Stochastic stability of random stacking of blocks

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Abstract

The paper explores the stability of a tower obtained by stacking identical rectangular blocks on top of each other with an inherent randomness due to slight positional offsets between successive blocks. With probabilistic modeling techniques, the diffusive behavior of the stacking process is studied and the collapse is seen as a first passage time problem. In the considered model, alignment errors are idealized as independent Gaussian random variables with zero mean and given standard deviation. We derive expressions for the joint probability density functions of block positions, and analyze their correlation. The study extends to the stochastic stability of a stack of given height, by exploring the statistical characteristics of the center of gravity of the part of tower above each block. Eventually, the probabilistic analysis of collapse is developed to quantify the statistics of the number of blocks that can be heaped up before the tower topples. Although this problem may initially appear playful, it offers an illustrated introduction to first passage problems on a non homogenous process. From a practical standpoint, this analysis offers a simple understanding of the influence of alignment errors on the overall stability of a stack, which may find several fields of application.

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1 Introduction

The intricate dynamics of random stacking is a seemingly whimsical yet inherently complex problem. Since very young age, humans are naturally captivated by topping blocks on each other. Scientists recognize that games related to block building activities have a central role in introducing the early-age spatial reasoning [1, 2]. Random stacking is pervasive in our daily lives, it extends from the haphazard arrangement of a child's wooden blocks, to books on a shelf, and through blocks on a construction site, dishes on a waiter's forearm, or even stored garden furnitures. In some fields stability of piled up objects is so important that it has been standardized [3], for instance as to falling foreign objects from shelves into food [4].

It has also relevance in various fields and potential impact on real-world scenarios. Storage relies on automated sorting and packing of objects [5]. The construction of dry masonry walls [6] needs to be optimized to cope for the imperfect nature of the stacked objects. It also extends to the random packing of larger objects, such as large containers in logistic terminals [7]. Understanding the probabilistic nature of this process is crucial for optimizing stacking methodologies, enhancing structural stability, and minimizing risks associated with unintended collapses.

There is a vast literature on the stackability of objects and the study of geometric shapes that are more amenable to stacking [8]. The stability and optimal height of structures composed of random elements has been a subject of interest in applied mathematics, such as the stacking of pans with random shapes [9], and the methods to minimize the height of the stack. At small scales, the stacking of carbon nanofibres is known to provide advantageous biological properties when designed and filtered in an appropriate manner [10].

Previous studies have also addressed stability concerns in a deterministic context, aiming for instance at the strategy that would be maximizing the overhang of a stack of perfect blocks [11], with single or multi-wide stacks relying on friction or not [12]. Although many of these problems were idealized and appeared first as theoretical curiosities, they eventually served real-life applications, for instance in additive manufacturing where overhangs are created by offsetting raw material on multiple layers [13].

It appears that the problem of random stacking with blocks exhibiting stochastic misalignments has not been deeply investigated yet. Therefore, the problem considered in this paper consists in adding blocks (assumed perfectly rectangular), on top of each other, and on top of a flat surface, but to pile them up with a certain imperfection. The ultimate aim is to determine the maximum height of the stack, in a probabilistic sense. In the course of our exploration, we uncovered that the overturning condition takes an unusual form of a first passage problem [14], as this process is non homogenous [15]. Also, for some given tower height at collapse, it appears that there are two preferred sections where failure could occur: either at the bottom of the stack, either at an interface that is located some distance away from the top of the stack. This conditional bi-modality is illustrated in this paper, after the mathematical model is posed in Section 2 and analyzed in Section 3.

2 Mathematical model

We consider the stability of a stack of $n+1$ blocks numbered from $i=0$ to $i=n$. The problem is 2-D and the blocks are assumed to take a perfect rectangular shape with given height h and width $2b$. Let x_i represent the transverse position of block i with respect to a vertical reference line and use the convention that $x_0 = 0$. Blocks are placed on top of each other with some uncertainty on the transverse placement of the block, see Figure 2.1.

