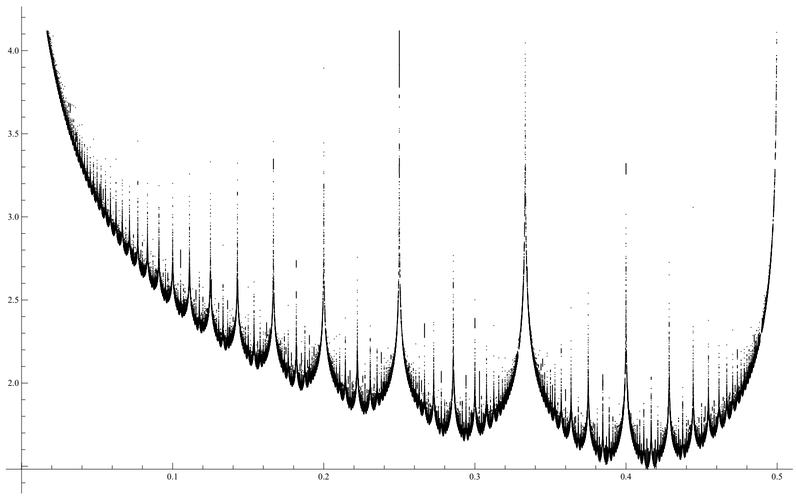


Figure 5.4: Brjuno-Yoccoz function $\mathfrak{B}$.

Consequently, $B_\alpha(x)$ is finite if and only if x is a Brjuno number and all generalized Brjuno functions differ from each other by an L^∞ function (compare for example B_1 and $\mathfrak{B}$ on $[0, 1/2]$ in Figure 5.3 and Figure 5.4, or in Figure 6.1).

Remark 5.2.3. A Brjuno number is usually defined as a number x for which
$$\sum_{n=0}^{\infty} \frac{\log Q_{n+1}}{Q_n} < \infty.$$

Remark 5.2.4. All diophantine numbers are Brjuno numbers. Equivalently, all Cremer numbers are Liouville numbers. Indeed, suppose that $\tau(x) < \infty$, the sequence $(\tau_n^{(\alpha)}(x))_{n\in\mathbb{N}}$ is thus bounded. We get

$$B_\alpha(x) = \sum_{n=0}^{\infty} \beta_{n-1}(x) \log(1/x_n) \le \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \le \sum_{n=0}^{\infty} \frac{\log q_n^{\tau_n^{(\alpha)}(x)-1}}{q_n} \le C \sum_{n=0}^{\infty} \frac{\log q_n}{q_n} < \infty,$$

where we have applied (5.6).

We get directly that almost every number is a Brjuno number. Nevertheless, not all irrational numbers are Bruno numbers. For example (see [72]), consider $x = [a_0, a_1, a_2, ...]$, where

$$a_j = \begin{cases} 10 & \text{if } \; j \in \{0,1\}, \\ Q_{j-1}^{Q_{j-1}} & \text{otherwise.} \end{cases}$$

One can easily show that any irrational number whose continued fraction expansion ends with a string of 1's is a Brjuno number (see [72]).

Remark 5.2.5. For all $\alpha \in [1/2, 1]$, B_α is a BMO function [85]. In particular, B_α belongs to L^p_{loc}. Let us observe that the fact that B_α belongs to L^p_{loc} can be determined directly : let $k \in \mathbb{N}_0$, $p \in [1, \infty)$, using (5.9), (5.8) and (5.1), one gets

$$\begin{aligned}
\int_0^\alpha \gamma_k(t)^p dt &\le \alpha^p \min(1, g^{p(k-1)}) \int_0^\alpha (\log \frac{1}{x_k(t)})^p \, dt \\
&\le \alpha^p \min(1, g^{p(k-1)}) \, c_0 \int_0^\alpha (\log \frac{1}{x_k(t)})^p \, m_\alpha(dt) \\
&= \alpha^p \min(1, g^{p(k-1)}) \, c_0 \int_0^\alpha (\log \frac{1}{t})^p \, m_\alpha(dt) \\
&\le C \, g^{p(k-1)},
\end{aligned}$$

hence the conclusion. We thereby recover the fact that almost every number is a Brjuno number.

These functions are clearly not locally bounded because of the logarithmic singularities centered at all rational points. We can easily check it explicitly:

Proposition 5.2.6. *If r is a rational number in $(0, \alpha)$, then*

$$\lim_{\substack{x \to r \\ x \in \text{BN}}} B_\alpha(x) = \infty.$$

Proof. Since $B_\alpha(x) \ge \log 1/x$ for all $x \in \text{BN}$, the result is true for $r = 0$. The result is also true for $1/q = \alpha$, since for all $x \in (1/(q+\alpha), \alpha]$, one has

$$B_\alpha(x) = \log \frac{1}{x} + x B_\alpha(A_\alpha(x)) \ge \frac{1}{4} B_\alpha(\frac{1}{x} - q),$$

which brings us back to the case $r = 0$.

If r is a rational in $(0, \alpha)$, then we can write

$$r = [0, (a_1, \varepsilon_1), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n] = \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cdots + \cfrac{\varepsilon_{n-1}}{a_n}}}.$$

It is easy to check that

$$\eta : s \mapsto [0, (a_1, \varepsilon_1), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n + s]$$

is a homeomorphism from $(-\alpha, \alpha)$ to a neighborhood of r in $(0, \alpha)$. For t in $(0, \alpha) \setminus \mathbb{Q}$, $\eta(\pm t)$ belongs to $(0, \alpha) \setminus \mathbb{Q}$ and $A_\alpha^n(\eta(\pm t))$ belongs to $\{-t, t\}$. Moreover, for all $t \in (0, \alpha) \setminus \mathbb{Q}$, we have

$$\beta_{k-1}(\pm t) \geq \frac{1}{q_k + q_{k-1}},$$

where q_k is such that

$$\frac{p_k}{q_k} = [0, (a_1, \varepsilon_1), \ldots, (a_{k-1}, \varepsilon_{k-1}), a_k],$$

with $k \in \{1, \ldots, n\}$. By iterating the functional equation, we get

$$B_\alpha(\eta(s)) \geq \beta_{n-1}(\eta(s)) B_\alpha(A_\alpha^n(\eta(s))),$$

for all $s \in (-1, 1) \setminus \mathbb{Q}$. The conclusion follows from the cases $r = 0$ and $r = \alpha$. □

Since $B_\alpha \in L^1_{\mathrm{loc}}$, we may define

$$\Psi_\alpha(t) = \int_0^t B_\alpha(t)dt \quad (t \in \mathbb{R}). \tag{5.13}$$

5.3 Description of α-cells

5.3.1 Elementary cells

First, we introduce the elementary cells – or cylinders – of A_α. For $a \in \mathbb{N}$ and $\varepsilon \in \{-1, 1\}$, we define

$$\mathfrak{c}(a, \varepsilon) = \begin{cases} \left(\frac{1}{a+\alpha}, \frac{1}{a}\right) \cap (0, \alpha) \text{ if } \varepsilon = 1, \\ \left(\frac{1}{a}, \frac{1}{a+\alpha-1}\right) \cap (0, \alpha) \text{ if } \varepsilon = -1. \end{cases} \tag{5.14}$$

We have

$$\mathfrak{c}(a, \varepsilon) = \begin{cases} \emptyset & \text{if } \varepsilon = -1 \text{ and } a = 1, \\ (1/(1+\alpha), \alpha) & \text{if } \varepsilon = 1 \text{ and } a = 1, \\ (1/2, \min(\alpha, 1/(1+\alpha)) & \text{if } \varepsilon = -1 \text{ and } a = 2, \\ (1/(a+\alpha), 1/a) & \text{if } \varepsilon = 1 \text{ and } a \geq 2, \\ (1/a, 1/(a+\alpha-1)) & \varepsilon = -1 \text{ and } a \geq 3. \end{cases} \tag{5.15}$$

We observe that if $1/2 \le \alpha \le g$, then $\min(\alpha, 1/(1+\alpha)) = \alpha$ and $\mathfrak{c}(1,1) = \emptyset$.

Proposition 5.3.1. *If $x \in (0,\alpha) \setminus \mathscr{D}(\alpha)$, then for every $a \in \mathbb{N}$, $\varepsilon \in \{\pm 1\}$,*

$$a_1(x) = a \text{ and } \varepsilon_1(x) = \varepsilon \Leftrightarrow x \in \mathfrak{c}(a,\varepsilon). \tag{5.16}$$

Proof. Let $x \in X$, $a \in \mathbb{N}$. We have

$$\begin{aligned} a_1(x) = a \text{ and } \varepsilon_1(x) = 1 &\Leftrightarrow \left[\frac{1}{x}\right]_\alpha = a \text{ and } \frac{1}{x} \ge a \\ &\Leftrightarrow a \le \frac{1}{x} - \alpha + 1 < a + 1 \text{ and } \frac{1}{x} \ge a \\ &\Leftrightarrow \frac{1}{a+\alpha} < x \le \frac{1}{a+\alpha-1} \text{ and } x \le \frac{1}{a} \\ &\Leftrightarrow \frac{1}{a+\alpha} < x \le \frac{1}{a}. \end{aligned}$$

Similarly,

$$\begin{aligned} a_1(x) = a \text{ and } \varepsilon_1(x) = -1 &\Leftrightarrow \left[\frac{1}{x}\right]_\alpha = a \text{ and } \frac{1}{x} < a \\ &\Leftrightarrow a \le \frac{1}{x} - \alpha + 1 < a + 1 \text{ and } \frac{1}{x} < a \\ &\Leftrightarrow \frac{1}{a+\alpha} < x \le \frac{1}{a+\alpha-1} \text{ and } x > \frac{1}{a} \\ &\Leftrightarrow \frac{1}{a} < x \le \frac{1}{a+\alpha-1}. \end{aligned}$$

□

Remark 5.3.2. Note that if we only suppose $x \in (0,\alpha)$, the implication

$$x \in \mathfrak{c}(a,\varepsilon) \Rightarrow a_1(x) = a \text{ and } \varepsilon_1(x) = \varepsilon$$

holds, whereas the converse fails because of the endpoints of the cells $\mathfrak{c}(a,\varepsilon)$ which are precisely the elements of $\mathscr{D}(\alpha)$.

On each non empty interval $\mathfrak{c}(a,\varepsilon)$, the map A_α is a strictly monotone, C^∞ function. The image of $\mathfrak{c}(a,\varepsilon)$ under A_α,

$$J(a,\varepsilon) = A_\alpha\big(\mathfrak{c}(a,\varepsilon)\big),$$

is an open interval, and the inverse of A_α on $\mathfrak{c}(a,\varepsilon)$ is given by

$$\begin{aligned} \psi_{(a,\varepsilon)} \colon J(a,\varepsilon) &\to \mathfrak{c}(a,\varepsilon) \\ t &\mapsto \frac{1}{a+\varepsilon t}. \end{aligned}$$

It follows from (5.15) that

- if $g < \alpha \leq 1$, then

$$J(a,\varepsilon) = \begin{cases} \emptyset & \text{if } a = 1 \text{ and } \varepsilon = -1, \\ (1/\alpha - 1, \alpha) & \text{if } a = \varepsilon = 1, \\ (0, \alpha) & \text{if } a \geq 2 \text{ and } \varepsilon = 1, \\ (0, 1-\alpha) & \text{if } a \geq 2 \text{ and } \varepsilon = -1, \end{cases} \tag{5.17}$$

- if $1/2 \leq \alpha \leq g$, then

$$J(a,\varepsilon) = \begin{cases} \emptyset & \text{if } a = 1, \\ (0, 2 - 1/\alpha) & \text{if } a = 2 \text{ and } \varepsilon = -1, \\ (0, \alpha) & \text{if } a \geq 2 \text{ and } \varepsilon = 1, \\ (0, 1-\alpha) & \text{if } a \geq 3 \text{ and } \varepsilon = -1. \end{cases} \tag{5.18}$$

5.3.2 The α-cells of depth n

We adopt some notations and concepts from words combinatorics. We define $\mathcal{A} = \mathbb{N} \times \{\pm 1\}$ and denote by $\mathcal{A}^*$ the set of all finite words (or strings) over $\mathcal{A}$:

$$\mathcal{A}^* = \{(a_1,\varepsilon_1)(a_2,\varepsilon_2)...(a_n,\varepsilon_n) \,:\, n \geq 1, \forall j \in \{1,\ldots,n\}, (a_j,\varepsilon_j) \in \mathcal{A}\}.$$

A strict total order is defined on $\mathcal{A}$ by the relation

$$(a,\varepsilon) < (a',\varepsilon') \Leftrightarrow a < a' \text{ or } \Big(a = a' \text{ and } (\varepsilon,\varepsilon') = (-1,1)\Big).$$

We write $(a,\varepsilon) \leq (a',\varepsilon')$ if $(a,\varepsilon) = (a',\varepsilon')$ or $(a,\varepsilon) < (a',\varepsilon')$. Finally, we write $(a,\varepsilon) \preccurlyeq (a',\varepsilon')$ if $(a,\varepsilon) < (a',\varepsilon')$ and if there is no (a'',ε'') such that $(a,\varepsilon) < (a'',\varepsilon'') < (a',\varepsilon')$. Hence, we have

$$(1,-1) \preccurlyeq (1,1) \preccurlyeq (2,-1) \preccurlyeq (2,1) \preccurlyeq (3,-1) \ldots$$

Definition 5.3.3 (α-cells of depth n)**.** For $n \in \mathbb{N}$, $(a_1,\varepsilon_1)...(a_n,\varepsilon_n) \in \mathcal{A}^*$, we define

$$\mathfrak{c}[(a_1,\varepsilon_1)...(a_n,\varepsilon_n)] = \mathfrak{c}(a_1,\varepsilon_1) \cap A_\alpha^{-1}(\mathfrak{c}(a_2,\varepsilon_2)) \cap \ldots \cap A_\alpha^{-(n-1)}(\mathfrak{c}(a_n,\varepsilon_n)). \tag{5.19}$$

If $\mathfrak{c}[(a_1,\varepsilon_1)...(a_n,\varepsilon_n)]$ is non empty, then we say that $\mathfrak{c}[(a_1,\varepsilon_1)...(a_n,\varepsilon_n)]$ is an *α-cell of depth n*.

Note that when $n = 1$, we have $\mathfrak{c}[(a_1,\varepsilon_1)] = \mathfrak{c}(a_1,\varepsilon_1)$, so that cells of depth 1 match with non-empty elementary cells. We will systematically use the notation $\mathfrak{c}(a_1,\varepsilon_1)$. Moreover, a word $(a_1,\varepsilon_1)...(a_n,\varepsilon_n)$ is called *admissible* if $\mathfrak{c}[(a_1,\varepsilon_1)...(a_n,\varepsilon_n)]$ is a cell. We denote by $\mathcal{L}_n(\alpha)$ the set of all admissible words of length n and we define

$$\mathcal{L}(\alpha) = \bigcup_{n \geq 1} \mathcal{L}_n(\alpha).$$

We gather some elementary properties of α-cells in the following lemma.

Lemma 5.3.4. *Let $n \in \mathbb{N}$ and $(a_1, \varepsilon_1)...(a_n, \varepsilon_n) \in \mathcal{L}(\alpha)$.*

1. *The cell $\mathfrak{c} = \mathfrak{c}[(a_1, \varepsilon_1), ..., (a_n, \varepsilon_n)]$ is a non-empty open interval.*

2. *For all $x \in X$, we have*

$$\forall j \in \{1, \ldots, n\}, a_j(x) = a_j, \varepsilon_j(x) = \varepsilon_j \Leftrightarrow x \in \mathfrak{c}. \tag{5.20}$$

3. *Let $(p_j)_{1 \leq j \leq n}$ and $(q_j)_{1 \leq j \leq n}$ be defined by (5.3). For all $x \in \mathfrak{c}$, $1 \leq j \leq n$, we have $p_j(x) = p_j$ and $q_j(x) = q_j$.*

Proof. 1. The map A_α is continuous and strictly monotone on every open interval $\mathfrak{c}(a, \varepsilon)$. Therefore, the intersection in (5.19) is an intersection of open intervals.

2. By induction on $n \geq 1$. If $n = 1$, then this is exactly (5.16). Now suppose that the assertion is true for some n. Let $(a_1, \varepsilon_1)...(a_{n+1}, \varepsilon_{n+1}) \in \mathcal{L}(\alpha)$ and write

$$\mathfrak{c}_n = \mathfrak{c}[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)]$$

and

$$\mathfrak{c}_{n+1} = \mathfrak{c}[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)(a_{n+1}, \varepsilon_{n+1})].$$

Since

$$\mathfrak{c}_{n+1} = \mathfrak{c}_n \cap A_\alpha^{-n}\big(\mathfrak{c}(a_{n+1}, \varepsilon_{n+1})\big)$$

we have

$$\begin{aligned} x \in \mathfrak{c}_{n+1} &\Leftrightarrow x \in \mathfrak{c}_n \text{ and } A_\alpha^n(x) \in \mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \\ &\Leftrightarrow x \in \mathfrak{c}_n \text{ and } \big(a_1(A_\alpha^n(x)), \varepsilon_1(A_\alpha^n(x))\big) = (a_{n+1}, \varepsilon_{n+1}) \\ &\Leftrightarrow \forall j \in \{1, \ldots, n\}, a_j(x) = a_j, \varepsilon_j(x) = \varepsilon_j \\ &\qquad \text{and } \big(a_{n+1}(x), \varepsilon_{n+1}(x)\big) = (a_{n+1}, \varepsilon_{n+1}). \end{aligned}$$

To prove the second equivalence, we had to remark that $A_\alpha^n(x) \in X$ because of the hypothesis $x \in X$. Hence (5.16) can be applied to $A_\alpha^n(x)$.

3. This follows from Remark 5.3.5 and the definition (5.3) of the p_n and q_n.

□

Remark 5.3.5. If we suppose $x \in (0, \alpha)$, then only the implication

$$x \in \mathfrak{c} \Rightarrow \forall j \in \{1, \ldots, n\}, a_j(x) = a_j, \varepsilon_j(x) = \varepsilon_j \tag{5.21}$$

holds. This is actually the same proof, using Remark 5.3.2.

Of course, if $(a_1, \varepsilon_1)...(a_n, \varepsilon_n) \in \mathcal{L}_n(\alpha)$ then there exists an irrational x such that

$$x = [(a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n), \ldots]$$

since $\mathfrak{c}[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)]$ is a non-empty open interval. However, the converse is not true in general, see example 5.3.6. In particular, our definition of $\mathcal{L}_n(\alpha)$ does not coincide with the one used in [7].

Example 5.3.6. Let $\alpha = (\sqrt{10}-1)/3$ which has been chosen so that

$$1-\alpha = \frac{1}{3+\frac{1}{1+\alpha}}.$$

Note that we have $g < \alpha < 1$. Now, let $x_0 = \frac{1}{1+\alpha}$. We have $A_\alpha(x_0) = 1-\alpha$, hence $A_\alpha^2(x_0) = \frac{1}{1+\alpha}$ and

$$\begin{aligned}\rho_1(x_0) &= (2,-1),\\ \rho_2(x_0) &= (3,1),\\ \rho_3(x_0) &= (2,-1).\end{aligned}$$

Now, since $x_0 \in \mathscr{D}_2(\alpha)$, we have

$$\lim_{x\to x_0^-} \rho(x) = \rho(x_0) \text{ and } \lim_{x\to x_0^+} \rho(x) = (1,1) \neq \rho(x_0).$$

Since A_α is left-continuous at x_0 and increasing in $\mathfrak{c}(2,-1)\cup x_0$, we have

$$\lim_{x\to x_0^-} A_\alpha(x) = A_\alpha(x_0)^- = (1-\alpha)^-.$$

We can check that $1-\alpha \in \mathfrak{c}(3,1)$. It follows that

$$\lim_{x\to x_0^-} \rho_2(x) = \lim_{x\to x_0^-} \rho(A_\alpha(x)) = \rho(1-\alpha) = (3,1) = \rho_2(x_0).$$

The map A_α is continuously decreasing in a neighborhood of $1-\alpha$, hence

$$\lim_{y\to(1-\alpha)^-} A_\alpha(y) = A_\alpha(1-\alpha)^+ = \left(\frac{1}{1+\alpha}\right)^+.$$

It follows that

$$\lim_{x\to x_0^-} \rho_3(x) = \lim_{x\to x_0^-} \rho(A_\alpha(A_\alpha(x))) = \lim_{y\to x_0^+} \rho_1(y) = (1,1) \neq \rho_3(x_0).$$

In conclusion, x_0 is the only point of $(0,\alpha)$ such that

$$x_0 = [(2,-1),(3,1),(2,-1),\ldots],$$

so that $\mathfrak{c}[(2,-1),(3,1),(2,-1)]$ is not a non-empty open interval and thus

$$(2,-1)(3,1)(2,-1) \notin \mathcal{L}(\alpha).$$

Using (5.15), (5.17) and (5.18), it is fairly easy to characterize the set of admissible words for $\alpha \in \{1/2, g, 1\}$. Precisely, let $\mathbf{s} = (a_1,\varepsilon_1)...(a_n,\varepsilon_n) \in \mathcal{A}^*$.

- $\mathbf{s} \in \mathcal{L}(1) \Leftrightarrow \forall j \in \{1,\ldots,n\}, \varepsilon_j = 1$;
- $\mathbf{s} \in \mathcal{L}(1/2) \Leftrightarrow \forall j \in \{1,\ldots,n\}, a_j \geq 2$ and $(a_j = 2 \Rightarrow \varepsilon_j = 1)$;
- $\mathbf{s} \in \mathcal{L}(g) \Leftrightarrow \forall j \in \{1,\ldots,n\}, a_j \geq 2$ and $\forall j \in \{1,\ldots,n-1\}, \big(\varepsilon_j = -1 \Rightarrow a_{j+1} \geq 3\big)$.

However, up to your knowledge, no easy characterization of $\mathcal{L}(\alpha)$ can be found for an abritrary $\alpha \in [1/2, 1]$.

5.3.3 α-cells

We observe that for all $(a,\varepsilon) \in \mathbb{N} \times \{\pm 1\}$, we have

$$\mathfrak{c}(a,\varepsilon) = \psi_{a,\varepsilon}\big(J(a,\varepsilon)\big). \tag{5.22}$$

In order to extend this formula, we introduce more generally the Möbius transformations

$$\psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)} = \psi_{(a_1,\varepsilon_1)} \circ \psi_{(a_2,\varepsilon_2)} \circ \cdots \circ \psi_{(a_n,\varepsilon_n)},$$

so that

$$\begin{aligned}\psi_{(a_1,\varepsilon_1),\dots,(a_n,\varepsilon_n)}(t) &= [(a_1,\varepsilon_1),\dots,(a_n,\varepsilon_n),t] \\ &= \frac{p_n + t\varepsilon_n p_{n-1}}{q_n + t\varepsilon_n q_{n-1}},\end{aligned}$$

where $q_n, p_n, q_{n-1}, p_{n-1}$ are defined by (5.3). Every function $\psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)}$ is continuous and strictly monotone on $(0,\alpha)$.

Lemma 5.3.7. *Let $n \geq 1$, $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n) \in \mathcal{A}^*$. We have*

$$\mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big] = \bigcap_{j=1}^{n} \psi_{(a_1,\varepsilon_1)\dots(a_j,\varepsilon_j)}\big(J(a_j,\varepsilon_j)\big) \tag{5.23}$$

Proof. We prove (5.23) by induction on $n \geq 1$. If $n = 1$, then (5.23) matches with (5.22). Let $n \geq 1$ and $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)(a_{n+1},\varepsilon_{n+1}) \in \mathcal{A}^*$ such that (5.23). By definition, we have

$$\mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)(a_{n+1},\varepsilon_{n+1})\big] = \mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big] \cap A_\alpha^{-n}\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big).$$

Let $x \in \mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big]$. It follows from (5.4) that

$$x = \psi(x_n)$$

with $\psi = \psi_{(a_1,\varepsilon_1),\dots,(a_n,\varepsilon_n)}$ and $x_n = A_\alpha^n(x)$. If $x \in A_\alpha^{-n}\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big)$, then $x_n \in \mathfrak{c}\big(a_{n+1},\varepsilon_{n+1}\big)$, hence $x \in \psi\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big)$. Conversely, if $x \in \psi\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big)$, then we have

$$x = \psi(x_n) = \psi(t)$$

with $t \in \mathfrak{c}(a_{n+1},\varepsilon_{n+1})$. It follows that $x_n = t$, hence $x \in A_\alpha^{-n}\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big)$. This shows that

$$\mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)(a_{n+1},\varepsilon_{n+1})\big] = \mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big] \cap \psi\big((\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big).$$

To conclude the proof, we observe that

$$\begin{aligned}\psi\big(\mathfrak{c}(a_{n+1},\varepsilon_{n+1})\big) &= \psi\big(\psi_{(a_{n+1},\varepsilon_{n+1})}(J(a_{n+1},\varepsilon_{n+1})\big) \\ &= \psi_{(a_1,\varepsilon_1),\dots,(a_{n+1},\varepsilon_{n+1})}\big(J(a_{n+1},\varepsilon_{n+1})\big).\end{aligned}$$

□

Remark 5.3.8. Since we have

$$\begin{aligned}\psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)}\big(J(a_n,\varepsilon_n)\big) &= \psi_{(a_1,\varepsilon_1)\dots(a_{n-1},\varepsilon_{n-1})}\big(\psi_{(a_n,\varepsilon_n)}\big(J(a_n,\varepsilon_n)\big) \\ &= \psi_{(a_1,\varepsilon_1)\dots(a_{n-1},\varepsilon_{n-1})}\big(\mathfrak{c}(a_n,\varepsilon_n)\big),\end{aligned} \tag{5.24}$$

we also have

$$\mathfrak{c}[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)] = \mathfrak{c}[(a_1,\varepsilon_1)\dots(a_{n-1},\varepsilon_{n-1})] \cap \psi_{(a_1,\varepsilon_1)\dots(a_{n-1},\varepsilon_{n-1})}\big(\mathfrak{c}(a_n,\varepsilon_n)\big). \tag{5.25}$$

5.3.4 Advantageous numbers

When $\alpha = 1$, knowing the endpoints of a given cell $\mathfrak{c}\big((a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big)$ appears to be essential, as highlighted in [9] and [48]. But for an arbitrary value of α, it looks difficult to control the intersections in (5.19) or (5.23).

However, it turns out that for some α, the sets $\big(\psi_{(a_1,\varepsilon_1)\dots(a_j,\varepsilon_j)}(J(a_j,\varepsilon_j))\big)_{1\le j\le n}$ appearing in (5.23) are always decreasing. This leads to the following definition.

Definition 5.3.9. A number $\alpha \in [1/2, 1]$ is called *advantageous* if for all $n \ge 1$ and for all $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n) \in \mathcal{L}_n(\alpha)$,

$$\mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big] = \psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)}\big(J(a_n,\varepsilon_n)\big). \tag{5.26}$$

Example 5.3.10. When $\alpha = 1$, we always have $J(a_n,\varepsilon_n) = (0,1)$. Hence, the sequence of sets $\big(\psi_{(a_1,\varepsilon_1)\dots(a_j,\varepsilon_j)}\big(J(a_j,\varepsilon_j)\big)\big)_{j=1}^n$ in (5.23) is decreasing. Thus 1 is advantageous and we have for every $(a_1,\dots,a_n) \in \mathbb{N}^n$,

$$\begin{aligned}\mathfrak{c}\big[(a_1,1)\dots(a_n,1)\big] &= \psi_{(a_1,1)\dots(a_n,1)}\big((0,1)\big) \\ &= \Big\{\frac{p_n + tp_{n-1}}{q_n + qp_{n-1}} \,:\, t \in (0,1)\Big\}.\end{aligned}$$

The situation is similar for $\alpha = 1/2$ since we always have $J(a_n,\varepsilon_n) = (0,1/2)$. It follows that $1/2$ is advantageous and that for every admissible $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)$,

$$\mathfrak{c}\big[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)\big] = \Big\{\frac{p_n + t\varepsilon_n p_{n-1}}{q_n + t\varepsilon_n p_{n-1}} \,:\, t \in (0,1/2)\Big\}.$$

Example 5.3.11. If $g < \alpha < 1$ is advantageous, then for every $(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n) \in \mathcal{L}_n(\alpha)$,

$$\mathfrak{c}[(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)] = \Big\{\frac{p_n + tp_{n-1}}{q_n + qp_{n-1}} \,:\, t \in J(a_n,\varepsilon_n)\Big\},$$

with

$$J(a_n,\varepsilon_n) = \begin{cases}(1/\alpha - 1, 1) & \text{if } a_n = \varepsilon_n = 1, \\ (0,\alpha) & \text{if } a_n \ge 2 \text{ and } \varepsilon_n = 1, \\ (0, 1-\alpha) & \text{if } a_n \ge 2 \text{ and } \varepsilon_n = -1.\end{cases}$$

However there are few advantageous numbers.

Proposition 5.3.12. *A number $\alpha \in [1/2, 1]$ is advantageous if and only if*

$$\alpha \in \{1/2, g, 1\} \cup \Big\{1 - \frac{1}{k}, k \geq 3\Big\} \cup \Big\{\frac{-k + \sqrt{k^2 + 4k}}{2}, k \geq 2\Big\}.$$

Proof. We have already treated $\alpha = 1$, $\alpha = 1/2$. Let $\alpha \in]1/2, 1[$ be advantageous and let $(a_1, \varepsilon_1) \in \mathcal{L}_1(\alpha)$. For every (a_2, ε_2) such that $(a_1, \varepsilon_1)(a_2, \varepsilon_2) \in \mathcal{L}_2(\alpha)$, we have

$$\mathfrak{c}[(a_1, \varepsilon_1)(a_2, \varepsilon_2))] = \psi_{(a_1,\varepsilon_1)}\big(\mathfrak{c}(a_2, \varepsilon_2)\big). \tag{5.27}$$

On the other hand, by (5.25) and (5.24), and since $\psi_{(a_1,\varepsilon_1)}$ is continuously strictly monotonic,

$$\begin{aligned}\mathfrak{c}[(a_1, \varepsilon_1)(a_2, \varepsilon_2)] &= \psi_{(a_1,\varepsilon_1)}\big(J(a_1, \varepsilon_1)\big) \cap \psi_{(a_1,\varepsilon_1)}\big(\mathfrak{c}(a_2, \varepsilon_2)\big)\\ &= \psi_{(a_1,\varepsilon_1)}\big(J(a_1, \varepsilon_1) \cap \mathfrak{c}(a_2, \varepsilon_2)\big).\end{aligned}$$

It follows that necessarily, for all $(a_1, \varepsilon_1), (a_2, \varepsilon_2) \in \mathcal{L}_1(\alpha)$,

$$\mathfrak{c}(a_2, \varepsilon_2) \subseteq J(a_1, \varepsilon_1) \text{ or } \mathfrak{c}(a_2, \varepsilon_2) \cap J(a_1, \varepsilon_1) = \emptyset$$

Suppose $g < \alpha < 1$. First, let us consider $(a_1, \varepsilon_1) = (1, 1)$. We have $J(1, 1) = (\frac{1}{\alpha} - 1, 1)$, hence we must have for all $(a_2, \varepsilon_2) \geq (1, 1)$,

$$\mathfrak{c}(a_2, \varepsilon_2) \subseteq \Big(\frac{1}{\alpha} - 1, 1\Big) \text{ or } \mathfrak{c}(a_2, \varepsilon_2) \cap \Big(\frac{1}{\alpha} - 1, 1\Big) = \emptyset,$$

and consequently, $\frac{1}{\alpha} - 1$ must be an endpoint of some cell $\mathfrak{c}(k, \varepsilon)$. Since $g < \alpha < 1$, we have $0 < \frac{1}{\alpha} - 1 < \frac{1}{1+\alpha}$. Then, observing that $(\frac{1}{2}, \frac{1}{1+\alpha}) = \mathfrak{c}(2, -1)$, we must have:

$$\begin{aligned}&\Big(\exists k \geq 2, \frac{1}{\alpha} - 1 = \frac{1}{k}\Big) \vee \Big(\exists k \geq 2, \frac{1}{\alpha} - 1 = \frac{1}{k + \alpha}\Big)\\ \Leftrightarrow &\Big(\exists k \geq 2, \alpha = 1 - \frac{1}{k+1}\Big) \vee \Big(\exists k \geq 2, \alpha = \frac{-k + \sqrt{k^2 + 4k}}{2}\Big)\\ \Leftrightarrow &\Big(\exists k \geq 3, \alpha = 1 - \frac{1}{k}\Big) \vee \Big(\exists k \geq 2, \alpha = \frac{-k + \sqrt{k^2 + 4k}}{2}\Big)\end{aligned} \tag{5.28}$$

Similarly, taking $(a_1, \varepsilon_1) = (a, -1)$ with any $a \geq 2$ and noting that $J(a, -1) = (0, 1-\alpha)$, we must have for all $(a_2, \varepsilon_2) \geq (1, 1)$,

$$\mathfrak{c}(a_2, \varepsilon_2) \subseteq (0, 1 - \alpha) \text{ or } \mathfrak{c}(a_2, \varepsilon_2) \cap (0, 1 - \alpha) = \emptyset.$$

Since $0 < 1 - \alpha < \frac{1}{2+\alpha}$, this forces

$$\begin{aligned}&\Big(\exists k \geq 3, 1 - \alpha = \frac{1}{k}\Big) \vee \Big(\exists k \geq 3, 1 - \alpha = \frac{1}{k + \alpha}\Big)\\ \Leftrightarrow &\Big(\exists k \geq 3, \alpha = 1 - \frac{1}{k}\Big) \vee \Big(\exists k \geq 3, \alpha = \frac{-(k-1) + \sqrt{(k-1)^2 + 4(k-1)}}{2}\Big)\\ \Leftrightarrow &\Big(\exists k \geq 3, \alpha = 1 - \frac{1}{k}\Big) \vee \Big(\exists k \geq 2, \alpha = \frac{-k + \sqrt{k^2 + 4k}}{2}\Big).\end{aligned} \tag{5.29}$$

Note that taking $(a_1, \varepsilon_1) = (a, 1)$ with $a \geq 2$ does not lead to any other restriction on α: since $J(a, -1) = (0, \alpha)$, we have $\mathfrak{c}(a_2, \varepsilon_2) \cap J(a_1, \varepsilon_1) = \mathfrak{c}(a_2, \varepsilon_2)$ for all $(a_2, \varepsilon_2) \in \mathcal{L}_2(\alpha)$.

Finally, α must satisfy both conditions (5.28) and (5.29). This proves that

$$\alpha = 1 - 1/k \text{ with } k \geq 3$$

or

$$\alpha = \frac{-k + \sqrt{k^2 + 4k}}{2} \text{ with } k \geq 2.$$

Conversely, if $\alpha = 1 - 1/k$ with $k \geq 3$, then

$$\frac{1}{\alpha} - 1 = \frac{1}{k-1} \text{ and } 1 - \alpha = \frac{1}{k},$$

and we can prove by induction on $n \geq 1$ that for all $\mathbf{s} = (a_1, \varepsilon_1)...(a_n, \varepsilon_n)$, $\mathbf{s} \in \mathcal{L}_n(\alpha)$ if and only if for all $j \in \{1, \dots n\}$, $(a_j, \varepsilon_j) \geq (1, 1)$

$$\forall j \in \{1, \dots n-1\}, \begin{cases} (a_j, \varepsilon_j) = (1,1) \Rightarrow (1,1) \leq (a_{j+1}, \varepsilon_{j+1}) \leq (k-1, -1) \\ (a_j \geq 2 \text{ and } \varepsilon_j = -1) \Rightarrow (a_{j+1}, \varepsilon_{j+1}) \geq (k, 1), \end{cases} \tag{5.30}$$

and in that case, we have

$$\mathfrak{c}\big[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)\big] = \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}\big(J(a_n, \varepsilon_n)\big). \tag{5.31}$$

It is obvious when $n = 1$. Now let $n \geq 1$ such that the hypothesis holds and let $(a_1, \varepsilon_1)...(a_n, \varepsilon_n)(a_{n+1}, \varepsilon_{n+1}) \in \mathcal{A}^*$. We can assume that for all $j \in \{1, \dots n+1\}$, $(a_j, \varepsilon_j) \geq (1, 1)$ and that conditions (5.30) hold, otherwise it is clear that $(a_1, \varepsilon_1)...(a_n, \varepsilon_n)(a_{n+1}, \varepsilon_{n+1})$ is not admissible. According to (5.25) and the inductive assumption, we have

$$\begin{aligned} \mathfrak{c}[(a_1, \varepsilon_1)...(a_{n+1}, \varepsilon_{n+1})] &= \mathfrak{c}[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)] \cap \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}\big(\mathfrak{c}(a_{n+1}, \varepsilon_{n+1})\big) \\ &= \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}(J(a_n, \varepsilon_n)) \cap \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}\big(\mathfrak{c}(a_{n+1}, \varepsilon_{n+1})\big) \\ &= \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}\big(J(a_n, \varepsilon_n) \cap \mathfrak{c}(a_{n+1}, \varepsilon_{n+1})\big), \end{aligned}$$

where the last equality follows from the fact that $\psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}$ is continuous and strictly monotonic. There are three cases:

1. $(a_n, \varepsilon_n) = (1, 1)$. Then $J(a_n, \varepsilon_n) = (\frac{1}{\alpha} - 1, 1)$. If $(a_{n+1}, \varepsilon_{n+1}) > (k-1, -1)$ then, $\mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \cap (\frac{1}{\alpha} - 1, 1) = \emptyset$ and $(a_1, \varepsilon_1)(a_{n+1}, \varepsilon_{n+1}) \notin \mathcal{L}_{n+1}(\alpha)$. Conversely, if $(a_{n+1}, \varepsilon_{n+1}) \leq (k-1, -1)$, then $\mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \cap (\frac{1}{\alpha} - 1, 1) = \mathfrak{c}(a_{n+1}, \varepsilon_{n+1})$. This yields $(a_1, \varepsilon_1)(a_{n+1}, \varepsilon_{n+1}) \in \mathcal{L}_{n+1}(\alpha)$ and

$$\begin{aligned} \mathfrak{c}[(a_1, \varepsilon_1)...(a_{n+1}, \varepsilon_{n+1})] &= \psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}\big(\mathfrak{c}(a_{n+1}, \varepsilon_{n+1})\big) \\ &= \psi_{(a_1,\varepsilon_1)...(a_{n+1},\varepsilon_{n+1})}(J(a_{n+1}, \varepsilon_{n+1})). \end{aligned} \tag{5.32}$$

2. $a_n \geq 2, \varepsilon_n = -1$. Then $J(a_n, \varepsilon_n) = (0, 1-\alpha)$. If $(a_{n+1}, \varepsilon_{n+1}) < (k, 1)$ then, $\mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \cap (\frac{1}{\alpha} - 1, 1) = \emptyset$ and $(a_1, \varepsilon_1)(a_{n+1}, \varepsilon_{n+1}) \notin \mathcal{L}_{n+1}(\alpha)$. Conversely, if $(a_{n+1}, \varepsilon_{n+1}) \geq (k-1, -1)$, then $\mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \cap (\frac{1}{\alpha} - 1, 1) = \mathfrak{c}(a_{n+1}, \varepsilon_{n+1})$. This yields $(a_1, \varepsilon_1)(a_{n+1}, \varepsilon_{n+1}) \in \mathcal{L}_{n+1}(\alpha)$ and (5.32).

3. $a_n \geq 2, \varepsilon_n = 1$. Then $J(a_n, \varepsilon_n) = (0, \alpha)$ so that $\mathfrak{c}(a_{n+1}, \varepsilon_{n+1}) \cap (0, \alpha) = \mathfrak{c}(a_{n+1}, \varepsilon_{n+1})$ for every $(a_{n+1}, \varepsilon_{n+1}) \in \mathcal{L}_1(\alpha)$ and again we have (5.32).

In the same way, if $\alpha = \frac{-k+\sqrt{k^2+4k}}{2}$ with $k \geq 2$, then

$$\frac{1}{\alpha} - 1 = \frac{1}{k+\alpha} \text{ and } 1 - \alpha = \frac{1}{k+1+\alpha},$$

and we obtain that $\mathbf{s} = (a_1, \varepsilon_1)...(a_n, \varepsilon_n)... \in \mathcal{L}_n(\alpha)$ if and only if for all $j \in \{1, \dots n\}$, $(a_j, \varepsilon_j) \geq (1, 1)$ and

$$\forall j \in \{1, \dots n-1\}, \begin{cases} (a_j, \varepsilon_j) = (1,1) \Rightarrow (1,1) \leq (a_{j+1}, \varepsilon_{j+1}) \leq (k, 1) \\ (a_j \geq 2 \text{ and } \varepsilon_j = -1) \Rightarrow (a_{j+1}, \varepsilon_{j+1}) \geq (k+2, -1). \end{cases}$$

Now let $1/2 < \alpha < g$ be advantageous. The same reasoning leads us to conclude that $1 - \alpha$ must be the endpoint of some elementary cell $\mathfrak{c}(k, \varepsilon)$. But this cannot occur, since

$$1/2 < \alpha < g \Rightarrow \frac{1}{2+\alpha} < 1 - \alpha < \frac{1}{2}.$$

Hence, there is no advantageous number in $]1/2, g[$.