We assume that the alignment error is expressed as $b\,u_i$, where b is the block halfwidth, and u_i , with $i \in [1, n]$, is the dimensionless alignment error incurred when placing block i. We assume that this dimensionless alignment error is independent of the tower's height and can be modeled as a Gaussian random variable with zero mean and standard deviation σ_u. The transverse position of block i can be estimated from the position of the previous block, with $x_0 = 0$ and

$$x_i = x_{i-1} + b\,u_i, \qquad i \in [1, n]. \tag{2.1}$$

Using recurrence it is straightforward to show that

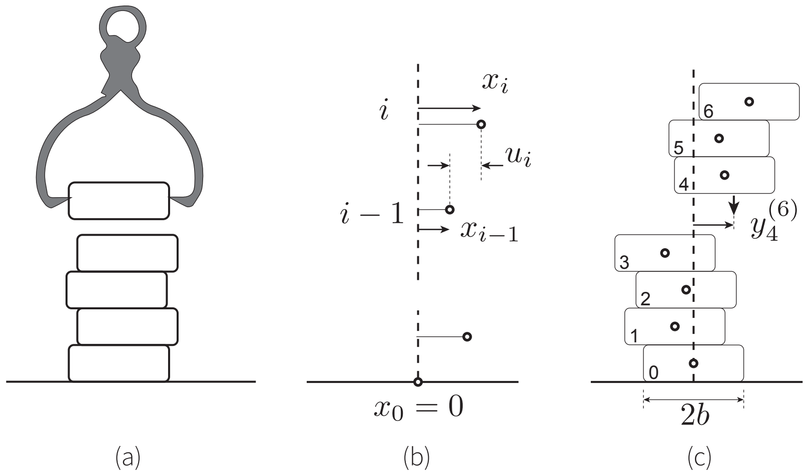

Fig. 2.1: Description of the problem. (a) Random stacking process (b) Description of block positions (c) Illustration of the position of the centroid of the upper part of the tower above level 3 ($n = 6, i = 4$).

$$x_i = b \sum_{j=1}^{i} u_j, \qquad i \in [1, n]. \tag{2.2}$$

Let $y_i^{(n)}$ be the transverse position of the center of gravity of the portion of the tower above block i, including block i. This is the position of the resultant force at the interface between blocks i and $i-1$. We express that

$$y_i^{(n)} = \frac{1}{n+1-i} \sum_{j=i}^{n} x_j, \qquad i \in [0, n] \tag{2.3}$$

and note that the case $i = 0$ is less relevant since block 0 is assumed to be placed on a continuous flat surface and therefore cannot overturn. On the other end, starting from the very top of the stack,

$$y_n^{(n)} = x_n \quad ; \quad y_{n-1}^{(n)} = \frac{1}{2}(x_n + x_{n-1}) \quad ; \quad ... \tag{2.4}$$

Overturning takes place, in a stack of height n, if and only if there exists a value of $i \in [1; n]$ such that $\left| y_i^{(n)} - x_{i-1} \right| > b$ which translates that the centroid of the upper part of the stack lies beyond the block's vertical face. Blocks are assumed as perfectly rigid so that the concept of central core is not useful [16]. The stability of the stack is therefore assessed by means of the statistics of

$$z_i^{(n)} = \frac{y_i^{(n)} - x_{i-1}}{b}. \tag{2.5}$$

Ultimately, the maximum stack height h is defined with the concepts of a first passage problem, as the smallest number n of stacked blocks such that the stack was stable in all configurations before the addition of the last block, and collapse precisely occurred when adding this block. Stable in all prior configurations means for all previous stack heights $m < n$, and at all interfaces $\forall i \in [0, m]$. Mathematically,

$$h = \arg\min_{n} \left\{ n \in \mathbb{N}_0 \,|\, \left(\forall m < n, \forall i \in [1, m], \left| z_i^{(m)} \right| \leq 1 \right) \cap \left(\exists i \in [1, n] \,|\, \left| z_i^{(n)} \right| > 1 \right) \right\}. \tag{2.6}$$

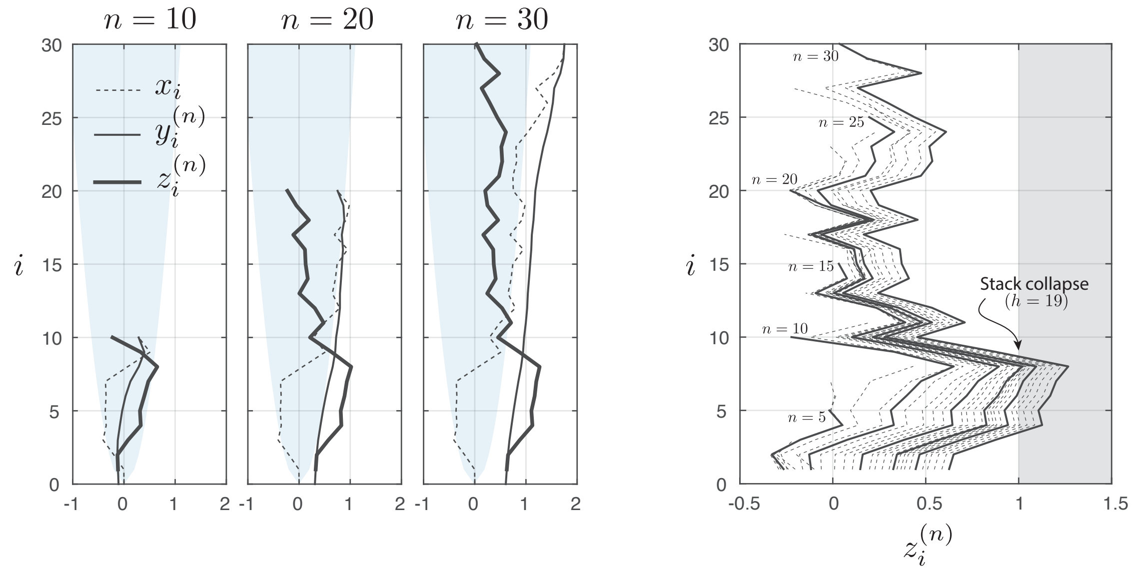

Fig. 2.2: Samples of the considered problem. Left: illustration of x_i, $y_i^{(n)}$ and $z_i^{(n)}$ for three stack heights. Right: detailed view of the sequence of overturning conditions $z_i^{(n)}$. Collapse takes places at the first crossing of $z_i^{(n)}$ with the vertical axis $|z| = 1$. In this example, all stacks for $n > 19$ are unstable. Numerical simulation with $b = 1$ and $\sigma_u = 0.2$.

The breaking level ℓ is another quantity of interest. It represents the level where overturning occurs as the stack collapses. Since there might be several levels where the overturning condition is satisfied, the breaking level is defined as the most critical one, i.e. with the largest overturning condition,

$$\ell = \arg\max_i \left| z_i^{(h)} \right| . \tag{2.7}$$

Since $z_i^{(n)}$ is a random process, the maximum stack height h and the breaking level ℓ are random variables, and the problem consists in determining their distributions. The paper describes the solution of this problem with an exact analytical approach when applicable, and validates, then explores the results by Monte Carlo simulations of the original problem.