It remains to consider the case $\alpha = g$. We have $\mathcal{L}_1(g) = \{(k, \varepsilon) \in \mathcal{A} : (k, \varepsilon) \geq (2, -1)\}$ and

$$J(k, \varepsilon) = \begin{cases} (0, 1-g) & \text{if } \varepsilon = -1 \\ (0, g) & \text{if } \varepsilon = 1. \end{cases}$$

Since $1 - g = \frac{1}{2+g}$, $1 - g$ is the right endpoint of the cell $\mathfrak{c}(3, -1)$. Then the same reasoning as before shows that g is advantageous and that $\mathbf{s} = (a_1, \varepsilon_1)...(a_n, \varepsilon_n) \in \mathcal{L}_n(g)$ if and only for all $j \in \{1, \dots, n\}$, $a_j \geq 2$ and

$$\forall j \in \{1, \dots, n-1\}, (a_j \geq 2 \text{ and } \varepsilon_j = -1) \Rightarrow a_{j+1} \geq 3. \qquad \square$$

Let us provide some examples.

Example 5.3.13. Consider the case $\alpha = 3/4$ and explicitly describe the cells of depth 1 in $(0, 3/4)$. For $\varepsilon_1 = -1$, we have

$$\frac{1}{k} < x \leq \frac{4}{4k-1}$$

and for $\varepsilon_n = 1$, we get

$$\frac{4}{4k+3} < x \leq \frac{1}{k}.$$

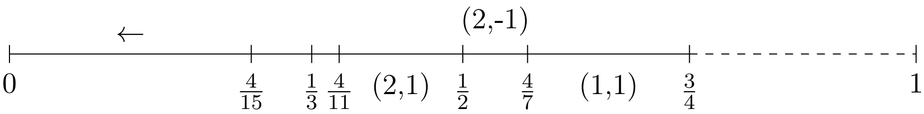


Figure 5.5: Firsts 3/4-cells of depth 1.

For the 3/4-cells of depth 2 in $(1/2, 4/7)$, we have

$$\frac{k}{2k-1} < x \leq \frac{4k-1}{8k-6}$$

for $\varepsilon_2 = -1$ and

$$\frac{4k+3}{8k+2} < x \leq \frac{k}{2k-1}$$

for $\varepsilon_2 = 1$.

Figure 5.6: Firsts 3/4-cells of depth 2 in $(1/2, 4/7)$.

In $(4/11, 1/2)$, for $\varepsilon_2 = -1$, we have

$$\frac{4k-1}{8k+2} \leq x \leq \frac{k}{2k+1}$$

and for $\varepsilon_2 = 1$, we get

$$\frac{k}{2k+1} \leq x \leq \frac{4k+3}{8k+10}.$$

Let us remark that in this interval $(4/11, 1/2)$, the intersection with the previous cell is mandatory, since $x \in (1/3, 4/11)$ checks $(a_2, \varepsilon_2) = (1, -1)$, see Figure 5.7.

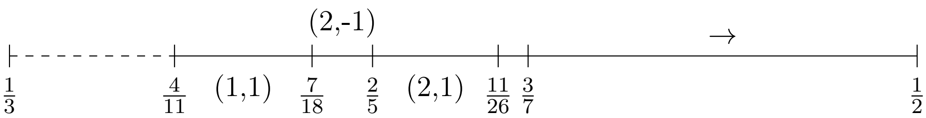


Figure 5.7: The 3/4-cells of depth 2 for $(4/11, 1/2)$.

Example 5.3.14. Let us now consider the case $\alpha = 7/9$ and first examine the cells of depth 1 in $(0, 7/9)$. For $\varepsilon_1 = -1$, we have

$$\frac{1}{k} < x \leq \frac{9}{9k-2}$$

and for $\varepsilon_1 = 1$, we get

$$\frac{9}{9k+7} < x \leq \frac{1}{k}.$$

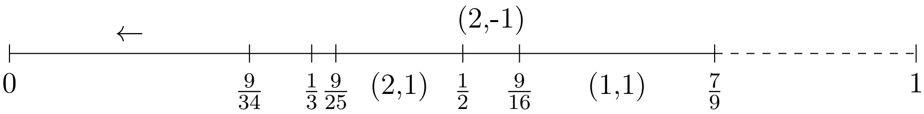


Figure 5.8: Firsts 7/9-cells of depth 1.

Let us find the subcells of $\mathfrak{c}(2, -1) = (1/2, 9/16)$. Since,

$$J(a, \varepsilon) = \begin{cases} (2/7, 7/9) & \text{if } a = \varepsilon = 1, \\ (0, 7/9) & \text{if } a \geq 2 \text{ and } \varepsilon = 1, \\ (0, 2/9) & \text{if } a \geq 2 \text{ and } \varepsilon = -1, \end{cases}$$

we have for $a \geq 2$,

$$\psi_{(2,-1)(1,1)}(J(1,1)) = \left\{ \frac{1}{2 - \frac{1}{1+t}} : 2/7 < t < 7/9 \right\}$$
$$\psi_{(2,-1)(a,1)}(J(a,1)) = \left\{ \frac{1}{2 - \frac{1}{a+t}} : 0 < t < 7/9 \right\}$$
$$\psi_{(2,-1)(a,-1)}(J(a,-1)) = \left\{ \frac{1}{2 - \frac{1}{a-t}} : 0 < t < 2/9 \right\}.$$

Then, we can see that $\mathfrak{c}((2, -1)(a, \varepsilon)) \neq \emptyset$ if and only if $(a, \varepsilon) \geq (4, 1)$ but the sets appearing in the intersection coming from (5.23) are not decreasing:

$$\begin{aligned} \mathfrak{c}((2,-1)(4,1)) &= (1/2, 9/16) \cap \psi_{(2,-1)(4,1)}(J(4,1)) = (1/2, 9/16) \cap (43/77, 4/7) \\ &= (43/77, 9/16). \end{aligned}$$

For $(9/25, 1/2)$, taking the intersection is also necessary, as $x \in (1/3, 9/25)$ checks $(a_2, \varepsilon_2) = (4, 1)$. We find

$$\frac{9k-2}{18k+5} \leq x < \frac{k}{2k+1}$$

and

$$\frac{k}{2k+1} \leq x < \frac{9k+7}{18k+23}$$

for $\varepsilon_2 = -1$ and $\varepsilon_2 = 1$, respectively.

Figure 5.9: The 7/9-cells of depth 2 for $(52/95, 9/16)$.

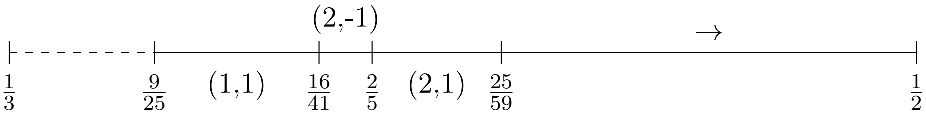


Figure 5.10: The 7/9-cells of depth 2 for $(9/25, 1/2)$.

5.4 Nearest integer continued fraction

We will primarily focus on the nearest integer continued fraction expansion, corresponding to $\alpha = 1/2$ in the generalized continued fraction expansion previously presented. We present here results that will be useful in Chapter 6.

Let us recall that

$$\mathcal{L}_1(1/2) = \{(a, \varepsilon) \in \mathbb{Z}_+ \times \{\pm 1\} \,:\, a \geq 2\} \setminus \{(2, -1)\},$$

and that

$$\mathcal{L}_n(1/2) = \{(a_1, \varepsilon_1)...(a_n, \varepsilon_n) \,:\, \forall j \in \{1, \ldots, n\}, (a_j, \varepsilon_j) \in \mathcal{L}_1\}.$$

In the sequel, we abbreviate $\mathcal{L}_n(1/2) = \mathcal{L}_n$ and $\mathcal{L}(1/2) = \mathcal{L}$.

Let $(a_1, \varepsilon_1)...(a_n, \varepsilon_n) \in \mathcal{L}$. The cell $\mathfrak{c} = \mathfrak{c}[(a_1, \varepsilon_1)...(a_n, \varepsilon_n)]$ is the open interval

$$\mathfrak{c} = \left\{ \frac{p_n + t\varepsilon_n p_{n-1}}{q_n + t\varepsilon_n q_{n-1}} \,:\, t \in (0, 1/2) \right\}.$$

The determinant of the homography $\psi_{(a_1,\varepsilon_1)...(a_n,\varepsilon_n)}(t) = \frac{p_n + t\varepsilon_n p_{n-1}}{q_n + t\varepsilon_n q_{n-1}}$ is given by (cf. (5.5))

$$(-1)^n \varepsilon_0 \cdots \varepsilon_n = s_n.$$

Hence,

- for $s_n > 0$ and $\varepsilon_n = 1$, we have

$$\mathfrak{c}_n = \left(\frac{p_n}{q_n}, \frac{2p_n + p_{n-1}}{2q_n + q_{n-1}} \right),$$

- for $s_n > 0$ and $\varepsilon_n = -1$, we have

$$\mathfrak{c}_n = \left(\frac{p_n}{q_n}, \frac{2p_n - p_{n-1}}{2q_n - q_{n-1}} \right),$$

- for $s_n < 0$ and $\varepsilon_n = 1$, we have

$$\mathfrak{c}_n = \Big(\frac{2p_n + p_{n-1}}{2q_n + q_{n-1}}, \frac{p_n}{q_n}\Big),$$

- for $s_n < 0$ and $\varepsilon_n = -1$, we have

$$\mathfrak{c}_n = \Big(\frac{2p_n - p_{n-1}}{2q_n - q_{n-1}}, \frac{p_n}{q_n}\Big).$$

It is convenient to define the following notation.

Notation 5.4.1. For any real numbers a and b, we define the interval

$$\mathcal{I}(a,b) = (\min\{a,b\}, \max\{a,b\}).$$

With this notation, we always have

$$\mathfrak{c}_n = \mathcal{I}\Big(\frac{p_n}{q_n}, \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}}\Big),$$

The subcells of $\mathfrak{c}$ of depth $n+1$ are given by

$$\begin{aligned}\mathfrak{c}_{(k,\varepsilon)} &:= \psi_{(a_1,\varepsilon_1)\dots(a_n,\varepsilon_n)}(\mathfrak{c}(k,\varepsilon)) \\ &= \mathcal{I}\Big(\frac{kp_n + \varepsilon_n p_{n-1}}{kq_n + \varepsilon_n q_{n-1}}, \frac{(k+\varepsilon/2)p_n + \varepsilon_n p_{n-1}}{(k+\varepsilon/2)q_n + \varepsilon_n q_{n-1}}\Big), \quad (k,\varepsilon) \in \mathcal{L}_1.\end{aligned}$$

Two cells $\mathfrak{c}_{k,\varepsilon}$ and $\mathfrak{c}_{k',\varepsilon'}$ are adjacent if and only if $(k,\varepsilon) \preccurlyeq (k',\varepsilon')$ or $(k',\varepsilon') \preccurlyeq (k,\varepsilon)$. Note that $\frac{2p_n+\varepsilon_n p_{n-1}}{2q_n+\varepsilon_n q_{n-1}}$ is an endpoint of $\mathfrak{c}_{(2,1)}$ whereas p_n/q_n is not an endpoint of any cell $\mathfrak{c}_{k,\varepsilon}$. The endpoints of $\mathfrak{c}_{k,\varepsilon}$ tend toward p_n/q_n when k tends to infinity.

5.4.1 Distance from an irrational to the edges of a cell of depth n

In [9, 48], it is useful to know the distance from an irrational to the edges of a cell of depth n that contains it. Let $x = [(0,1),(a_1,\varepsilon_1),\dots,(a_j,\varepsilon_j),\dots]$ an element of $]0,1/2[\setminus\mathbb{Q}$. Given an integer $n \geq 0$, there exists a unique cell $\mathfrak{c}_n$ of depth n containing x, given by

$$\mathfrak{c}_n = \mathcal{I}\Big(\frac{p_n}{q_n}, \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}}\Big).$$

We define $\delta_n = \delta_n(x)$ as the distance from x to the boundary of $\mathfrak{c}_n$. The proof of Proposition 4 in [9], concerning the case $\alpha = 1$, can be easily adapted to get the following result:

Proposition 5.4.2. *For $n \in \mathbb{N}_0$, we have*

$$\delta_n \leq \frac{2}{q_n q_{n+1}} \text{ and } \delta_n \geq \begin{cases} \frac{4}{45 q_n q_{n+1}} & \text{if } a_{n+1} \geq 3, \\ \frac{2}{3 q_{n+1} q_{n+2}} & \text{if } a_{n+1} = 2. \end{cases} \tag{5.33}$$

Proof. We have

$$x - \frac{p_n}{q_n} = \frac{s_n \beta_n(x)}{q_n},$$

and

$$\begin{aligned} x - \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}} &= \frac{p_n + \varepsilon_n p_{n-1} x_n}{q_n + \varepsilon_n q_{n-1} x_n} - \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}} \\ &= \frac{2s_n(x_n - 1/2)}{(q_n + q_{n-1}\varepsilon_n x_n)(2q_n + \varepsilon_n q_{n-1})}. \end{aligned}$$

We already know by (5.7) that

$$\frac{2}{3q_n q_{n+1}} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{2}{q_n q_{n+1}}.$$

This yields

$$\delta_n(x) \leq \frac{2}{q_n q_{n+1}}.$$

If $a_{n+1} \geq 3$, then $x_n < 2/5$ since $x_n = (a_{n+1} + \varepsilon_{n+1} x_{n+1})^{-1}$. It follows that

$$\left| x - \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}} \right| \geq \frac{1}{10(q_n + q_{n-1}/2)^2} \geq \frac{2}{45q_n^2}.$$

This yields

$$\delta_n(x) \geq \min\left(\frac{2}{45q_n^2}, \frac{2}{3q_{n+1}q_n} \right) \geq \frac{4}{45q_n q_{n+1}},$$

where we used the fact that $a_{n+1} \geq 3$ which gives $q_{n+1} \geq 2q_n$.

If $a_{n+1} = 2$ then $\varepsilon_{n+1} = -1$ and p_{n+1}/q_{n+1} lies between x and $\frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}}$. Consequently,

$$\left| x - \frac{2p_n + \varepsilon_n p_{n-1}}{2q_n + \varepsilon_n q_{n-1}} \right| \geq \frac{\beta_{n+1}}{q_{n+1}} \geq \frac{2}{3q_{n+1}q_{n+2}}$$

and

$$\delta_n(x) \geq \frac{2}{3q_{n+1}q_{n+2}}.$$

□

5.4.2 Representation of rationals

Let us examine the expansion for rational numbers. If the expansion ends with 2, the rational number admits two possible representations. This observation leads us to define

$$E_{1/2} = \bigcup_{n \geq 1} A_{1/2}^{-n}(\{1/2\})$$

and

$$E_0 = (\bigcup_{n \geq 1} A_{1/2}^{-n}(\{0\})) \backslash E_{1/2}.$$

In this Section, we investigate which cells each rational number appears as an endpoint of.

The set $E_{1/2}$

For every $r \in E_{1/2}$, we can associate an expansion of the form

$$r = [(a_1, \varepsilon_1), \dots, (a_k, \varepsilon_k), 2]$$

with $k \geq 1$. It is also associated with a representation of the form

$$r = [(a_1, \varepsilon_1), \dots, (a_k + \varepsilon_k, -\varepsilon_k), 2].$$

The representation given by the algorithm is the one involving $\max\{a_k, a_k + \varepsilon_k\}$. We define the canonical representation of such a number r as

$$r = [(a_1, \varepsilon_1), \dots, (a_k, -1), 2] \quad \text{with } a_k \geq 3.$$

The other representation is therefore given by

$$r = [(a_1, \varepsilon_1), \dots, (a_k - 1, 1), 2].$$

For all $j \in \{1, \dots, k\}$, let $p_j/q_j = [(a_0, \varepsilon_0), \dots, (a_{j-1}, \varepsilon_{j-1}), a_j]$ be the convergents of r and let

$$p = 2p_k - p_{k-1} \text{ and } q = 2q_k - q_{k-1}$$

so that $r = p/q$ is the reduced form of r. We also observe that r is an endpoint of two cells of depth k which are

$$\mathfrak{c}_{-1} := \mathfrak{c}[(a_0, \varepsilon_0) \dots (a_k, -1)] = \mathcal{I}\Big(r, \frac{p_k}{q_k}\Big)$$
$$\mathfrak{c}_1 := \mathfrak{c}[(a_0, \varepsilon_0) \dots (a_k - 1, 1)] = \mathcal{I}\Big(r, \frac{p_k - p_{k-1}}{q_k - q_{k-1}}\Big).$$

Note that

$$\frac{p_k}{q_k} = [(a_0, \varepsilon_0), \dots, (a_{k-1}, \varepsilon_{k-1}), a_k],$$
$$\frac{p_k - p_{k-1}}{q_k - q_{k-1}} = [(a_0, \varepsilon_0), \dots, (a_{k-1}, \varepsilon_{k-1}), a_k - 1)].$$

In these two cells, r is the endpoint which is not the limit point of subcells of depth $k+1$.

We have

$$\Big|r - \frac{p_k}{q_k}\Big| = \frac{1}{q_k q} > \frac{1}{q^2},$$

and

$$\Big|r - \frac{p_k - p_{k-1}}{q_k - q_{k-1}}\Big| = \frac{1}{q(q_k - q_{k-1})} > \frac{1}{q^2}.$$

This proves that

$$\Big(r - \frac{1}{q^2}, r + \frac{1}{q^2}\Big) \setminus \{r\} \subseteq \mathfrak{c}_1 \cup \mathfrak{c}_{-1}.$$

The set E_0

It can be easily verified that each $r \in E_0$ admits a unique representation,

$$r = [(a_1, \varepsilon_1), \ldots, (a_{k-1}, \varepsilon_{k-1}), a_k]$$

with $n \geq 2$ and $a_n \geq 3$. For all $j \in \{1, \ldots, k\}$, denote by $p_j/q_j = [(a_0, \varepsilon_0), \ldots, (a_{j-1}, \varepsilon_{j-1}), a_j]$ be the convergents of r. Hence, if we write $p = p_k$, $q = q_k$, we have the reduced form $r = p/q$.

Here, r is the endpoint of the two adjacent cells of depth k

$$\mathfrak{c}_1 := \mathfrak{c}[(a_1, \varepsilon_1) \ldots (a_k, 1)] = \mathcal{I}\Big(r, \frac{2p_k + p_{k-1}}{2q_k + q_{k-1}}\Big)$$
$$\mathfrak{c}_{-1} := \mathfrak{c}[(a_1, \varepsilon_1) \ldots (a_k, -1)] = \mathcal{I}\Big(r, \frac{2p_k - p_{k-1}}{2q_k - q_{k-1}}\Big).$$

We have

$$\Big|r - \frac{2p_k + p_{k-1}}{2q_k + q_{k-1}}\Big| = \frac{1}{q(2q_k + q_{k-1})} > \frac{1}{3q^2}$$

and

$$\Big|r - \frac{2p_k - p_{k-1}}{2q_k - q_{k-1}}\Big| = \frac{1}{q(2q_k - q_{k-1})} > \frac{1}{2q^2}.$$

It follows that

$$\Big(r - \frac{1}{3q^2}, r + \frac{1}{3q^2}\Big) \setminus \{r\} \subseteq \mathfrak{c}_1 \cup \mathfrak{c}_{-1}.$$

Chapter 6

Regularity of functions defined through continued fractions, focus on the Brjuno-Yoccoz function $\mathfrak{B}$ and Thomae's function

In the first Section, we aim to conduct a multifractal study of the function $\mathfrak{B}$, which is the function considered in [117] to simplify calculations. It is worth noting that the standard methods for multifractal studies cannot be applied here because, like B, $\mathfrak{B}$ is not assimilable to a locally bounded function. The natural approach involves replacing the usual L^∞ norm by an L^p norm and thus utilizing the spaces T_p^u originally introduced by Calderón and Zygmund [26, 79]. This leads to the concept of p-exponent. We establish that the p-exponent $h_{\mathfrak{B}}^{(p)}(x)$ of $\mathfrak{B}$ at x is the inverse of the irrationality exponent $\tau(x)$ of x:

Theorem 6.0.1. *Let $p \in [1, \infty)$; the p-exponents of $\mathfrak{B}$ are given by*

$$h_{\mathfrak{B}}^{(p)}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$$

This Theorem can be proved using the fact that the difference $\mathfrak{B} - B_1$ can be extended from $\mathbb{R}\setminus\mathbb{Q}$ to $\mathbb{R}$ as a 1/2-Hölder continuous periodic function with period one (see [85]) and a result from [9] concerning the local behavior of B at rational points. Since the regularity of $B - B_\alpha$ is not generally known, the previous argument does not extend to determining the pointwise regularity of B_α. In Chapter 5, we generalized some of the results from [9, 48] to α-continued fractions. This led to the development of a metric theory for these continued fraction expansions, which allowed us to observe that the notions of cells associated with these expansions are easier to handle for certain values of α. For $\alpha = 1/2$, this adaptation proceeds smoothly, allowing us to recover Theorem 6.0.1. In this context, we have taken care to explicitly specify whether the results apply to B_α or $\mathfrak{B}$ in particular, allowing the reader to discern the missing steps needed to achieve a multifractal study of B_α. To achieve this, we adapt the methods employed in [48]. We conclude with remarks on the potential extension of this work to B_α and other generalizations of B_1.