Figure 2.2 shows a example of the sequential evolution of x_i, $y_i^{(n)}$ and $z_i^{(n)}$ for three values of n, i.e. for increasing stack heights. Although the ordinate values correspond to discrete integers with no inherent physical meaning between them, the lines connecting the points are used to enhance visual clarity and help discern trends and relationships within the data more easily. In particular, on the right, it is possible to visualize how the overturning condition $z_i^{(n)}$ evolves as the number n of stacked blocks grows. It is even possible to push the simulation beyond a critical state: in this case, the stack should have collapsed after having added the 19^{th} block, but the simulation was pushed to $n = 30$. The greyed area on the left corresponds to the range $[-\sigma_{x_i}, +\sigma_{x_i}]$, where σ_{x_i} is the (analytical expression of the) standard deviation of x_i, which is derived in Section 3. It helps visualize the diffusive nature of process x_i.

3 Analysis of the model

3.1 Statistics of the lateral block position x_i

Being a sum of Gaussian random variables, see (2.2), the block position x_i is also a Gaussian random variable. Its mean is zero, and its standard deviation is

$$\sigma_{x_i} = b\,\sigma_u\sqrt{i}. \tag{3.1}$$

This translates the drifting of the block positions, similar to the transient in diffusion processes [17]. The random variables x_i, $i = 1, \cdots, n$, representing the transverse position of each block, are not independent, by the way they are constructed, see (2.1). The covariance between the two zero-mean variables x_i and x_{i-1} is expressed as

$$\mathrm{cov}\,(x_i x_{i-1}) = \iint_{-\infty}^{+\infty} x_i x_{i-1} p_{x_i x_{i-1}}(x_i, x_{i-1})\mathrm{d}x_{i-1}\mathrm{d}x_i. \tag{3.2}$$

Based on the formula (9.1) given in the Appendix, it yields $\mathrm{cov}\,(x_i x_{i-1}) = (i-1)b^2\sigma_u^2$, and finally, considering the expression (3.1) for the standard deviations, the correlation coefficient between x_i and x_{i-1} is

$$\rho_{x_{i,i-1}} = \frac{\mathrm{cov}\,(x_i x_{i-1})}{\sigma_{x_{i-1}}\sigma_{x_i}} = \sqrt{1-\frac{1}{i}}. \tag{3.3}$$

The positions of two successive blocks at the bottom of the stack, at level i_1, are less correlated than the positions of two successive blocks located higher in the stack, at level $i_2 > i_1$.

Following similar arguments, defining $\mathbf{x} = (x_1, ...x_n)$ as the vector of transverse block positions, it is possible to demonstrate that the elements of the covariance matrix of $\mathbf{x}$ are given by $\mathrm{cov}_{\mathbf{x}_{i,j}} = \min\,(i,j)\,b^2\sigma_u^2$, i.e. its structure looks like

$$\mathrm{cov}_{\mathbf{X}} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & \ddots & & \vdots \\ \vdots & \vdots & & n-1 & n-1 \\ 1 & 2 & \cdots & n-1 & n \end{bmatrix} b^2\sigma_u^2, \tag{3.4}$$

so that

$$\rho_{x_{i,j}} = \frac{\min\,(i,j)}{\sqrt{ij}} = \sqrt{\frac{\min\,(i,j)}{\max\,(i,j)}} = \sqrt{1-\frac{|i-j|}{\max\,(i,j)}}, \tag{3.5}$$

which indicates that the correlation drops as i and j go away from each other, as expected. Furthermore, since $\rho_{x_{i,j}} \geq 0$, $\forall i, j \in [1, n]$, the conditional position of a block has the same sign as the block used for the condition. In other words, the position x_j of block j, is positive on average if the position of another block $i < j$ is known to be positive, and vice versa. It is a consequence of the independent nature of increments u_i which does not allow the natural compensations that could be observed in a more human-like stacking aiming at reaching the highest height.

3.2 Statistics of the position $y_i^{(n)}$ of the centroid of the upper part of the stack

The transverse position of the center of gravity of the portion of the tower above block i, noted $y_i^{(n)}$, is a linear combination of the transverse positions x_j of blocks located above interface i. Therefore, the average position of the centroid is null and its variance is obtained from the covariance matrix of block positions (3.4), so that

$$\sigma^2_{y_i^{(n)}} = \frac{b^2}{(n+1-i)^2} \sum_{j_1=i}^{n} \sum_{j_2=i}^{n} \min(j_1, j_2)\, \sigma_u^2. \tag{3.6}$$

After some developments, see Appendix, this expression is evaluated as

$$\sigma^2_{y_i^{(n)}} = \frac{2n^2 + 2in + n - 4i^2 + 5i}{6\,(n+1-i)} b^2 \sigma_u^2. \tag{3.7}$$

It is observed that the standard deviation of $y_i^{(n)}$ depends on the interface level i, but also on the stack height n. This is naturally expected since it represents the centroid of an upper part of the tower.

For a given tower height, the previous expression has the two limit values, as $n \to \infty$,

$$\sigma^2_{y_1^{(n)}} = \frac{(2n+1)(n+1)}{6n} b^2 \sigma_u^2 \sim \frac{1}{3} n b^2 \sigma_u^2. \quad ; \quad \sigma^2_{y_n^{(n)}} = n b^2 \sigma_u^2 \tag{3.8}$$

representing the standard deviation of the position of the centroid of the whole tower ($i = 1$), and of the top block ($i = n$), respectively. This shows that for high stacks, the position of the centroid of parts of the tower above a given level is about $\sqrt{3}$ larger at the top than at the base. This is because our definition of this centroid is made respect to x_0, and not to the block just below. Therefore, this also includes the fact that the blocks themselves are also more likely to be located further away from the central line above the bottom block $x_0 = 0$.

3.3 Statistics of the overturning condition $z_i^{(n)}$

To conclude about possible instability of the tower, the overturning condition $z_i^{(n)}$ must be analyzed. Since the definition of $z_i^{(n)}$ involves $y_i^{(n)}$ and x_{i-1}, see (2.5), it is necessary to establish their joint statistics. Being two linear combinations of the same Gaussian process u_i, they are jointly Gaussian and therefore characterized by their variances, and covariance. With details given in the Appendix, the latter is expressed as

$$\mathrm{cov}\left(y_i^{(n)}, x_{i-1}\right) = \mathbb{E}\left[\frac{x_{i-1}}{n+1-i} \sum_{j=i}^{n} x_j\right] = (i-1)\, b^2 \sigma_u^2 \tag{3.9}$$

while the variances of $y_i^{(n)}$ and x_{i-1} are respectively given in (3.7) and (3.1).

This result could have been anticipated by considering the fact that $y_i^{(n)}$ and x_{i-1} just interact by means of x_{i-1}, the transverse position at level $i-1$. Indeed, on one hand x_{i-1} depends on the increments u_j for $j \leq i-1$, see (2.2). On the other hand, $y_i^{(n)}$ refers to the position of the upper part of the tower, therefore on the increments u_j for $j \geq i$.

Based on the definition (2.5), we obtain

$$b^2 \sigma^2_{z_i^{(n)}} = \sigma^2_{y_i^{(n)}} - (i-1)\, b^2 \sigma_u^2. \tag{3.10}$$

In particular, when $i = 1$, the second term vanishes and only the first remains, which is given in (3.7). The probability that instability occurs at the base of the tower ($i = 1$) depends the standard deviation

$$\sigma_{z_1^{(n)}} = \sigma_u \sqrt{\frac{(n+\frac{1}{2})(n+1)}{3n}} \tag{3.11}$$

which asymptotically scales with the square root of the stack height, $\sigma_{z_1^{(n)}} \sim \sigma_u \sqrt{n/3}$ as $n \to +\infty$. Furthermore, in the more specific case $n = 1$, the previous expression gives $\sigma_{z_1^{(1)}} = \sigma_u$, as expected, since in that case only one block is susceptible of overturning.