We also explore the pointwise regularity of another function related to Diophantine approximation: Thomae's function [71], a classic example of a function that is continuous at the irrationals and discontinuous elsewhere. Defined for a parameter $\theta \in (0, 2]$, it displays a complex self-similar structure and intriguing regularity properties. After revisiting its key features, we examine its Hölder continuity, focusing on the interaction between its discrete spikes and its behavior on dense subsets of the real line. This analysis offers a refined understanding of Thomae's function's irregularity, using classical analytical tools to highlight its fractal nature.

For the Brjuno-Yoccoz function and Thomae's function, the regularity exponent at a given point is inversely proportional to the irrationality measure of that point. This means that the better the point is approximable by rationals, the less regular these functions are at that point. One may then wonder whether all known multifractal functions linked to diophantine approximation satisfy this kind of relationship. To investigate, we introduce the Takagi function, a monofractal with regularity 1, and the Minkowski function, whose pointwise regularity remains unknown but exhibits multifractal behavior distinct from the inverse of the irrationality measure.

Finally, we provide some perspectives for a follow-up to this Chapter, aiming to establish the multifractal analysis of a whole class of functions, including Brjuno functions in particular.

6.1 Multifractal analysis of $\mathfrak{B}$

This Section was written in collaboration with Bruno Martin and Samuel Nicolay. Our collaboration is ongoing, and this text reflects my personal interpretation of our discussions. Some results will not appear in future articles, while others remain essential.

6.1.1 Difference $B_1 - \mathfrak{B}$

As mentioned in the introduction, we can deduce the local behavior of $\mathfrak{B}$ from results on B_1 and from the very nice Theorem 4.6 of [85]: the difference $\mathfrak{B} - B_1$ can be extended from $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{R}$ as an 1/2-Hölder continuous periodic function with period one. As a first consequence, $B_{1/2}$ and B_1 have the same Lebesgue points, which are Brjuno numbers. Another easy consequence is the following:

$$h_{\mathfrak{B}}^{(p)}(x) \geq \frac{1}{\tau(x)}.$$

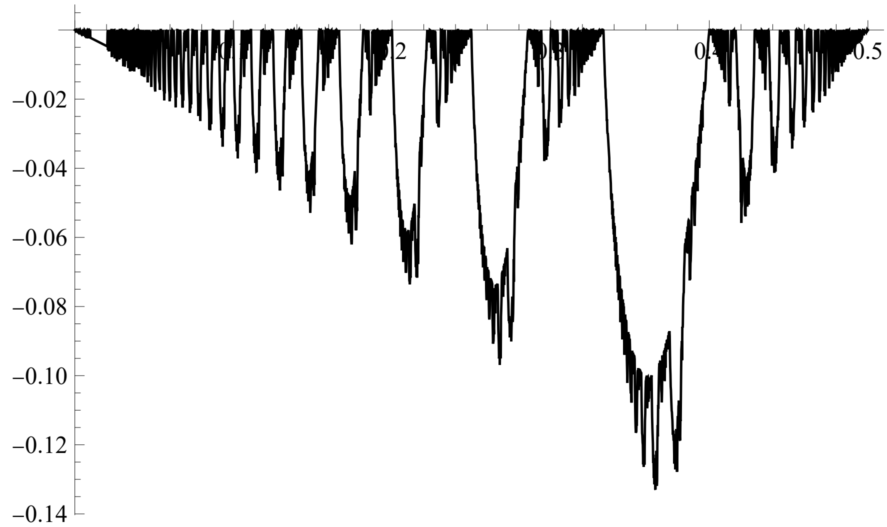


Figure 6.1: $B_1 - B_{1/2}$.

Indeed, since $\tau(x) \geq 2$, one has

$$\begin{aligned} h^{-1/p}\|\mathfrak{B}(x)-\mathfrak{B}\|_{L^p([x-\frac{h}{2},x+\frac{h}{2}])} &\leq h^{-1/p}(\|B(x)-B\|_{L^p([x-\frac{h}{2},x+\frac{h}{2}])} \\ &\quad + \|(\mathfrak{B}(x)-B(x))-(\mathfrak{B}-B)\|_{L^p([x-\frac{h}{2},x+\frac{h}{2}])}) \\ &\leq Ch^{\frac{1}{\tau(x)}} + Ch^{-1/p}h^{1/2}(\int_{x-h/2}^{x+h/2} dt)^{1/p} \\ &\leq Ch^{\frac{1}{\tau(x)}}. \end{aligned}$$

This also proves that

$$h_{\mathfrak{B}}^{(p)}(x) < 1/2 \Rightarrow h_{\mathfrak{B}}^{(p)}(x) = h_{B_1}^{(p)}(x).$$

Now, given a rational $r \in]0,1[$ with reduced form $r = p/q$, we know from [9] that for $0 < |h| < 1/(3q^2)$, we have

$$\frac{1}{h}\int_r^{r+h} B_1(t)\,dt = \frac{\log(e/q^2|h|)}{q} + \widetilde{B_1}(r) + O(qh\log\frac{1}{q^2|h|}),$$

where the O is uniform in q and h and

$$\widetilde{B_1}(r) = \sum_{n=0}^{K-1} \beta_{j-1}(r)\log 1/x_j(r),$$

with K being the rank where the 1-continuous fraction expansion of x stops. It follows that

$$\frac{1}{h}\int_r^{r+h}\mathfrak{B}(t)\,dt = \frac{\log(e/q^2|h|)}{q} + \widetilde{B_1}(r) + (\mathfrak{B} - B_1)(r) + O\Big(h^{1/2} + qh\log\frac{1}{q^2|h|}\Big). \quad (6.1)$$

Following Jaffard and Martin's approach in [48, §2.1], we can consider the Haar wavelet (see Section 6.1.5):

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

and deduce from (6.1) applied to h and $h/2$ that

$$\frac{1}{h}\int_{\mathbb{R}}\mathfrak{B}(x)\psi\Big(\frac{x-r}{h}\Big)dx = \frac{\log 2}{q} + O\Big(h^{1/2} + qh\log\Big(\frac{1}{q^2h}\Big)\Big).$$

Now, following the proof of [48, Proposition 1], we can show that there exists $A > 0$ such that θ defined by $\theta(\rho) = A\rho^{3/2}$ is not a 1-modulus of continuity of $\mathfrak{B}$ (cf. Definition 6.1.15). In particular, this proves that the 1-exponent of $\mathfrak{B}$ satisfies $h^{(1)}_{\mathfrak{B}}(x) \le 1/2$ for all $x \in \mathbb{R}\setminus\mathbb{Q}$.

We thus obtain Theorem 6.0.1.

6.1.2 Preliminary estimates on $\mathfrak{B}$

Results here hold for $\alpha \in [1/2, 1]$.

Proposition 6.1.1. *For $k \in \mathbb{N}_0$, there exists a constant $C > 0$ independent of α and k such that*

$$\gamma_k \le C\frac{\log q_{k+1}}{q_k}.$$

Proof. Remark that since $x \in (0, \alpha)$, $x_0 = x$, $\varepsilon_0 = 1$, $p_0 = 0$, $q_0 = 1$. If $k = 0$, then

$$\gamma_0 - \frac{\log q_1}{q_0} = \log(\frac{a_1 + \varepsilon_1 x_1}{a_1}) \le \log(1 + \alpha).$$

For all k, one has

$$\gamma_k = \beta_{k-1}\log(\frac{\beta_{k-1}}{\beta_k}) \le \frac{1}{\alpha q_k}\log(\frac{(1+\alpha)q_{k+1}}{\alpha q_k}) \le C\frac{\log q_{k+1}}{q_k}.$$

□

In the next proposition, we use the existence of the invariant measure ρ_α to derive a general estimate for $\int_I \log^p \frac{1}{x_k(t)}\,dt$ on any interval $I \subseteq (0, \alpha)$.

Proposition 6.1.2. *Let $p \geq 1$ and I be an interval of length $h \leq e^{-2p}$ included in $(0, \alpha)$. For $k \in \mathbb{N}_0$, we have*

$$\int_I \log^p \frac{1}{x_k(t)} \, dt \leq \frac{c_0^2}{2} e p^p h \log^p \frac{1}{h}.$$

Proof. For $q \geq 2$ and $r = q/(q-1)$, Hölder's inequality gives

$$\int_I \log^p \frac{1}{x_k(t)} \, dt \leq h^{1/r} \Big(\int_0^\alpha \log^{qp} \frac{1}{x_k(t)} \, dt \Big)^{1/q}.$$

Using the properties of A_α, we obtain

$$\begin{aligned}\int_0^\alpha \log^{qp}(1/x_k(t)) \, dt &\leq c_0 \int_0^\alpha \log^{qp}(1/x_k(t)) \, m_\alpha(dt) \\ &= c_0 \int_0^\alpha \log^{qp}(1/t) \, m_\alpha(dt) \\ &\leq c_0^2 \int_0^1 \log^{qp}(1/t) \, dt = c_0^2 \Gamma(qp+1).\end{aligned}$$

For $s \in [2, \infty)$, we have $2\Gamma(s+1) \leq s^s$ (cf. [9, Lemme 1]). Therefore, if $qp \geq 2$ we have

$$\int_I \log^p \frac{1}{x_k(t)} \, dt \leq \frac{c_0^2}{2} h^{1/r} (qp)^{(qp)/q} = \frac{c_0^2}{2} h^{1-1/q} q^p p^p.$$

For $q = \log 1/h$, we have $q \geq 2$, then

$$\int_I \log^p \frac{1}{x_k(t)} \, dt \leq \frac{c_0^2}{2} e p^p h \log^p \frac{1}{h},$$

as desired. □

Let us denote by ω_α the modulus of continuity of the function Ψ_α defined in (5.13), that is

$$\omega_\alpha(h) = \sup\{|\Psi_\alpha(t) - \Psi_\alpha(s)| : 0 \leq t, s \leq \alpha, \, |t-s| \leq h\}.$$

Proposition 6.1.3. *For $h \in (0, e^{-2}]$, we have*

$$\omega_\alpha(h) \leq 10 h \log \frac{1}{h}.$$

Proof. Using Proposition 6.1.2 and bounds (5.8) and (5.9) we get

$$\begin{aligned}\Psi_\alpha(t+h) - \Psi_\alpha(t) = \int_t^{t+h} B_\alpha(s) \, ds &= \sum_{k \geq 0} \int_t^{t+h} \gamma_k(s) \, ds \\ &\leq \alpha \sum_{k \geq 0} \min(1, g^{k-1}) \int_t^{t+h} \log \frac{1}{x_k(s)} \, ds \\ &\leq 3.62 e h \log \frac{1}{h},\end{aligned}$$

which is sufficient to conclude. □

6.1.3 Average behavior of $\mathfrak{B}$ at irrationals

In [9], the authors describe the average behaviour of the Brjuno function B in the neighbourhood of any given point in the unit interval. Particularly, they establish that the Lebesgue set of B is the set of Brjuno numbers and elucidate the asymptotic behavior of the modulus of continuity of the integral of B. In this Section, we transpose these findings to the context of the Brjuno-Yoccoz function $\mathfrak{B}$. In particular, we show that the Lebesgue set of $\mathfrak{B}$ is the set of Brjuno numbers.

Unless it is explicitly stated, results here hold for $\alpha = 1/2$, since we explicitly use the characterization of cells of Section 5.4 (adapting this Section for advantageous α should be possible).

Let us recall that on every cell $\mathfrak{c}_n = \mathfrak{c}[(a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n)]$ of depth n, $(a_k(x), \varepsilon_k(x)) = (a_k, \varepsilon_k)$, $(p_k(x), q_k(x)) = (p_k, q_k)$ for all $1 \le k \le n$. It follows that for $t \in \mathfrak{c}_n$, one has

$$x_n(t) = -\varepsilon_n \frac{q_n t - p_n}{q_{n-1} t - p_{n-1}} \quad \text{and} \quad \beta_n(t) = s_n(q_n t - p_n),$$

which implies

$$x_n'(t) = \frac{\varepsilon_n (q_n p_{n-1} - q_{n-1} p_n)}{(q_{n-1} t - p_{n-1})^2} = \frac{s_n}{\beta_{n-1}^2(t)},$$

and $\beta_n'(t) = s_n q_n$. Therefore, we get

$$\begin{aligned} \gamma_n'(t) &= \beta_{n-1}'(t) \log \frac{1}{x_n(t)} - \frac{\beta_{n-1}(t) x_n'(t)}{x_n(t)} \\ &= s_{n-1} q_{n-1} \log \frac{1}{x_n(t)} - \frac{s_n}{x_n(t) \beta_{n-1}(t)} \\ &= s_{n-1} q_{n-1} \log \frac{1}{x_n(t)} - \frac{s_n}{\beta_n(t)}. \end{aligned}$$

For $x \in X$ and $h \in (0, e^{-2})$ such that $x - h/2$ and $x + h/2$ both belong to X, we set $I_x = (x - h/2, x + h/2)$ and define $K = K(I_x)$ as the largest integer such that

$$I_x \subset \mathfrak{c}[(a_1, \varepsilon_1) \ldots (a_K, \varepsilon_K)].$$

Let us select $u, v \in (2, \infty)$ with $u < v$ such that

$$x \pm h/2 \in \left\{ \frac{u p_K + \varepsilon_K p_{K-1}}{u q_K + \varepsilon_K q_{K-1}}, \frac{v p_K + \varepsilon_K p_{K-1}}{v q_K + \varepsilon_K q_{K-1}} \right\}.$$

We define m and n as the integer parts of $u + \frac{1}{2}$ and $v + \frac{1}{2}$, respectively. It follows that, due to the maximality of K,

$$2 \le m \le a_{K+1}(x) \le n.$$

Moreover, we can express h as follows:

$$\begin{aligned} h &= |\frac{up_K + \varepsilon_K p_{K-1}}{uq_K + \varepsilon_K q_{K-1}} - \frac{vp_K + \varepsilon_K p_{K-1}}{vq_K + \varepsilon_K q_{K-1}}| \\ &= \frac{(v-u)}{(uq_K + \varepsilon_K q_{K-1})(vq_K + \varepsilon_K q_{K-1})} \\ &\geq \frac{v-u}{q_K^2 (u+1)(v+1)} \geq \frac{v-u}{6q_K^2 mn}, \end{aligned}$$

where the last inequality arises from the relations $u+1 \leq 3m$ and $v+1 \leq 2n$.

Proposition 6.1.4. *We have*

$$\|\gamma_K\|_{L^p(I_x)} \leq \frac{c_0^2 e(1+\alpha)}{\alpha^3} p h^{1/p} \gamma_K(x) + p \frac{c_0^2 h^{1/p}}{\alpha^3 q_K}(6 \log q_K + 4).$$

Proof. Since

$$\beta_{K-1} \leq \frac{1}{\alpha q_K},$$

we get

$$\begin{aligned} \|\gamma_K\|_{L^p(I_x)} &= (\int_{I_x} (\beta_{K-1}^3(t) \beta_{K-1}^{-2}(t) \log \frac{1}{x_k(t)})^p \, dt)^{1/p} \\ &\leq (\alpha q_K)^{-3} (\int_{I_x} (\beta_{K-1}^{-2}(t) \log \frac{1}{x_K(t)})^p \, dt)^{1/p}. \end{aligned}$$

Since $x_K'(t) = s_K \beta_{K-1}^{-2}(t)$, we get

$$\|\gamma_K(t)\|_{L^p(I_x)} \leq (\alpha q_K)^{-3} (\int_{x_K(I_x)} \log^p \frac{1}{u} \, du)^{1/p}.$$

Now, the endpoints of the interval $x_K(I_x)$ are $x_K(x - h/2)$ and $x_K(x + h/2)$. Since

$$\beta_{K-1} \geq \frac{1}{q_K + q_{K-1}},$$

we get

$$|x_K(x \pm \frac{h}{2}) - x_K(x)| \leq \frac{h}{2} \sup_{\xi \in I} \beta_{K-1}^{-2}(\xi) \leq \frac{h}{2}(q_K + q_{K-1})^2 \leq 2\, q_K^2 h.$$

We must distinguish two cases. If $4q_K^2 h \leq x_K(x)$, for all $u \in x_K(I_x)$, we have

$$\frac{1}{u} \leq \frac{2}{x_K(x)}$$

and thus

$$\begin{aligned}
\|\gamma_k(t)\|_{L^p(I_x)} &\leq (\alpha q_K)^{-3} \log \frac{2}{x_K(x)} (\int_{x_K(I_x)} du)^{1/p} \\
&\leq (\alpha q_K)^{-3} \log \frac{2}{x_K(x)} 4 q_K^2 h^{1/p} \\
&\leq 4 \log \frac{1}{x_K(x)} h^{1/p} \frac{1+\alpha}{\alpha^3} \beta_{K-1}(x) + \frac{4h^{1/p}}{\alpha^3 q_K} \log 2 \\
&\leq 4 \frac{1+\alpha}{\alpha^3} h^{1/p} \gamma_K(x) + \frac{4h^{1/p}}{\alpha^3 q_K}.
\end{aligned}$$

On the other hand, if $4q_K^2 h \geq x_K(x)$, using Proposition 6.1.2, one has

$$\begin{aligned}
\|\gamma_K\|_{L^p(I_x)} &\leq \sup_{t\in I} \beta_{K-1}(t) (\frac{c_0^2 e}{2})^{1/p} p h^{1/p} \log \frac{1}{h} \\
&\leq \frac{c_0^2 e}{\alpha q_K} p h^{1/p} \log \frac{4q_K^2}{x_K(x)} \\
&= \frac{c_0^2 e}{\alpha q_K} p h^{1/p} \log \frac{1}{x_K(x)} + \frac{c_0^2 p h^{1/p}}{\alpha q_K} (2e \log 2 + 2e \log q_K) \\
&\leq c_0^2 e p \frac{1+\alpha}{\alpha} h^{1/p} \gamma_K(x) + \frac{c_0^2 p h^{1/p}}{\alpha q_K} (6 \log q_K + 4),
\end{aligned}$$

which leads to the conclusion. □

Lemma 6.1.5. *Let $p \geq 1$ and $m, n \in \mathbb{N}$ be such that $1 \leq m < n$; we have*

$$\sum_{l=m}^{n} \frac{1}{l^{p+2}} \leq 3 \frac{n-m}{m^{p+1} n}.$$

Proof. We directly obtain

$$\begin{aligned}
\sum_{l=m}^{n} \frac{1}{l^{p+2}} \leq \frac{1}{m^{p+2}} + \int_m^n \frac{dt}{t^{p+2}} &= \frac{1}{m^{p+2}} + \frac{1}{p+1} (\frac{1}{m^{p+1}} - \frac{1}{n^{p+1}}) \\
&= \frac{1}{m^{p+2}} + \frac{n^{p+1} - m^{p+1}}{(p+1) m^{p+1} n^{p+1}} \\
&= \frac{1}{m^{p+2}} + \frac{(n-m) \sum_{j=0}^{p} n^{p-j} m^j}{(p+1) m^{p+1} n^{p+1}} \\
&\leq \frac{1}{m^{p+2}} + \frac{n-m}{m^{p+1} n}.
\end{aligned}$$

Since $1 \leq m \leq n-1$, we have

$$\frac{1}{m^{p+2}} \leq 2 \frac{n-m}{m^{p+1} n},$$

which leads to the conclusion. □

Proposition 6.1.6. *For $k > K$, we have the following relations:*

- *if $n = m$ or $n = m + 1$, then*

$$\|\gamma_k\|_{L^p(I_x)} \leq \frac{C}{q_{K+1}F_{k-K}} h^{1/p} \log \frac{1}{h},$$

- *if $n \geq m + 2$, then*

$$\|\gamma_k\|_{L^p(I_x)} \leq C \frac{(n-m)^{1/p}}{q_K^{1+2/p} F_{k-K}\,(m-1)^{1+1/p}(n-1)^{1/p}}.$$

Proof. First, remark that using relation (5.6), we obtain

$$\beta_{j_1+j_2} \leq \frac{1}{\alpha^2\, q_{j_1+1} F_{j_2+1}}, \tag{6.2}$$

which is valid for all $j_1 \in \mathbb{N}_0 \cup \{-1\}$ and $j_2 \in \mathbb{N}_0$.
Let us deal with the case $n = m$. Then, one has

$$x \pm \frac{h}{2} \in \left\{\overline{\mathfrak{c}_{(m,-1)}}, \overline{\mathfrak{c}_{(m,1)}}\right\}.$$

For $t \in I_x \setminus \partial(\overline{\mathfrak{c}_{(m,-1)}} \cup \overline{\mathfrak{c}_{(m,1)}})$, one has

$$q_{K+1}(t) = m q_K + \varepsilon_K q_{K-1}.$$

Therefore, we get

$$\begin{aligned}
\|\gamma_k\|_{L^p(I_x)} &= \Big(\int_{I_x} (\beta_{k-1}(t) \log \frac{1}{x_k(t)})^p\, dt\Big)^{1/p} \\
&= \Big(\int_{I_x} (\frac{q_{K+1}(t)}{q_{K+1}(x)} \beta_{k-1}(t) \log \frac{1}{x_k(t)})^p\, dt\Big)^{1/p} \\
&\leq \frac{1}{\alpha^2}\Big(\int_{I_x} (\frac{q_{K+1}(t)}{q_{K+1}(x)} \frac{1}{q_{K+1}(t) F_{k-K}} \log \frac{1}{x_k(t)})^p\, dt\Big)^{1/p} \\
&\leq \frac{C_1}{q_{K+1}(x) F_{k-K}} h^{1/p} \log \frac{1}{h},
\end{aligned}$$

where the last inequality results from Proposition 6.1.2.
For the other cases, by the definition of m and n, we have

$$x \pm \frac{h}{2} \in \left\{\overline{\mathfrak{c}_{(m,\varepsilon_1)}}, \overline{\mathfrak{c}_{(n,\varepsilon_2)}}\right\}.$$

Moreover, since I_x is included in the union of the sets $\overline{\mathfrak{c}_{(l,\varepsilon)}}$ for $m \leq l \leq n$, $\varepsilon \in \{\pm 1\}$, we get

$$\|\gamma_k\|_{L^p(I_x)} \leq \Big(\sum_{\substack{m \leq l \leq n \\ \varepsilon \in \{\pm 1\}}} \int_{\mathfrak{c}_{(l,\varepsilon)}} \beta_{k-1}^p(t) \log^p \frac{1}{x_k(t)}\, dt\Big)^{1/p}. \tag{6.3}$$

If $n-m=1$, since $I_x \subset \overline{\mathfrak{c}_{(m,1)}} \cup \overline{\mathfrak{c}_{(m+1,-1)}}$, for $t \in I_x \setminus \partial\mathfrak{c}_{(m,\varepsilon)}$, we get

$$q_{K+1}(t) \in \{mq_K + \varepsilon_K q_{K-1}, (m+1)q_K + \varepsilon_K q_{K-1}\}.$$

In particular, for $t \in I_x \setminus \partial(\overline{\mathfrak{c}_{(m,1)}} \cup \overline{\mathfrak{c}_{(m+1,-1)}})$, we have

$$\frac{q_{K+1}(t)}{q_{K+1}(x)} \geq \frac{mq_K + \varepsilon_K q_{K-1}}{(m+1)q_K + \varepsilon_K q_{K-1}} \geq \frac{1}{2}.$$

Consequently, we get

$$\begin{aligned}
\|\gamma_k\|_{L^p(I_x)} &= 2\Big(\int_{I_x} \big(\frac{1}{2}\beta_{k-1}(t)\log\frac{1}{x_k(t)}\big)^p\,dt\Big)^{1/p} \\
&\leq 2\Big(\int_{I_x} \big(\frac{q_{K+1}(t)}{q_{K+1}(x)}\beta_{k-1}(t)\log\frac{1}{x_k(t)}\big)^p\,dt\Big)^{1/p} \\
&\leq \frac{2}{\alpha^2}\Big(\int_{I_x} \big(\frac{q_{K+1}(t)}{q_{K+1}(x)}\frac{1}{q_{K+1}(t)F_{k-K}}\log\frac{1}{x_k(t)}\big)^p\,dt\Big)^{1/p} \\
&\leq \frac{C_1}{q_{K+1}(x)F_{k-K}} h^{1/p}\log\frac{1}{h},
\end{aligned}$$

where the last inequality results from Proposition 6.1.2.

If $n-m \geq 2$, using (6.3) and (6.2), we have

$$\begin{aligned}
\|\gamma_k\|_{L^p(I_x)} &\leq \frac{1}{\alpha^2 F_{k-K}}\Big(\sum_{\substack{m\leq l\leq n\\ \varepsilon\in\{\pm1\}}} \int_{\mathfrak{c}_{(l,\varepsilon)}} \frac{\log^p 1/x_k(t)}{q_{K+1}(t)^p}\,dt\Big)^{1/p} \\
&\leq \frac{1}{\alpha^2 F_{k-K} q_K}\Big(\sum_{\substack{m\leq l\leq n\\ \varepsilon\in\{\pm1\}}} \int_{\mathfrak{c}_{(l,\varepsilon)}} \frac{1}{(l-1)^p}\log^p\frac{1}{x_k(t)}\,dt\Big)^{1/p},
\end{aligned}$$

where the second inequality follows from the fact that for all $t \in \mathfrak{c}_{(l,\varepsilon)}$,

$$q_{K+1}(t) = lq_K + \varepsilon_K q_{K-1} \geq (l-1)q_K.$$

Remembering that

$$x'_{K+1} = s_{K+1}(q_{K+1} + \varepsilon_{K+1}x_{K+1}q_K)^2,$$

the change of variable $w = x_{K+1}(t)$ gives

$$\begin{aligned}
\|\gamma_k\|_{L^p(I_x)} &\leq \frac{1}{\alpha^2 q_K F_{k-K}}\Big(\sum_{\substack{m\leq l\leq n\\ \varepsilon\in\{\pm1\}}} \frac{1}{(l-1)^p}\int_0^1 \frac{\log^p 1/x_{k-K-1}(w)}{(lq_K + \varepsilon\, w\, q_K)^2}\,dw\Big)^{1/p} \\
&\leq \frac{C}{\alpha^2 q_K^{1+2/p} F_{k-K}}\Big(\int_0^1 \log^p\frac{1}{x_{k-K-1}(w)}\,dw \sum_{l=m}^{n}\frac{1}{(l-1)^{p+2}}\Big)^{1/p} \\
&\leq \frac{C}{q_K^{1+2/p} F_{k-K}}\Big(\int_0^1 \log^p\frac{1}{x_{k-K-1}(w)} m_\alpha(dw) \sum_{l=m}^{n}\frac{1}{(l-1)^{p+2}}\Big)^{1/p}.
\end{aligned}$$

Using the previous lemma and the invariance of m_α by x_{k-K-1}, we then have

$$\begin{aligned}\|\gamma_k\|_{L^p(I_x)} &\leq \frac{C(n-m)^{1/p}}{q_K^{1+2/p} F_{K-k}(m-1)^{1+1/p}(n-1)^{1/p}} (\int_0^1 \log^p \frac{1}{w} m_\alpha(dw))^{1/p} \\ &\leq C \frac{(n-m)^{1/p}}{q_K^{1+2/p} F_{K-k}(m-1)^{1+1/p}(n-1)^{1/p}},\end{aligned}$$

as desired. □

Lemma 6.1.7. *There exists a positive constant C independent of K such that the following inequalities hold: for all $k < K$,*

$$\|\gamma_k - \gamma_k(x)\|_{L^p(I_x)} \leq C q_{k+1} h^{1+1/p}, \tag{6.4}$$

$$\|\gamma_K - \gamma_K(x)\|_{L^p(I_x)} \leq C q_{K+1} h^{1+1/p} \log q_{K+1} \tag{6.5}$$

and for all $k > K$,

$$\|\gamma_k\|_{L^p(I_x)} \leq C \frac{h^{1/p}}{F_{k-K}} (\frac{\log 1/h}{q_{K+1}} + h^{1/2}). \tag{6.6}$$

Proof. We begin by establishing (6.4). Since γ_k is differentiable for $k \leq K$, we know that we can express its derivative as :

$$\gamma_k' = (-1)^{k-1} \varepsilon_0 \cdots \varepsilon_{k-1} q_{k-1} \log \frac{1}{x_k} + \frac{(-1)^{k-1} \varepsilon_0 \cdots \varepsilon_k}{\beta_k}.$$

Consequently, for $k < K$, we can derive the following inequality:

$$\begin{aligned}|\gamma_k'(t)| &\leq q_{k-1} \log(a_{k+1} + \frac{1}{2}) + (1 + \frac{1}{2}) q_{k+1} \\ &\leq (2 + \frac{1}{2}) q_{k+1} + (\frac{1}{2} - \varepsilon_k) q_{k-1} \\ &\leq 4 q_{k+1}.\end{aligned}$$

Now, considering that we have

$$(\int_{I_x} |t - x|^p \, dt)^{1/p} \leq \frac{h^{1+1/p}}{2},$$

choosing $C = 2$ in (6.4), we establish the conclusion using the mean value Theorem.

Let us prove that (6.5) holds. We need to consider two cases. Suppose that $n \geq 2m+1$; then, we have

$$v - u \geq \frac{n-m}{2}$$

and since

$$\frac{n-m}{mn} \geq \frac{1}{2m},$$

we obtain

$$h \geq \frac{1}{24\, m q_K^2} \geq \frac{1}{24 a_{K+1}(x) q_K^2} \geq \frac{1}{24\, q_K\, q_{K+1}}. \tag{6.7}$$

By Proposition 6.1.4 and Proposition 6.1.1, we know that there exists a positive constant C_1 such that

$$\begin{aligned}\|\gamma_K\|_{L^p(I_x)} &\leq \frac{c_0^2 e(1+\frac{1}{2})}{(\frac{1}{2})^3} p h^{1/p} \gamma_K(x) + p \frac{c_0^2 h^{1/p}}{\frac{1}{8} q_K}(6 \log q_K + 4) \\ &\leq C_1 h^{1/p} \frac{\log q_{K+1}}{q_K}.\end{aligned}$$

By Minkowski inequality, using Proposition 6.1.1 again and (6.7), we get

$$\begin{aligned}\|\gamma_K(t) - \gamma_K(x)\|_{L^p(I_x)} &\leq \|\gamma_K\|_{L^p(I_x)} + h^{1/p}\ \gamma_K(x) \\ &\leq C_2 \frac{h^{1/p}}{q_K} \log q_{K+1} \\ &\leq C_3 h^{1+1/p} q_{K+1} \log q_{K+1}.\end{aligned}$$

Now, let us consider the case where $m \leq n \leq 2m$. For $t \in I_x$, we have

$$\gamma_K'(t) = (-1)^{K-1} \varepsilon_0 \cdots \varepsilon_{K-1} q_{K-1}(t) \log \frac{1}{A_{1/2}^K(t)} - s_K \beta_K^{-1}(t)$$

and thus

$$|\gamma_K'(t)| \leq C_4 q_{K+1}(t).$$

Moreover, for $t \in I_x$, there exists $l \in [m, n]$ such that t belongs to the cell

$$\mathfrak{c}[(a_1, \varepsilon_1) \ldots (a_K, \varepsilon_K)(l, \varepsilon)]$$

and thus such that

$$q_{K+1}(t) = l q_K + \varepsilon_K q_{K-1} \leq 2m q_K + \varepsilon_K q_{K-1} \leq 2 q_{K+1}.$$

By the mean value Theorem, we get

$$\begin{aligned}\|\gamma_K - \gamma_K(x)\|_{L^p(I_x)} &\leq 2 C_4 q_{K+1} \Big(\int_{I_x} |x-t|^p dt\Big)^{1/p} \\ &\leq C_4 q_{K+1} h^{1+1/p} \\ &\leq C_5 q_{K+1} h^{1+1/p} \log q_{K+1}.\end{aligned}$$

It remains to demonstrate that inequality (6.6) holds. Let $k > K$; once again, we distinguish two cases. Suppose that $n - m \in \{0, 1\}$. Using Proposition 6.1.6, we directly obtain

$$\|\gamma_k\|_{L^p(I_x)} \leq \frac{C}{q_{K+1} F_{k-K}} h^{1/p} \log \frac{1}{h}.$$

Now, if $n-m \geq 2$, we have

$$v-u \geq \frac{n-m}{2}$$

and thus

$$h \geq \frac{n-m}{12\, q_K^2 mn}.$$

Using Proposition 6.1.6 again, we obtain

$$\|\gamma_k\|_{L^p(I_x)} \leq C \frac{(n-m)^{1/p}}{F_{k-K}\,(m-1)^{1+1/p}(n-1)^{1/p}}\,\frac{1}{q_K^{1+2/p}}.$$

Therefore, we have

$$\begin{aligned}\|\gamma_k(t)^p\|_{L^p(I_x)} &\leq C'\,\frac{(n-m)^{1/p}}{F_{k-K}(m-1)^{1+1/p}(n-1)^{1/p}}\,\frac{h^{1/2+1/p}(mn)^{1/2+1/p}}{(n-m)^{1/2+1/p}}\\ &= C'\frac{h^{1/2+1/p}}{F_{k-K}}\Big(\frac{n}{(n-m)m}\Big)^{1/2}\Big(\frac{m}{m-1}\Big)^{1+1/p}\Big(\frac{n}{n-1}\Big)^{1/p}.\end{aligned}$$

Since for $2 \leq m \leq n$, we have

$$\frac{n}{(n-m)m} \leq 1, \quad \frac{m}{m-1} \leq 2, \quad \frac{n}{n-1} \leq 2,$$

the conclusion follows. □

Lemma 6.1.8. *For $\alpha \in [1/2,1]$, there exists $C>0$ such that*

$$\sum_{j=0}^{k} q_j \leq C q_k.$$

Proof. If $\alpha = 1$, we obtain

$$\sum_{j=0}^{k} q_j \leq \sum_{j=2}^{k}(q_j - q_{j-1}) + q_{k-1} + q_k \leq 3q_k.$$

For $\alpha < 1$, we have

$$\sum_{j=0}^{k} q_j = q_k \sum_{j=0}^{k}\frac{q_j}{q_k} \leq \frac{(1+\alpha)q_k}{\alpha}\sum_{j=0}^{k}\frac{\beta_{k-1}}{\beta_{j-1}} = \frac{(1+\alpha)q_k}{\alpha}\sum_{j=0}^{k} x_{k-1}\cdots x_j \leq \frac{(1+\alpha)q_k}{\alpha}\sum_{j=0}^{k}\alpha^{k-j}.$$

□

Proposition 6.1.9. *The Lebesgue points of $\mathfrak{B}$ are exactly the Brjuno numbers.*

Proof. The necessity of the condition follows from Proposition 6.1.12 and Proposition 6.1.13.

The condition is sufficient. Let x be a Brjuno number; we need to show that

$$\lim_{h\to 0^+} \frac{1}{h}\|\mathfrak{B}-\mathfrak{B}(x)\|_{L^1([x-\frac{h}{2},x+\frac{h}{2}])}=0.$$

By the properties of irrational numbers, we can suppose that $x-h/2$ and $x+h/2$ are irrational numbers. Again, set $I_x=[x-h/2,x+h/2]$ and let K be the greatest integer such that I_x is included in $\mathfrak{c}((a_1,\varepsilon_1),\dots,(a_K,\varepsilon_K))$. We have

$$\|\mathfrak{B}-\mathfrak{B}(x)\|_{L^1(I_x)} \le \sum_{k<K}\|\gamma_k-\gamma_k(x)\|_{L^1(I_x)}+\|\gamma_K\|_{L^1(I_x)}+\sum_{k>K}\|\gamma_k\|_{L^1(I_x)}+h\sum_{k\ge K}\gamma_k(x).$$

Using (6.4) with $p=1$, we get

$$\frac{1}{h}\sum_{k<K}\|\gamma_k-\gamma_k(x)\|_{L^1(I_x)}\le Ch\sum_{k<K}q_{k+1}\le Chq_K\le\frac{C}{q_{K+1}},$$

where the last inequality results from Proposition 5.4.2.

Now, using Proposition 6.1.4 with $p=1$, we get

$$\frac{1}{h}\|\gamma_K\|_{L^1(I_x)}\le C\frac{\log q_{K+1}}{q_K}.$$

Using Proposition 6.1.6 and proceeding as for Lemma 6.1.7, we get

$$\frac{1}{h}\sum_{k>K}\|\gamma_k\|_{L^p(I_x)}\le C(\frac{1}{q_K}+\frac{\log q_{K+2}}{q_{K+1}}+\frac{\log q_{K+3}}{q_{K+2}}).$$

As we also have

$$\sum_{k\ge K}\gamma_k(x)\le\sum_{k\ge K}\frac{\log(2\,q_{k+1})}{q_k},$$

we derive that

$$\frac{1}{h}\|\mathfrak{B}-\mathfrak{B}(x)\|_{L^1(I_x)}\le C\sum_{k\ge K}\frac{\log q_{k+1}+1}{q_k}.$$

But, as x is a Brjuno number, we know that the right-hand side of the last inequality tends to zero as K tends to infinity. □

6.1.4 Average behavior of $\mathfrak{B}$ at rationals

In the sequel, we will abbreviate $A_{1/2}=A$.

Lemma 6.1.10. *For all $x\in[0,1/2]$, we have*

$$\int_0^x \mathfrak{B}(A(t))\,dt=O(x).$$

Proof. Suppose $x \in \overline{\mathfrak{c}(a_1, -1)}$ with $a_1 \geq 3$. Hence,

$$\frac{1}{a_1} \leq x \leq \frac{1}{a_1 - 1/2}.$$

We have

$$\begin{aligned}
\int_0^x \mathfrak{B}(A(t))\,dt &= \sum_{n \geq a_1} \Big(\int_{1/(n+1/2)}^{1/n} \mathfrak{B}\Big(\frac{1}{t} - n\Big)\,dt + \int_{1/n}^{1/(n-1/2)} \mathfrak{B}(n - \frac{1}{t})\,dt \Big) \\
&= \sum_{n \geq a_1} \Big(\int_0^{1/2} \mathfrak{B}(u) \Big(\frac{1}{(n+u)^2} + \frac{1}{(n-u)^2} \Big)\,du \\
&\leq \int_0^{1/2} \mathfrak{B}(u) \sum_{n \geq a_1} \Big(\frac{1}{(n+u)^2} + \frac{1}{(n-u)^2} \Big)\,du \\
&\leq \frac{5}{a_1} \int_0^{1/2} \mathfrak{B}(u)\,du = O(x).
\end{aligned}$$

A similar computation provides the same estimate when $x \in \overline{\mathfrak{c}(a_1, 1)}$ with $a_1 \geq 2$. $\square$

Now we investigate the average behavior of $\mathfrak{B}$ at 0 and 1/2. We abbreviate $\Psi_{1/2} = \Psi$.