3.4 Statistics of the stack height h at collapse

Last but not least, we can focus on the first passage problem consisting in determining the minimum height corresponding to first occurrence of $\left|z_i^{(n)}\right| > 1$, i.e. the smallest value of n such that there exists an $i \in [1, n]$ that satisfies this condition.

The survival probability S_n is the probability that the stack has not collapsed up to addition of block n. It is given by

$$S_n = \text{prob}\left(\left|z_j^{(m)}\right| \leq 1, \forall m \leq n, \forall j \in [1, m]\right). \tag{3.12}$$

Invoking Bayes' theorem, $S_n = S_{n-1}T_n$where

$$\mathcal{T}_n = \text{prob}\left(\left|z_i^{(n)}\right| \leq 1, \forall i \in [1, n] \middle| \left|z_j^{(m)}\right| \leq 1, \forall m < n, \forall j \in [1, m]\right) \tag{3.13}$$

is the conditional probability that the tower is still stable after addition of block n, knowing that it was still standing before addition of this last block.

In usual Markov chains, T_n is expressed as the transitional probability of the chain, and the first passage problem is solved with a straightforward recurrence [15]. However, in this problem, $z_i^{(n)}$ is not a usual sequence, indexed solely on n, since the addition of a block affects the states of the chain for all heights visited in the past. Therefore, the first passage problem does not only concern the last added sample in the sequence but requires revisiting the states of the entire sequence. This is particularly important as we can anticipate that collapse will most likely not occur directly under the last added block, but somewhere in the revisited states. This is well illustrated in Figure 2, where collapse eventually takes place at interface $i = 8$, a floor which was safe when the stack height was $n = 8$, even at $n = 10$, and so on until collapse happens for $n = 19$, in this example.

To solve this problem, the sequence $z_i^{(n)}$ should be considered as non homogenous (non constant transition matrix), or as a discrete stochastic field with two independent variables: i (space) and n (time). The first passage therefore becomes that of a random field instead of a chain. Early works on first passage times for stochastic fields, date back to late 1980s [18] under some strong assumptions, and have moved to problems such as the diffusion of many particles in a common stochastic environment [19]. This recent work is closely related to the concept of *extreme first passage times*, representing the shortest among several times, for many (e.g. diffusing) particles [20]. In our case, each particle i represents a space coordinate. Since, floor i is only reached after time n (addition of n blocks), particles are progressively introduced into the problem. This moving boundary feature makes it very specific and hard to cast into existing forms of first passage problems for random fields.

Since we are in the presence of a triangular array of Gaussian random vectors, another approach would be to focus on their maximum, by relying on specific theories for this class of problem, see e.g. [21], which provides bounds and asymptotic distributions, but not tractable description of the distribution.

Lacking general solutions, it is therefore proposed to develop an *ad hoc* solution to our problem, by focusing on the direct evaluation of S_n as defined in (3.12). The condition is expressed as a function of $z_j^{(m)}$ for all $m < n$ and for all $j \in [1, m]$. For instance, for $n = 2$,

$$\left|z_1^{(1)}\right| \leq 1 \quad ; \quad \left|z_1^{(2)}\right| \leq 1 \quad ; \quad \left|z_2^{(2)}\right| \leq 1, \tag{3.14}$$

and the overturning conditions $z_j^{(m)}$ are jointly Gaussian. It appears that the survival probability is obtained as the integral of this joint high-dimensional probability over a hypercube. Instead of integrating on such a simple domain the complicated joint PDF of z_i, the problem is rewritten as a function of the original independent quantities, u_i. To do so, the definitions of $y_i^{(n)}$ (split into two sums) and x_{i-1} are substituted in the definition of $z_i^{(n)}$, which yields

$$z_i^{(n)} = \frac{1}{n+1-i} \left(\sum_{j=i}^{n} \sum_{k=1}^{i-1} u_k + \sum_{j=i}^{n} \sum_{k=i}^{j} u_k \right) - \sum_{k=1}^{i-1} u_k \tag{3.15}$$