Lemma 6.1.11. *For all $x \in (0, 1/2)$, we have*

$$\Psi(x) = x \log \frac{1}{x} + x + O(x^2)$$

and

$$\Psi(1/2) - \Psi(1/2 - x) = \frac{x}{2} \log \frac{1}{x} + O\Big(x^2 \log \frac{1}{x}\Big).$$

Proof. Using the functional equation (5.12) of $\mathfrak{B}$ and Lemma 6.1.10, we obtain

$$\begin{aligned}
\Psi(x) &= \int_0^x \Big(\log \frac{1}{t} + t\mathfrak{B}(A(t)) \Big)\,dt \\
&= x \log \frac{1}{x} + x + O(x^2).
\end{aligned}$$

For the second equality, we may assume $x \in (0, 1/10)$ so that $1/2 - x \in \mathfrak{c}(2, 1) = (2/5, 1/2)$. Utilizing the functional equation again, we get

$$\begin{aligned}
\Psi(1/2) - \Psi(1/2 - x) &= \int_{1/2-x}^{1/2} \Big(t\mathfrak{B}(A(t)) + \log \frac{1}{t} \Big)\,dt \\
&= \int_{1/2-x}^{1/2} \Big(t\mathfrak{B}\Big(\frac{1}{t} - 2\Big) + \log \frac{1}{t} \Big)\,dt.
\end{aligned}$$

We get

$$\begin{aligned}\Psi(1/2)-\Psi(1/2-x) &= \int_0^{4x/(1-2x)} (\frac{\mathfrak{B}(u)}{2+u}+\log(2+u))\frac{du}{(2+u)^2}.\\ &= \int_0^{4x/(1-2x)} \Big(\frac{1}{8}\mathfrak{B}(u)+\frac{1}{4}(\log 2)u+O(u\,\mathfrak{B}(u))+O(u)\Big)\,du\\ &= \frac{1}{2}x\log\frac{1}{x}+O\Big(x^2\log\frac{1}{x}\Big).\end{aligned}$$

But for $x\to 0$, one has

$$\begin{aligned}4x/(1-2x) &= 4x+O(x^2),\\ \log(x+2) &= \log 2+\frac{x}{2}+O(x^2),\\ \frac{1}{2+x} &= \frac{1}{2}\big(1-\frac{x}{2}+O(x^2)\big)=\frac{1}{2}-x/4+O(x^2),\\ \frac{1}{(2+x)^2} &= \frac{1}{4}-x/4+O(x^2).\end{aligned}$$

It follows

$$\begin{aligned}\Big(\log(2+x)+\frac{\mathfrak{B}(x)}{2+x}\Big)\frac{1}{(2+x)^2} &= \Big(\log 2+\frac{x}{2}+O(x^2)+\mathfrak{B}(x)\Big(\frac{1}{2}-x/4+O(x^2)\Big)\Big)\\ &\quad\cdot\Big(\frac{1}{4}-x/4+O(x^2)\Big)\\ &= \Big(\log 2+O(x)+\frac{1}{2}\mathfrak{B}(x)+O(x\mathfrak{B}(x))\Big)\\ &\quad\cdot\Big(\frac{1}{4}+O(x)\Big)\\ &= \frac{1}{4}\log 2+\frac{1}{8}\mathfrak{B}(x)+O(x)+O(x\mathfrak{B}(x)).\end{aligned}$$

Then

$$\begin{aligned}&\int_0^{4x/(1-2x)} (\frac{\mathfrak{B}(u)}{2+u}+\log(2+u))\frac{du}{(2+u)^2}\\ &\quad= \int_0^{4x/(1-2x)} \frac{1}{4}\log 2+\frac{1}{8}\mathfrak{B}(u)+O(u)+O(u\mathfrak{B}(u))\,du\\ &\quad= x\log 2+\Big(\frac{1}{8}+O(x)\Big)\Big(\frac{4x}{1-2x}\log\frac{1-2x}{4x}+\frac{4x}{1-2x}+O(x^2)\Big)+O(x^2).\end{aligned}$$

We have

$$\begin{aligned}\frac{4x}{1-2x}\log\frac{1-2x}{4x} &= -(4x+O(x^2))\log(4x+O(x^2))\\ &= -(4x+O(x^2))\big(\log 4+\log x+O(x)\big)\\ &= 4x\log(1/x)-8(\log 2)x+O(x^2\log(1/x)).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{4x/(1-2x)} & (\frac{\mathfrak{B}(u)}{2+u} + \log(2+u))\frac{du}{(2+u)^2} \\ &= x\log 2 + \Big(\frac{1}{8} + O(x)\Big)\Big(4x\log(1/x) - 8(\log 2)x + O(x^2\log(1/x))\Big) + O(x^2) \\ &= \frac{1}{2}x\log(1/x) + \Big(\log 2 - \log 2\Big)x + O(x^2\log(1/x)) \\ &= \frac{1}{2}x\log(1/x) + O(x^2\log(1/x)).\end{aligned}$$

Finally, we obtain

$$\Psi(1/2) - \Psi(1/2 - x) = \frac{1}{2}x\log\frac{1}{x} + O\Big(x^2\log\frac{1}{x}\Big).$$

□

Proposition 6.1.12. *Let $x = p/q$ be a rational number in its irreducible form. If $|h| < 1/(3q^2)$, then*

$$\frac{1}{h}\int_x^{x+h}\mathfrak{B}(t)\,dt = \frac{\log(e/q^2|h|)}{q} + \widetilde{\mathfrak{B}}(x) + O(qh\log\frac{1}{q^2|h|}),$$

where the O is uniform in q and h and

$$\widetilde{\mathfrak{B}}(x) = -\sum_{n=0}^{N-1}\beta_{j-1}(x)\log x_j,$$

with N being the rank where the continuous fraction expansion of x stops.

Proof. • Suppose that $x \in E_0$, it can be written as

$$x = [(0,1),(b_1,\varepsilon_1),\dots,(b_{K-1},\varepsilon_{K-1}),b_K],$$

with $b_K \geq 3$. Remark that x is an extremity of two cells of depth K:

$$\mathfrak{c} = \mathfrak{c}[(0,1)(b_1,\varepsilon_1)\dots(b_{K-1},\varepsilon_{K-1})(b_K,1)] = \mathcal{I}\Big(r, \frac{2p_K + p_{K-1}}{2q_K + q_{K-1}}\Big)$$

of length

$$\frac{1}{q(2q + q_{K-1})} > \frac{1}{3q^2}$$

and

$$\mathfrak{c}' = \mathfrak{c}[(0,1)(b_1,\varepsilon_1)\dots(b_{K-1},\varepsilon_{K-1})(b_K,-1)] = \mathcal{I}\Big(r, \frac{2p_K - p_{K-1}}{2q_K - q_{K-1}}\Big)$$

of length

$$\frac{1}{q(2q-q_{K-1})} > \frac{1}{2q^2}.$$

Therefore, we have

$$(x, x+h) \subseteq (x - \frac{1}{3q^2}, x + \frac{1}{3q^2}) \setminus \{x\} \subseteq \mathfrak{c} \cup \mathfrak{c}'.$$

Remark that for $t \in \mathfrak{c} \cup \mathfrak{c}'$, $p_K(t) = p_K = p$ and $q_K(t) = q_K = q$. Now, we have

$$\begin{aligned}\Psi_{1/2}(x+h) - \Psi_{1/2}(x) &= \int_x^{x+h} \mathfrak{B}(t)\,dt \\ &= \sum_{k<K} \int_x^{x+h} \gamma_k(t)\,dt + \int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(A^K_{1/2}(t))\,dt.\end{aligned}$$

But, using (6.4) of Lemma 6.1.7 and Lemma 6.1.8, we get

$$\begin{aligned}|\int_x^{x+h} \sum_{k<K} \gamma_k(t)\,dt - h\widetilde{\mathfrak{B}}(x)| &= |\int_x^{x+h} \sum_{k<K} (\gamma_k(t) - \gamma_k(x))\,dt| \\ &\leq C\,h^2 \sum_{k<K} q_{k+1} \\ &\leq Ch^2 q.\end{aligned}$$

It remains to show that

$$\int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(A^K_{1/2}(t))\,dt = h\frac{\log(e/q^2|h|)}{q} + O(q\,h^2 \log\frac{1}{q^2|h|}). \tag{6.8}$$

Let us first consider the case when the interval of extremities x and $x+h$ is in $\bar{\mathfrak{c}}$, which is the case if and only if $s_K h > 0$. We have

$$x_K(t) = -\frac{qt-p}{q_{K-1}t - p_{K-1}}.$$

From there, we can compute the bounds of this last interval. We directly get $x_K(x) = 0$ and

$$\begin{aligned}x_K(x+h) &= \frac{q(p/q+h)-p}{-q_{K-1}(p/q+h)+p_{K-1}} = \frac{q^2h}{-q_{K-1}p - q_{K-1}qh + p_{K-1}q} \\ &= \frac{q^2h}{s_K - hqq_{K-1}} = \frac{q^2|h|}{1-|h|qq_{K-1}}.\end{aligned}$$

Therefore, $y = x_K(x+h)$ belongs to $(0,\alpha)$ and using Lemma 6.1.10 and Lemma 6.1.11, we get

$$\begin{aligned}\int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(x_K(t))\,dt &= \frac{s_K}{q^3}\int_0^y \mathfrak{B}(u)\frac{du}{(1+u\,q_{K-1}/q)^3}\\ &= \frac{s_K}{q^3} y\log\frac{1}{y} + \frac{s_K}{q^3}y + \frac{s_K}{q^3}O(y^2)\\ &\quad + \frac{s_K}{q^3}O(y^2\log\frac{1}{y}).\end{aligned}$$

Let us now consider the case when the interval of extremities x and $x+h$ is in $\bar{\mathfrak{c}}'$, which is the case if and only if $s'_K h > 0$. We have

$$x_K(t) = \frac{qt-p}{q_{K-1}t - p_{K-1}}.$$

Again, we can compute the bounds of the interval. We have $x_K(x) = 0$ and

$$\begin{aligned}x_K(x+h) &= \frac{q(p/q+h)-p}{q_{K-1}(p/q+h)-p_{K-1}} = \frac{q^2h}{q_{K-1}p + q_{K-1}qh - p_{K-1}q}\\ &= \frac{q^2h}{s'_K + hqq_{K-1}} = \frac{q^2|h|}{1+|h|qq_{K-1}}.\end{aligned}$$

Consequently, $y' = x_K(x+h)$ belongs to $(0,\alpha)$ and using Lemma 6.1.10 et Lemma 6.1.11, we get

$$\begin{aligned}\int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(x_K(t))\,dt &= \frac{s'_K}{q^3}\int_0^{y'} \mathfrak{B}(u)\frac{du}{(1+u\,q_{K-1}/q)^3}\\ &= \frac{s'_K}{q^3}\int_0^{y'} \mathfrak{B}(u)(1+O(u))\,du\\ &= \frac{s'_K}{q^3} y'\log\frac{1}{y'} + \frac{s'_K}{q^3}y' + \frac{s'_K}{q^3}O(y'^2)\\ &\quad + \frac{s'_K}{q^3}O(y'^2\log\frac{1}{y'}).\end{aligned}$$

By setting z equal to y or y', we have

$$\int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(x_K(t))\,dt = \frac{\operatorname{sgn} h}{q^3} z\log(\frac{1}{z}) + \frac{\operatorname{sgn} h}{q^3}z + O(q^{-3}z^2\log\frac{2}{z}).$$

Since $q_{K-1} < q$ and $|h| < 1/3\,q^2$,

$$\log\frac{1}{z} = \log\frac{1}{q^2|h|} + O(q^2h)$$

and

$$z = q^2|h| + O(h^2q^4).$$

We thus have established (6.8).

• Suppose that $x \in E_{1/2}$, it can be written as

$$x = [(b_1, \varepsilon_1), \dots, (b_{K-1}, \varepsilon_{K-1}), b_K]$$

with $\varepsilon_{K-1} = -1$ and $b_K = 2$. We also observe that r is an extremity of two cells of depth $K-1$ which are

$$\mathfrak{c} := \mathfrak{c}[(b_0, \varepsilon_0) \dots (b_{K-1}, -1)] = \mathcal{I}\Big(r, \frac{p_{K-1}}{q_{K-1}}\Big)$$

of length

$$\Big|r - \frac{p_{K-1}}{q_{K-1}}\Big| = \frac{1}{q_{K-1}q} > \frac{1}{q^2}$$

and

$$\mathfrak{c}' := \mathfrak{c}[(b_0, \varepsilon_0) \dots (b_{K-1} - 1, 1)] = \mathcal{I}\Big(r, \frac{p_{K-1} - p_{K-2}}{q_{K-1} - q_{K-2}}\Big)$$

of length

$$\Big|r - \frac{p_{K-1} - p_{K-2}}{q_{K-1} - q_{K-2}}\Big| = \frac{1}{q(q_{K-1} - q_{K-2})} > \frac{1}{q^2}.$$

Therefore, we have

$$(x, x+h) \subseteq \Big(r - \frac{1}{q^2}, r + \frac{1}{q^2}\Big) \setminus \{r\} \subseteq \mathfrak{c} \cup \mathfrak{c}'.$$

Now, we have

$$\begin{aligned}\Psi_{1/2}(x+h) - \Psi_{1/2}(x) &= \int_x^{x+h} \mathfrak{B}(t)\,dt \\ &= \sum_{k<K} \int_x^{x+h} \gamma_k(t)\,dt + \int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(A_{1/2}^K(t))\,dt.\end{aligned}$$

But, using (6.4) of Lemma 6.1.7 and Lemma 6.1.8, we get

$$\begin{aligned}\Big|\int_x^{x+h} \sum_{k<K} \gamma_k(t)\,dt - h\widetilde{\mathfrak{B}}(x)\Big| &= \Big|\int_x^{x+h} \sum_{k<K} (\gamma_k(t) - \gamma_k(x))\,dt\Big| \\ &\le C\,h^2 \sum_{k<K} q_{k+1} \\ &\le Ch^2 q.\end{aligned}$$

It remains to show that

$$\int_x^{x+h} \beta_{K-1}(t)\mathfrak{B}(A_{1/2}^K(t))\,dt = h\frac{\log(e/q^2|h|)}{q} + O(q\,h^2 \log \frac{1}{q^2|h|}). \tag{6.9}$$

Remark that cells of depth K in $\mathfrak{c}$ are given by

$$\mathfrak{c}_{(k,\varepsilon)} = \mathcal{I}\Big(\frac{kp_{K-1} + \varepsilon_{K-1}p_{K-2}}{kq_{K-1} + \varepsilon_{K-1}q_{K-2}}, \frac{(k+\varepsilon/2)p_{K-1} + \varepsilon_{K-1}p_{K-2}}{(k+\varepsilon/2)q_{K-1} + \varepsilon_{K-1}q_{K-2}}\Big).$$

Since

$$x = p/q = p_K/q_K = \frac{2p_{K-1} + \varepsilon_{K-1}p_{K-2}}{2q_{K-1} + \varepsilon_{K-1}q_{K-2}},$$

for $\varepsilon \in \{\pm 1\}$, p/q is the endpoint of a cell $\tilde{\mathfrak{c}}_\varepsilon$ of depth K, with the other endpoint given by

$$\frac{(2+\varepsilon/2)p_{K-1} + \varepsilon_{K-1}p_{K-2}}{(2+\varepsilon/2)q_{K-1} + \varepsilon_{K-1}q_{K-2}} = \frac{p + \varepsilon/2p_{K-1}}{q + \varepsilon/2q_{K-1}},$$

of length

$$\frac{1}{q(q+\varepsilon/2q_{K-1})} > \frac{2}{3q^2} > \frac{1}{3q^2},$$

so that $(x, x+h) \subseteq \tilde{\mathfrak{c}}_{-1} \cup \tilde{\mathfrak{c}}_1$.
In $\tilde{\mathfrak{c}}_\varepsilon$, we have

$$x_K(t) = -\varepsilon\frac{qt - p}{q_{K-1}t - p_{K-1}}.$$

From there, we can compute the bounds of this last interval. We directly get $x_K(x) = 0$ and

$$x_K(x+h) = \frac{q^2|h|}{1 - \varepsilon|h|qq_{K-1}}.$$

We get the same conclusion then.

□

Proposition 6.1.13. *If $x \in X$ is a Cremer number, then we have*

$$\lim_{h\to 0^+} \frac{\Psi_\alpha(x+h) - \Psi_\alpha(x)}{h} = \infty.$$

Proof. For $h > 0$ and $k_0 \in \mathbb{N}$, we have

$$\frac{\Psi_\alpha(x+h) - \Psi_\alpha(x)}{h} \geq \frac{1}{h}\int_x^{x+h} \sum_{k\leq k_0} \gamma_k(t)\,dt = \sum_{k\leq k_0} \frac{1}{h}\int_x^{x+h} \gamma_k(t)\,dt.$$

Since $x \in X$, the functions γ_k are continuous at x, and thus,

$$\liminf_{h\to 0^+} \frac{\Psi_\alpha(x+h) - \Psi_\alpha(x)}{h} \geq \sum_{k\leq k_0} \gamma_k(x).$$

The right-hand side tends to infinity as k_0 tends to infinity, since x is a Cremer number. □

6.1.5 Pointwise irregularity of the Brjuno-Yoccoz function

In this Section, we follow a methodology similar to the one proposed in [48] to establish the pointwise irregularity of $\mathfrak{B}$.

We will establish an upper bound for $h^{(1)}_{\mathfrak{B}}$ employing the continuous wavelet transform. We note that the use of the wavelet transform can be avoided by making similar choices for the sequence $(h_n)_n$ as in the proof of Proposition 6.1.17, but applied directly within the average asymptotic formula of Proposition 6.1.12, in analogy with the treatment of the pointwise irregularity of Riemann's function in [22, 33].

By wavelet [31], we mean a function ψ of $\mathbb{R}$ that has a compact support, is bounded by 1 in modulus and such that

$$\int \psi(x)\,dx = 0.$$

We specifically use the Haar wavelet:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

As usual, we set for $a > 0$ and $b \in \mathbb{R}$,

$$\psi_{a,b}(x) = \psi(\frac{x-b}{a})$$

and define

$$W_f(b,a) = \frac{1}{a}\int f(x)\psi_{a,b}(x)\,dx,$$

whenever this expression (seen as a tensor product) makes sense.

Definition 6.1.14. A function $\theta : [0,\infty) \to [0,\infty)$ satisfies hypothesis $(\mathcal{H})$ if it is continuous and non-decreasing in a neighborhood of 0.

Definition 6.1.15. Let θ be a function that satisfies $(\mathcal{H})$ and $f \in L^p_{\text{loc}}(\mathbb{R})$. We say that θ is a p-modulus of continuity of f at x if there exists a polynomial P such that, for all $r > 0$ small enough,

$$\|f - P(\cdot - x)\|_{L^p([x-r,x+r])} \leq \theta(r).$$

In [48], it is proven that for $p \in [1,\infty]$ and $f \in L^p_{\text{loc}}(\mathbb{R})$, if θ is a p-modulus of continuity of f at x for which there exists a positive constant C such that for every $r \in (0,1]$,

$$\theta(r) \geq C\, r^{1+1/p},$$

then we have

$$|W_f(b,a)| \leq C^{1-1/p}\theta(r)\frac{r^{1-1/p}}{a}, \tag{6.10}$$

for any r such that the support of $\psi_{a,b}$ is included in $(x-r, x+r)$.

Lemma 6.1.16. *Let $x = p/q$ be a rational number in its irreducible form; for $|h| < 1/3q^2$, we have*

$$|W_{\mathfrak{B}}(\frac{p}{q}, h) - \frac{\log 2}{q}| \leq Cqh \log \frac{1}{q^2|h|},$$

where the wavelet is the Haar wavelet and where the constant C is independent of q and h.

Proof. Utilizing Proposition 6.1.12 for h and $h/2$, the following expressions are obtained:

$$\begin{aligned} &\frac{2}{h}\int_x^{x+h/2} \mathfrak{B}(t)\,dt - \frac{1}{h}\int_x^{x+h} \mathfrak{B}(t)\,dt \\ &= \frac{\log(2e/q^2|h|)}{q} - \frac{\log(e/q^2|h|)}{q} + O(qh \log \frac{1}{q^2|h|}). \end{aligned}$$

However,

$$\begin{aligned} &\frac{2}{h}\int_x^{x+h/2} \mathfrak{B}(t)\,dt - \frac{1}{h}\int_x^{x+h} \mathfrak{B}(t)\,dt \\ &= \frac{1}{h}\int_x^{x+h/2} \mathfrak{B}(t)\,dt - \frac{1}{h}\int_{x+h/2}^{x+h} \mathfrak{B}(t)\,dt \\ &= \frac{1}{h}\int \mathfrak{B}(t)\psi(\frac{t-x}{h})\,dt, \end{aligned}$$

with ψ the Haar wavelet, leading to the stated conclusion. □

The subsequent result is a straightforward adaptation of the techniques outlined in [48].

Proposition 6.1.17. *Let C be the constant in the previous lemma and choose $\varepsilon \in (0, 1/2]$ such that*

$$C\varepsilon \log \frac{1}{\varepsilon} \leq 10^{-2}.$$

If θ is defined for all $r \geq 0$ as

$$\theta(r) = \frac{\varepsilon}{16} r^{3/2},$$

then θ is not a 1-modulus of continuity of $\mathfrak{B}$ at any irrational x.

Proof. Let x be irrational and set $c_n = p_n(x)/q_n(x)$. For all $n \in \mathbb{N}$, we define

$$h_n = \frac{\varepsilon}{q_n^2}.$$

In view of the previous lemma, we thus have

$$W_B(c_n, h_n) \geq \left(-10^{-2} + \log(2)\right) \frac{1}{q_n} \geq \frac{1}{4\, q_n}.$$

Proceeding by contradiction, let us assume that θ is a 1-modulus of continuity of $\mathfrak{B}$. For all $n \in \mathbb{N}$, we define

$$r_n = |x - c_n| + h_n.$$

Using (6.10), we obtain

$$|W_B(c_n, h_n)| \le C_1^0\, \theta(r_n)\, \frac{r_n^0}{h_n^1} = \frac{\theta(r_n)}{h_n}.$$

Thus, it follows that

$$\frac{1}{4\, q_n} \le \frac{\theta(r_n)}{h_n}.$$

Since we have $r_n \le 2/q_n^2$, it follows that

$$\frac{1}{4\, q_n} \le \frac{\theta(2/q_n^2)}{h_n} = \frac{\varepsilon}{16}\, \frac{2\sqrt{2}}{q_n^3}\, \frac{q_n^2}{\varepsilon},$$

which leads to a contradiction. □

Given the preceding proposition and considering that the regularity $T_1^u(x)$ corresponds to $\theta(r) = Cr^{u+1}$, it immediately follows that for all $x \in \mathbb{R}$, the 1-exponent of $\mathfrak{B}$ at x satisfies

$$h_{\mathfrak{B}}^{(1)}(x) \le \frac{1}{2}. \tag{6.11}$$

Consequently, the polynomials involved in the definition of the spaces $T_1^u(x)$ for $\mathfrak{B}$ will consistently be of degree zero.

To obtain a more precise upper bound for $h_{\mathfrak{B}}^{(1)}(x)$, we require the concept of a well approximable number.

Definition 6.1.18. An irrational number x is $(1/2, \tau)$-well approximable if

$$|x - \frac{p_n}{q_n}| \le \frac{1}{q_n^\tau},$$

for an infinity of natural numbers n.

Once again, we derive this straightforward result from [48]:

Proposition 6.1.19. *Let $\tau > 2$; if x is $(1/2, \tau)$-well approximable, then the 1-exponent of $\mathfrak{B}$ at x satisfies*

$$h_{\mathfrak{B}}^{(1)}(x) \le \frac{1}{\tau}. \tag{6.12}$$

Proof. We denote by $c_n = p_n/q_n$ the sequence of convergents of x. Let C_0 be the constant from the previous lemma. For all $n \in \mathbb{N}$, we define

$$h_n = \frac{\varepsilon}{q_n^\tau}.$$

Since $\tau > 2$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$q_n^2\, h_n \log\left(q_n^{-2} h_n^{-1}\right) \le \frac{10^{-2}}{C_0}.$$

By the previous lemma, we thus obtain for $n \geq N$,

$$\left| W_B(c_n, h_n) - \frac{\log 2}{q_n} \right| \leq C_0 \, q_n \, h_n \log \left(q_n^{-2} h_n^{-1} \right) \leq \frac{10^{-2}}{q_n},$$

and therefore,

$$W_B(c_n, h_n) \geq \frac{1}{4 \, q_n}.$$

Proceeding by contradiction, let us assume that $\theta(r) = \frac{1}{16} r^{1+1/\tau}$ is a modulus of continuity of $\mathfrak{B}$ at x. We define $r_n = |x - c_n| + q_n^{-\tau}$. Using (6.10), we have

$$|W_B(c_n, h_n)| \leq \frac{\theta(r_n)}{h_n}.$$

Since we have $r_n \leq 2/q_n^{\tau}$, it follows that

$$\frac{\theta(r_n)}{h_n} \leq \frac{2^{1+1/\tau} q_n^{-\tau(1+1/\tau)}}{q_n^{-\tau}} = \frac{2^{1+1/\tau}}{q_n}.$$

We thus obtain

$$\frac{1}{4} \leq \frac{2^{1+1/\tau}}{16},$$

which leads to a contradiction. □

Proposition 6.1.20. *For any irrational number x, then the 1-exponent of $\mathfrak{B}$ at x satisfies*

$$h_{\mathfrak{B}}^{(1)}(x) \leq \frac{1}{\tau(x)}.$$

Proof. We distinguish three cases. If $\tau^{(1/2)}(x) = 2$, this is a direct consequence of (6.11).

If $\tau^{(1/2)}(x)$ belongs to $(2, \infty)$, then for all $\varepsilon > 0$ small enough, x is $(1/2, \tau^{(1/2)}(x) - \varepsilon)$-well approximable, and the conclusion follows from (6.12).

If $\tau^{(1/2)}(x) = \infty$, then x is $(1/2, \tau)$-well approximable for all $\tau > 2$ and thus $h_{\mathfrak{B}}^{(1)}(x) \leq 0$, in view of (6.12). □

6.1.6 Pointwise regularity of the Bruno-Yoccoz function

We finally consider the pointwise regularity and the multifractal spectrum of the Brjuno-Yoccoz function $\mathfrak{B}$.

Proposition 6.1.21. *Given $p \in [1, \infty)$ and $\varepsilon > 0$, let $x \in X$ be diophantine. There exists $h_0 = h_0(x_0, \varepsilon) > 0$ such that*

$$h^{-1/p} \| \mathfrak{B}(x) - \mathfrak{B} \|_{L^p([x - \frac{h}{2}, x + \frac{h}{2}])} \leq C'_x h^{1/(\tau(x) + \varepsilon)} \log \frac{1}{h},$$

for $h \in (0, h_0)$, where the constant C'_x depends on x. In particular, the p-exponent of $\mathfrak{B}$ at x satisfies

$$h_{\mathfrak{B}}^{(p)}(x) \geq \frac{1}{\tau(x)}.$$

Proof. By the properties of irrational numbers, we can assume that $x - h/2$ and $x + h/2$ are not rational numbers. Moreover, since x is diophantine, there exists an integer K_0 such that for all $l \geq K_0$,

$$\tau_l^{(1/2)}(x) \leq \tau(x) + \varepsilon.$$

Let K be the greatest integer such that

$$I = (x - \frac{h}{2}, x + \frac{h}{2})$$

is included in $\mathfrak{c}((a_1, \varepsilon_1), \dots, (a_K, \varepsilon_K))$. We have

$$\frac{1}{2q_{K+2}q_{K+3}} \leq (1 + \frac{1}{2})\delta_{K+1} < (1 + \frac{1}{2})\frac{h}{2},$$

which implies that K tends to ∞ as h tends to 0.

Let $h_0 \in (0, e^{-2})$ be such that for all $h \in (0, h_0)$, we have $K \geq \max\{K_0, 1\}$. For $h < h_0$, we obtain

$$\begin{aligned} \|\mathfrak{B}(x) - \mathfrak{B}\|_{L^p(I)} &\leq \sum_{k \leq K} \|\gamma_k(x) - \gamma_k\|_{L^p(I)} + \sum_{k > K} \|\gamma_k(x) - \gamma_k\|_{L^p(I)} \\ &\leq \sum_{k \leq K} \|\gamma_k(x) - \gamma_k\|_{L^p(I)} + \sum_{k > K} \|\gamma_k\|_{L^p(I)} + h^{1/p} \sum_{k > K} \gamma_k(x). \end{aligned}$$

Since the sequence $(q_n)_{n \in \mathbb{N}}$ grows at least exponentially, and using both (6.4) and (6.5), we get

$$\begin{aligned} \sum_{k \leq K} \|\gamma_k(x) - \gamma_k\|_{L^p(I)} &\leq C h^{1+1/p} \; (\sum_{k < K} q_{k+1} + q_{K+1} \log q_{K+1}) \\ &\leq C' h^{1+1/p} q_{K+1} \log q_{K+1}. \end{aligned}$$

Now, using (6.6) and the exponential growth of the Fibonacci sequence $(F_k)_{k \in \mathbb{N}}$, we also have

$$\begin{aligned} \sum_{k > K} \|\gamma_k(t)\|_{L^p(I)} &\leq C h^{1/p} (\frac{\log 1/h}{q_{K+1}} + h^{1/2}) \sum_{k > K} \frac{1}{F_{k-K}} \\ &\leq C' h^{1/p} (\frac{\log 1/h}{q_{K+1}} + h^{1/2}). \end{aligned}$$

Since $\tau(x) < \infty$, the sequence $(\tau_n^{(1/2)}(x))_{n \in \mathbb{N}}$ is bounded. Using Proposition 6.1.9, we get

$$\gamma_k(x) \leq \frac{\log q_{k+1}}{q_k} \leq \frac{\log q_k^{\tau_k^{(1/2)}(x) - 1}}{q_k} \leq C_1 \frac{\log q_k}{q_k},$$

for $k > K$, where $C_1 > 0$ is a constant that depends on x. Since x is diophantine, we have

$$\sum_{k > K} \gamma_k(x) \leq C_2 \frac{\log q_{K+1}}{q_{K+1}}.$$

Moreover, since

$$q_{K+1} \le q_K^{\tau_K^{(1/2)}(x)-1} = |x - \frac{p_K}{q_K}|^{\frac{1-\tau_K^{(1/2)}(x)}{\tau_K^{(1/2)}(x)}} \le h^{-1+\frac{1}{\tau_K^{(1/2)}}(x)},$$

we can write

$$\log q_{K+1} \le C_3 \log \frac{1}{h}.$$

To bound q_{K+1}^{-1}, we need to distinguish two cases. If $a_{K+2} \ge 2$, using Proposition 5.4.2, we get

$$\frac{1}{q_{K+1}^{\tau_{K+1}(x)}} = |x - \frac{p_{K+1}}{q_{K+1}}| \le \frac{1}{q_{K+1}q_{K+2}} \le (1+\frac{1}{2})\delta_{K+1} < h$$

and thus

$$\frac{1}{q_{K+1}} \le h^{1/\tau_{K+1}^{(1/2)}(x)}.$$

If $a_{K+2} = 1$, we have

$$\frac{1}{q_{K+1}} \le \frac{2}{q_{K+1} + q_K} \le \frac{2}{q_{K+1} + \varepsilon_{K+1} q_K} = \frac{2}{q_{K+2}}.$$

Now, using Proposition 5.4.2 again, we know that

$$\frac{1}{q_{K+2}^{\tau_{K+2}^{(1/2)}(x)}} = |x - \frac{p_{K+2}}{q_{K+2}}| \le \frac{1}{q_{K+2}q_{K+3}} \le (1+\frac{1}{2})\delta_{K+1} < h,$$

hence

$$\frac{1}{q_{K+1}} \le 2\, h^{1/\tau_{K+2}^{(1/2)}(x)}.$$

Therefore, using the fact that for all $l \ge K_0$, we have

$$2 \le \tau_l^{(1/2)}(x) \le \tau(x) + \varepsilon,$$

we get

$$\begin{aligned}
\|\mathfrak{B}(x) - \mathfrak{B}\|_{L^p(I)} &\le C h^{1/p} (h q_{K+1} \log q_{K+1} + \frac{\log 1/h}{q_{K+1}} + h^{1/2} + \frac{\log q_{K+1}}{q_{K+1}}) \\
&\le C' h^{1/p} (h^{1/\tau_K^{(1/2)}(x)} + h^{1/\tau_{K+1}^{(1/2)}(x)} + h^{1/\tau_{K+2}^{(1/2)}(x)}) \log \frac{1}{h} \\
&\le C'' h^{1/p} h^{1/(\tau(x)+\varepsilon)} \log \frac{1}{h},
\end{aligned}$$

which concludes the proof. □

Using Proposition 6.1.3, for x and h small enough, given any constant D, we get

$$\frac{1}{h}\|\mathfrak{B} - D\|_{L^1(B(x,h/2))} \leq C \log \frac{1}{h}.$$

As a consequence, the 1-exponent of $\mathfrak{B}$ at x satisfies

$$h^{(1)}_{\mathfrak{B}}(x) \geq 0. \tag{6.13}$$

The regularity of $\mathfrak{B}$ at rational numbers is easy to establish: in view of Proposition 6.1.12, if x is a rational number of $(0, 1/2)$, then, for any constant D and h small enough, we have

$$\frac{1}{h}\|\mathfrak{B} - D\|_{L^1(B(x,h/2))} \geq C \log \frac{1}{h},$$

hence

$$h^{(1)}_{\mathfrak{B}}(x) = 0. \tag{6.14}$$

As we know that the Brjuno numbers are exactly the Lebesgue points of $\mathfrak{B}$, we conclude that the Cremer numbers also have a vanishing 1-exponent.

We are now able to prove Theorem 6.0.1.

Proof. We first notice that thanks to Proposition 6.1.2 and proceeding as in the proof of Proposition 6.1.3, for all $x \in [0, 1/2]$, we get

$$h^{(p)}_{\mathfrak{B}}(x) \geq 0.$$

Furthermore, we know that

$$h^{(p)}_{\mathfrak{B}}(x) \leq h^{(1)}_{\mathfrak{B}}(x).$$

If $x \in \mathbb{Q}$, using (6.14), we get

$$0 \leq h^{(p)}_{\mathfrak{B}}(x) \leq h^{(1)}_{\mathfrak{B}}(x) = 0,$$

hence $h^{(p)}_{\mathfrak{B}}(x) = 0$.

If $\tau(x) = \infty$, using (6.13) and Proposition 6.1.20, we get

$$0 \leq h^{(p)}_{\mathfrak{B}}(x) \leq h^{(1)}_{\mathfrak{B}}(x) = 0.$$

If $\tau(x) < \infty$, using Proposition 6.1.20 and Proposition 6.1.21, we get

$$\frac{1}{\tau(x)} \leq h^{(p)}_{\mathfrak{B}}(x) \leq h^{(1)}_{\mathfrak{B}}(x) \leq \frac{1}{\tau(x)}.$$

Consequently, we can conclude. □

Remark 6.1.22. We can interpret this result as follows: the more slowly the sequence $q_n(x)$ grows, the more regular the function $\mathfrak{B}$ is at x.

Let us now compute the p-spectrum of $\mathfrak{B}$. From Jarnik's Theorem [36, 48], we have

$$\dim\{x : \tau(x) = h\} = \frac{2}{h},$$

for $h \geq 2$. As a consequence, we have the following result:

Corollary 6.1.23. *Let $p \in [1, \infty)$; the p-spectrum of $\mathfrak{B}$ is given by*

$$D^{(p)}_{\mathfrak{B}}(h) = \begin{cases} 2h & \text{if } h \in [0, 1/2], \\ -\infty & \text{otherwise.} \end{cases}$$

6.1.7 Concluding remarks

We have shown that the regularity of $\mathfrak{B}$ is identical to that of B. It is therefore natural to inquire about the general regularity of B_α. Firstly, the description of cells related to the metrical theory for the α-continued fractions is more challenging in general for non-advantageous numbers (as remarked in Chapter 5). In summary, a natural question arises: Do we have $h^{(1)}_{B_\alpha}(x) = 1/\tau(x)$ for all $\alpha \in (1/2, 1)$? It is conceivable that B and $\mathfrak{B}$ represent the simplest cases in terms of regularity. For example, we do not have results concerning the regularity of $B - B_\alpha$ in general.

6.2 Other generalizations of B

In this Section, we propose further generalizations of the Brjuno function. The first builds upon the approach introduced in [14].

Definition 6.2.1. Let $\theta > 0$; the generalized Brjuno function B_θ is defined as

$$B_\theta : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} \beta_{n-1}(x)^\theta \log x_n.$$

One of the reasons for generalizing the Brjuno function in this manner is that it satisfies a new functional equation depending on θ: for all $x \in (0, 1]$,

$$B_\theta(x) = -\log x + x^\theta B_\theta(\frac{1}{x}) = -\log x + x^\theta B_\theta(A_1(x)), \tag{6.15}$$

It can also be proved (see [14]) that there exists a constant $C > 0$ such that for all $x \in \mathbb{R}\setminus\mathbb{Q}$, we have

$$|B_\theta(x) - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n^\theta}| \leq C.$$

By adapting the arguments from [48], the following result can be established with ease.

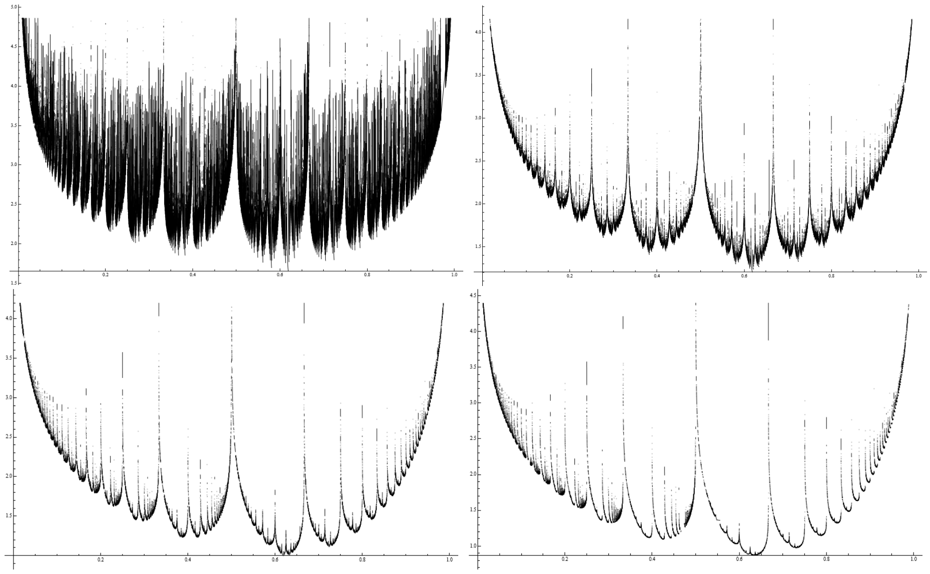


Figure 6.2: The θ-Brjuno functions with resp. $\theta = 0.5$, $\theta = 1$, $\theta = 1.5$ and $\theta = 2$.

Theorem 6.2.2. *Let $\theta \in [0, 2]$, $p \in [1, \infty)$; the p-exponents of B_θ are given by*

$$h^{(p)}_{B_\theta}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ \theta/\tau(x) & \text{otherwise.} \end{cases}$$

Corollary 6.2.3. *Let $\theta \in (0, 2]$, $p \in [1, \infty)$; the p-spectrum of B_θ is given by*

$$D^{(p)}_{B_\theta}(h) = \begin{cases} \frac{2h}{\theta} & \text{if } h \in [0, \theta/2], \\ -\infty & \text{otherwise.} \end{cases}$$

One can also deal with the Boyd functions : let $\phi \in \mathcal{B}$, we set

$$B_\phi : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} \phi(\beta_{n-1}(x)) \log x_n.$$

Nevertheless, we no longer have an exact functional equation but rather an approximate one, leaving the usefulness of this generalization unclear : for all $x \in (0, 1]$,

$$-\log x + \frac{1}{\overline{\phi}(1/x)} B_\phi\left(\frac{1}{x}\right) \le B_\phi(x) \le -\log x + \overline{\phi}(x) B_\phi\left(\frac{1}{x}\right).$$

Another possible generalization is to consider the alternating series associated with the series; this function is referred to as the Wilton function [85, 14, 8].

Definition 6.2.4. Let $\alpha \in [1/2, 1]$; the generalized Wilton function W_α is defined as

$$W_\alpha : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} (-1)^n \beta_{n-1}(x) \log x_n. \tag{6.16}$$

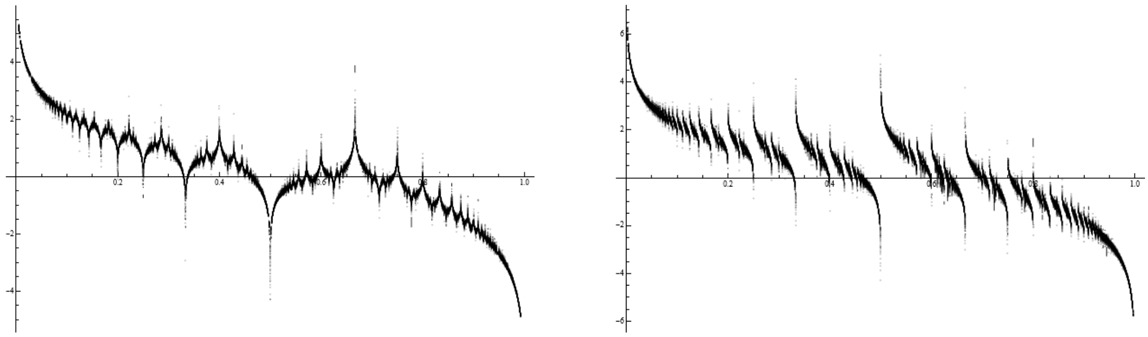


Figure 6.3: Wilton functions $W_{1/2}$ and W_1.