where the first double summation cancels out with the last summation, so that we are left with

$$z_i^{(n)} = \frac{1}{n+1-i} \sum_{j=i}^{n} \sum_{k=i}^{j} u_k := \sum_{k=i}^{n} \zeta_{i,k}^{(n)} u_k \tag{3.16}$$

where we have defined

$$\zeta_{i,k}^{(n)} := \frac{n+1-k}{n+1-i}. \tag{3.17}$$

Equation (3.15) seems to indicate that $z_i^{(n)}$ depends on u_k for k running from 1 to $i-1$ (through the first term), but at the end, it just depends on values of $k \in [i, n]$. This makes sense since $z_i^{(n)}$ translates the equilibrium of the top part of the stack, above floor i. The first few of these combinations are

$$\begin{aligned}
&z_1^{(1)} = u_1 \\
&z_1^{(2)} = u_1 + \frac{1}{2}u_2 \quad ; \quad z_2^{(2)} = u_2 \\
&z_1^{(3)} = u_1 + \frac{2}{3}u_2 + \frac{1}{3}u_3 \quad ; \quad z_2^{(3)} = u_2 + \frac{1}{2}u_3 \quad ; \quad z_3^{(3)} = u_3.
\end{aligned} \tag{3.18}$$

This notation makes explicit the triangular nature of the array of Gaussian random vectors involved in this problem. Having rewritten the problem under this format, the integral of the joint PDF of $\mathbf{z}$ over a hypercube turns into the integral of the joint PDF of all u_i's over the polytope defined by the hyperplanes (3.18). The difficulty has been shifted from expressing the joint PDF of $\mathbf{z}$ to integrating the (simple) joint PDF of u_i's over a more involved volume. There exist techniques for integration on polytopes in high dimensional spaces [22] but they still heavily rely on numerical algorithms, in their most general form. Fortunately, a specific approach to the solution of this problem is in fact available.

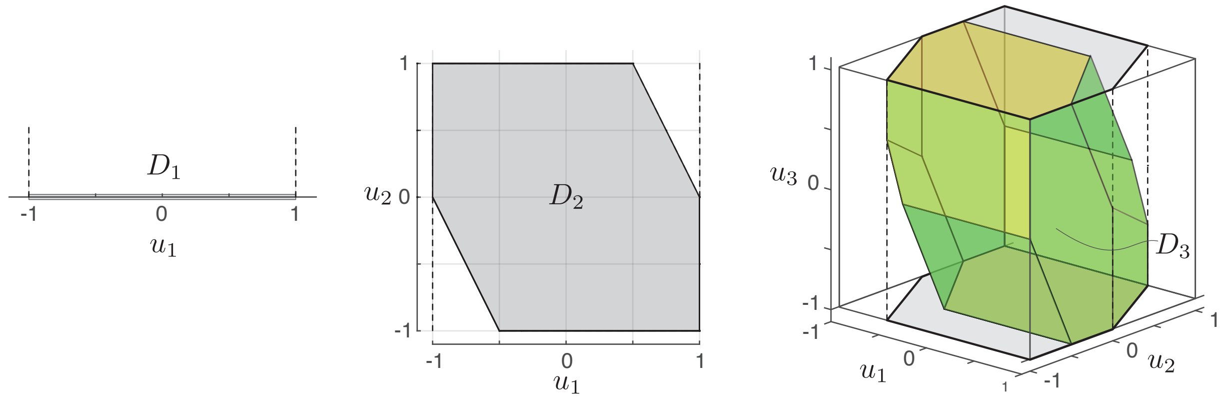

Fig. 3.1: Domains D_1, D_2 and D_3 over which integration of the joint PDF of (independent) u_i's need to be integration to determine the survival probabilities.