The series (6.16) for $\alpha = 1$ was initially introduced by Wilton to study trigonometric series of the form

$$\sum_{n=1}^{\infty} \frac{d(n)}{n} \sin(2\pi n x) \tag{6.17}$$

where $d(n)$ denotes the number of divisors of the natural number n. Specifically, Wilton demonstrated that the convergence of the series (6.17) is equivalent to the convergence of W_1.
It can be proved that there exists a constant $C > 0$ such that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, we have

$$|W_\alpha(x) - \sum_{n=0}^{\infty} \frac{(-1)^n \log q_{n+1}}{q_n}| \leq C.$$

It satisfies the following functional equation : for all $x \in (0, \alpha]$,

$$W_\alpha(x) = -\log x - x W_\alpha(A_\alpha(x)). \tag{6.18}$$

In [14], it is demonstrated that the functions W_α are BMO and that W_1 is not BMO. Results concerning the case $\alpha > g$ have been obtained in [8]. But it can be directly seen that $W_\alpha \in L^p_{\text{loc}}$ for all $\alpha \in [1/2, 1]$ using Remark 5.2.5.
In line with the established notation for the function $B_{1/2}$, let us denote $\mathfrak{W} = W_{1/2}$. The results from Section 6.1 and [48] can be adapted to establish the p-regularity of W_1 and $\mathfrak{W}$.

Theorem 6.2.5. *Let $p \in [1, \infty)$, then*

$$h^{(p)}_{W_1}(x) = h^{(p)}_{\mathfrak{W}}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$$

We have also looked into removing logarithmic singularities. Let us define

$$b : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto \sum_{n=0}^{\infty} \beta_{n-1}(x).$$

We easily establish the p-regularity in this case, including the case $p = \infty$.

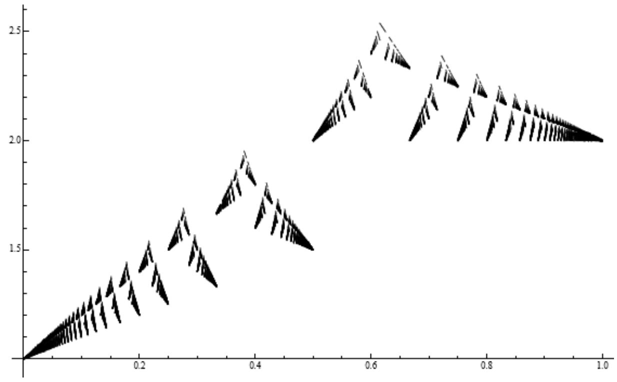


Figure 6.4: Function b.

Theorem 6.2.6. *Let $p \in [1, \infty]$, then*

$$h_b^{(p)}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$$

Remark 6.2.7. In [11], they prove that the minimum of B reaches a strict global minimum at g. In the case of b, without logarithmic singularities, we have a strict global maximum at g. Since g is a fixed point for A_1, its value is given by

$$b(g) = \sum_{n=0}^{\infty} g^n = \frac{1}{1-g} = g + 2.$$

More generally, in [10], they define for $\theta > 0$ and $|z| < g^\theta$, the function

$$b_{\theta,z} : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto \sum_{n=0}^{\infty} z^n \beta_{n-1}(x)^\theta.$$

This function has the same regularity as in Theorem 6.2.2, case $p = \infty$ included.

Remark 6.2.8. One may also replace logarithmic singularities by another function f satisfying certain conditions, as was done in [10, 82], although the pointwise regularity might depend on this new function f.

6.3 Thomae's function

The Thomae function has long served as a striking example in real analysis, showcasing the interplay between continuity and discontinuity. Introduced by Thomae in 1875 as a refinement of the Dirichlet function [112], within the framework of Riemann's concept of integration, it is defined as follows. Unless explicitly stated otherwise, any rational number

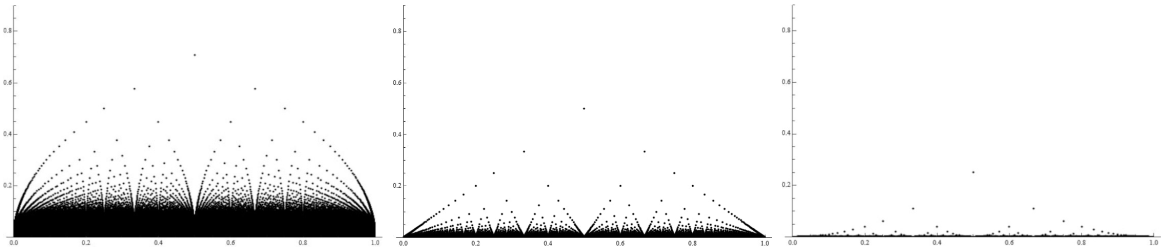


Figure 6.5: Representation of the function T_θ on $(0,1)$ for $\theta = 1/2$, 1 and 2.

x expressed as $x = p/q$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$) with p and q coprime. The Thomae function is then given by

$$T_\theta(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

with $\theta = 1$. The limiting case $\theta = 0$ corresponds to the Dirichlet function. For $\theta > 0$, the function exhibits the remarkable property of being continuous on the irrational numbers while discontinuous at every rational point. This duality, combined with its self-similar structure, renders the Thomae function an essential object of study for understanding irregular functions in analysis. Here, we will focus on the case where $\theta \in (0, 2]$, while more general functions are discussed in [13].

Beyond its classical role in real analysis, the Thomae function has found relevance in broader mathematical and applied contexts. Recent studies have highlighted analogies between its spiked structure and distributions observed in empirical datasets, particularly in biology and clinical research [115].

This Section focuses on the Hölder regularity of the Thomae function, a key aspect of its behavior. The first Subsection reviews its fundamental properties, offering a detailed account of its defining characteristics and self-similar nature. Then, we analyze the function's Hölder regularity, uncovering insights into its fractal-like properties through contemporary mathematical tools. By bridging its classical foundations with these contemporary perspectives, we aim to highlight both the theoretical elegance and the deeper structural nuances of this remarkable function. This section led to the paper [71].

6.3.1 Fundamental properties

We begin by recalling some well-known properties of the Thomae function.

The Thomae function is periodic with period 1.

Proposition 6.3.1. *For any $x \in \mathbb{R}$, $T_\theta(x+1) = T_\theta(x)$.*

Proof. If x is irrational, so is $x + 1$ and therefore $T_\theta(x+1) = T_\theta(x) = 0$.

If $x = p/q$, then $x + 1 = \frac{p+q}{q}$. Since a number divides p and q if and only if it divides $p + q$ and q, p and q coprime implies $(p + q)$ and q coprime. Thus, for such a rational x, $T_\theta(x + 1) = T_\theta(x) = q^{-\theta}$. □

Regarding continuity, the following result holds.

Proposition 6.3.2. *The function T_θ is discontinuous at rational points and continuous at irrational points.*

Proof. If x is rational, let s be an irrational number, and define $x_j = x + s/j$ for $j \in \mathbb{N}$. Clearly $x_j \to x$, but since x_j is irrational for all j, $T_\theta(x_j) \not\to T_\theta(x)$.

If x is irrational, assume $x \in (0, 1)$. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ be such that $n^{-\theta} < \varepsilon$. For $j \in \{1, \ldots , n\}$, define $m_j = \sup\{m \in \mathbb{N}_0 : m < jx\}$, and set

$$\delta_j = \inf\{|x - \frac{m_j}{j}|, |x - \frac{m_j + 1}{j}|\}.$$

Let $\delta = \inf_{1\leq j\leq n} \delta_j$. If $y = p/q$ is rational and $y \in (x - \delta, x + \delta)$, then $q > n$, so $T_\theta(y) < n^{-\theta} < \varepsilon$. If y is irrational, $T_\theta(y) = 0 < \varepsilon$. Thus, $|x - y| < \delta$ implies $|T_\theta(x) - T_\theta(y)| < \varepsilon$, proving that T_θ is continuous at x. □

From the preceding result, T_θ is not differentiable at rational points. Let us now establish that for $\theta \in (0, 2]$, the function is not differentiable at irrational points. A stronger result will be demonstrated in the next Subsection.

Proposition 6.3.3. *For $\theta \in (0, 2]$, the function f is not differentiable at any point.*

Proof. Let x be an irrational number. By Hurwitz's Theorem [46], there exists a sequence $(x_j)_{j\in\mathbb{N}}$ of rational numbers converging to x, such that $x_j = p_j/j$ with p_j and j coprime and $|x - x_j| < \frac{1}{\sqrt{5}j^2}$. Then

$$|\frac{T_\theta(x) - T_\theta(x_j)}{x - x_j}| > \frac{j^{-\theta}}{1/(\sqrt{5}j^2)} = \sqrt{5}j^{2-\theta}.$$

This ensures that $DT_\theta(x)$ cannot be equal to zero. However, by irrational approximation, if $DT_\theta(x)$ exists, it must be zero. □

The differentiability in the case $\theta > 2$ is considered in [13], where a more general scenario is studied. Whether T_θ is differentiable at x or not involves conditions on the irrationality exponent of x.

6.3.2 Hölder regularity

Proposition 6.3.4. *For $\theta \in (0, 2]$, the Hölder exponents of the Thomae function T_θ are given by*

$$h^{(\infty)}_{T_\theta}(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ \theta/\tau(x) & \text{if } x \text{ is irrational.} \end{cases}$$

Proof. Since T_θ is not continuous at rational numbers, we can suppose that $x \in (0,1)$ is irrational. If $y \in (0,1)$ is also irrational, we naturally have $T_\theta(x) - T_\theta(y) = 0$. If $y_n = \frac{p_n}{q_n} \in (0,1)$ with p_n and q_n coprime, then there exists $N \in \mathbb{N}$ such that for all $\varepsilon > 0$ and for all $n \geq N$, one has

$$\frac{|T_\theta(y_n) - T_\theta(x)|}{|y_n - x|^{\theta/\tau(x)}} = \frac{1/q_n^\theta}{|y_n - x|^{\theta/\tau(x)}} \leq q_n^{-\theta}\, q_n^{(\tau(x)+\varepsilon\tau(x))\theta/\tau(x)} = q_n^{-\theta} q_n^{\theta+\varepsilon\theta},$$

so that $T_\theta \in \Lambda^{\theta/\tau(x)}(x)$. If $\theta/\tau(x) = 1$ (which can only occur when $\theta = 2$), it follows that $h_{T_\theta}^{(\infty)}(x)$ is equal to 1, as T_θ is not differentiable at x. Otherwise, let $\varepsilon > 0$ be such that $\varepsilon + \theta/\tau(x) < 1$ and consider the convergents p_j/q_j of x. For sufficiently large j, we have

$$\frac{|T_\theta(x) - T_\theta(p_j/q_j)|}{|x - p_j/q_j|^{\frac{\theta}{\tau(x)}+\varepsilon}} = \frac{q_j^{-\theta}}{|x - p_j/q_j|^{\frac{\theta}{\tau(x)}+\varepsilon}} \geq \frac{q_j^{(\tau(x)-\varepsilon)(\frac{\theta}{\tau(x)}+\varepsilon)}}{q_j^\theta} = q_j^{\beta_\varepsilon},$$

with $\beta_\varepsilon > 0$. As $q_j \to \infty$, we get $T_\theta \notin \Lambda^{\frac{\theta}{\tau(x)}+\varepsilon}(x)$. □

Remark 6.3.5. The function studied here can be generalized using Boyd functions. Relationships 1.2 enable us to adapt the preceding results to the function

$$T_\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \phi(1/q) & \text{if } x = p/q, \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

where $\underline{b}(\phi) = \overline{b}(\phi) = \theta \in (0,2]$.

Remark 6.3.6. For $\mathfrak{B}$ and T_θ, the regularity exponent at a given point is inversely proportional to the irrationality measure of that point. This means that the better the point is approximable by rationals, the less regular these functions are at that point. One may then wonder whether all known multifractal functions linked to diophantine approximation satisfy this kind of relationship.

To explore this, let us consider the Takagi function, first introduced by T. Takagi in 1903 ([111]), which is a continuous but nowhere differentiable function that has fascinated mathematicians ever since. Beyond its intrinsic singular properties, it plays a crucial role in various fields, including real analysis, multifractal analysis, combinatorics, and number theory [2, 65, 24]. The Takagi function is defined by

$$\mathfrak{T}(x) = \sum_{n\geq 0} \frac{\varphi(2^n x)}{2^n}, \quad x \in [0,1],$$

where $\varphi(y) = \operatorname{dist}(y, \mathbb{Z})$. We extend $\mathfrak{T}$ on $\mathbb{R}$ by periodicity. Remark that for $x \in [0,1]$, $A_{1/2}(x) = \varphi(1/x)$.
The regularity of $\mathfrak{T}$ has been established [106]:

$$\mathfrak{T} \in \cap_{u\in[0,1)} \Lambda^u(\mathbb{R}) \backslash \Lambda^1(\mathbb{R}),$$

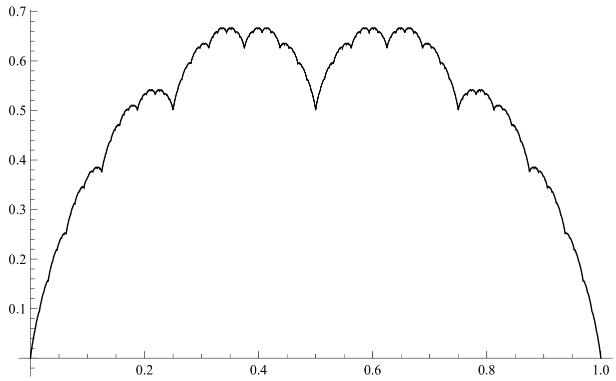


Figure 6.6: Tagaki function $\mathfrak{T}$.

so that $h^{(\infty)}_{\mathfrak{T}}(x) = 1$ for all $x \in \mathbb{R}$. As we know, Boyd functions can be used to better characterize regularity. One may wonder whether there exists $\phi \in \mathcal{B}$ with Boyd indices equal to 1 such that, for certain x, we have $\mathfrak{T} \in \Lambda^{\phi}(x)$. In fact, if you take $\phi \in \mathcal{B}$ such that $\phi(t) = t(|\log t| + 1)$ for $t \in (0, 1)$, then for all $x \in \mathbb{R}$, $\mathfrak{T} \in \Lambda^{\phi}(x)$ [4].

Another example is the Minkowski function, whose pointwise regularity remains unknown but which is multifractal with also a different regularity behavior than the inverse of the irrationality measure. This function was defined by Minkowski for the purpose of establishing a one-to-one correspondence between the rational numbers of $(0, 1)$ and the quadratic irrationals of $(0, 1)$. This function is a rare example of being strictly increasing, with a derivative equal to zero almost everywhere [103].
If $x = [a_0, a_1, a_2, ...]$ is irrational, then we define

$$M(x) = a_0 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+...+a_k}}.$$

If $x = [a_0, a_1, a_2, ..., a_m]$ is rational, then we define

$$M(x) = a_0 + 2\sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{a_1+...+a_k}}.$$

For example,

$$M(g) = M([0, 1, 1, ...]) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{2^{k+1}} = \frac{2}{3}.$$

One can show ([103, 32]) that M belongs to the space $\Lambda^{\frac{\log 2}{2\log G}}(\mathbb{R})$. Therefore, for all $x \in \mathbb{R}$,

$$h^{(\infty)}_M(x) \geq \frac{\log 2}{2\log G}.$$

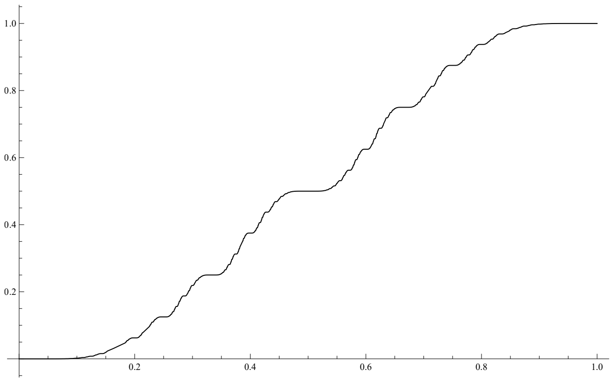


Figure 6.7: Minkowski Question Mark function M.

This lower bound cannot be improved since $h_M^{(\infty)}(g) = \frac{\log 2}{2\log G}$. More precisely, let $x \in (0,1)\backslash\mathbb{Q}$, if

$$a_1(x) + ... + a_n(x) < \frac{n \log 2}{2\log G}$$

for sufficiently large n, then

$$h_M^{(\infty)}(x) \in [\frac{\log 2}{2\log G}, 1].$$

If

$$a_1(x) + ... + a_n(x) > \kappa_2 n$$

for sufficiently large n, then

$$h_M^{(\infty)}(x) \geq 1,$$

where $\kappa_2 \simeq 4.401$.

6.4 Perspectives

In the literature, one can often find functions defined in the following manner:

$$f(x) = \sum_{n\geq 0} f_1(x,n) f_2(x,n),$$

where f_1 is related to an iteration of a transformation that involves the position of real numbers relative to nearby integers and f_2 plays the role of a singularity.

Example 6.4.1. If $f_1(\cdot, n)$ is the product $A_1...A_1^{n-1}$ and $f_2(\cdot, n) = \log(1/A_1^n)$, then f is the usual Brjuno function B. In this case, $h_B^{(1)}(x) = 1/\tau(x)$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$.

Example 6.4.2. If $f_1(\cdot, n)$ is the product $A_{1/2}...A_{1/2}^{n-1}$ and $f_2(\cdot, n) = \log(1/A_{1/2}^n)$, then f is the Brjuno-Yoccoz function $\mathfrak{B}$. In this case, we also have $h_{\mathfrak{B}}^{(1)}(x) = 1/\tau(x)$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$.

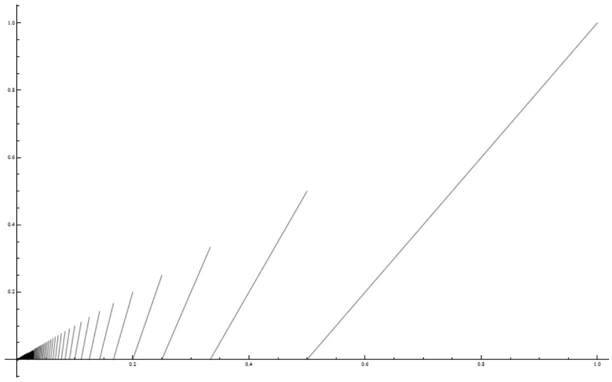


Figure 6.8: Transformation S.

The pointwise regularity of the generalized Riemann function [105, 22] or of Davenport series [47] also involves the inverse of exponents of irrationality, defined according to the context.
A possible direction for future work is therefore the following: establish certain hypotheses on the functions f_1 and f_2 so that the regularity exponent involves the inverse of an irrationality exponent defined through the function f_1.
For example, one can define Brjuno functions using the notion of Engel series [73]: we introduce the transformation

$$S:(0,1]\to(0,1]:x\mapsto a(x)x-1,$$

where $a(x)=[\frac{1}{x}]_1+1$. We can thus write

$$x=\frac{1}{a(x)}+\frac{S(x)}{a(x)}.$$

Set $x_0=|x-[x]_1|$ and for $n\in\mathbb{N}_0$,

$$\begin{cases} x_{n+1}=S(x_n)\\ a_{n+1}(x)=a(S^n(x))=a(x_n).\end{cases}$$

One gets

$$x=\frac{1}{a_1(x)}+\frac{1}{a_1(x)a_2(x)}+...+\frac{1}{a_1(x)...a_n(x)}+\frac{S^n(x)}{a_1(x)...a_n(x)},$$

so that if x is irrational, one has

$$x=\sum_{n\geq 0}\frac{1}{a_1(x)...a_n(x)}.$$

One can then examine the pointwise regularity of the function

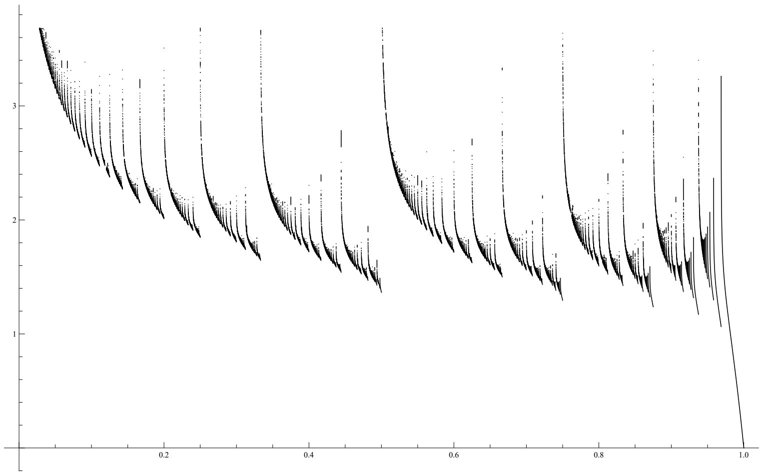


Figure 6.9: First terms of the series B_E.

$$B_E : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n\geq 0} xS(x)...S^{n-1}(x) \log S^n(x),$$

which fulfills the functional equation

$$B_E(x) = -\log x + xB_E(S(x)).$$

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