Indeed it turns out that an affordable recurrence can be developed to compute the survival probabilities. For $n = 1$,

$$S_1 = \text{prob}\left(\left|z_1^{(1)}\right| \le 1\right) = \text{prob}\left(|u_1| \le 1\right) = 2\Phi\left(\frac{1}{\sigma_u}\right) - 1 = \text{erf}\left(\frac{1}{\sqrt{2}\sigma_u}\right) \tag{3.19}$$

where $\Phi(\cdot)$ represents the cumulative density function of the standard normal distribution. For $n = 2$, the survival probability S_2 corresponds to the probability that the first three overturning conditions $z_1^{(1)}$, $z_1^{(2)}$ and $z_2^{(2)}$, in (3.18), are below 1 in absolute value. In terms of the independent increments, this means $\left|u_1 + \frac{1}{2}u_2\right| \le 1$, $|u_2| \le 1$, and $|u_1| \le 1$. The corresponding domain in the plane (u_1, u_2) is illustrated by the grayed zone D_2 in Figure 3.1.

Interestingly, because of the triangular nature of the problem, the survival probability S_n can be estimated by splitting the conditions on the overturning condition into one set of conditions on $z_i^{(n)}$ ($\forall i \in [1, n]$) and another one on conditions on $z_j^{(m)}$ ($\forall m < n, \forall j \in [1, m]$), i.e. associated to prior stack heights. The later contains $n(n-1)/2$ conditions, which depend on values of the increments u_i with $i = 1, \cdots, n-1$. They are the same as those used to determine S_{n-1}. The former involves n new conditions combining all increments u_i with $i = 1, \cdots, n$, and more importantly, all of them involve u_n. As a consequence, the survival probability S_n can be estimated by integration over a domain that extrudes, in a space having a single additional dimension, the domain of integration pertaining to the estimation of S_{n-1}. This extrusion is limited by the conditions on the last independent increment.

For instance, for $n = 2$,

$$S_2 = \iint\limits_{D_2} \phi(u_1)\, \phi(u_2)\, \mathrm{d}u_2 \mathrm{d}u_1 \tag{3.20}$$

where

$$D_2 = \left\{(u_1, u_2) \mid u_1 \in D_1, u_2 \in \left[\max\left\{-1; -2(1 + u_1)\right\}; \min\left\{1; 2(1 - u_1)\right\}\right]\right\}, \tag{3.21}$$

and $D_1 = [-1; 1]$. The condition on u_2 in the definition of domain D_2 comes from the two conditions $\left|u_1 + \frac{1}{2}u_2\right| \le 1$ and $|u_2|$. In (3.20), the integrand has been factorized because the increments u_i are independent.

For $n = 3$, a threefold integral of the product of marginals is computed over the domain

$$\begin{aligned} D_3 &= \{(u_1, u_2, u_3) \mid (u_1, u_2) \in D_2, u_3 \in [\inf_3; \sup_3]\}. \\ \inf_3 &= \max\left\{-1; -2\,(1+u_2)\,; -3\left(1+u_1+\frac{2}{3}u_2\right)\right\} \\ \sup_3 &= \min\left\{1; 2\,(1-u_1)\,; 3\left(1-u_1-\frac{2}{3}u_2\right)\right\} \end{aligned} \tag{3.22}$$

where the new bounds are obtained from the last three conditions in (3.18). This domain is illustrated on the right in Figure 3.1, where the thick and dashed lines illustrate the extrusion of domain D_2 to generate domain D_3. More generally, the recurrence is built with the sequence of domains

$$\begin{aligned} D_n &= \{(u_1, \cdots, u_n) \mid (u_1, \cdots, u_{n-1}) \in D_{n-1}, u_n \in [\inf_n; \sup_n]\}. \\ \inf_n &= \max_{i=1,\cdots,n}\left\{-\,(n+1-i)\left(1+\sum_{k=i}^{n-1}\zeta_{i,k}^{(n)}u_k\right)\right\} \\ \sup_n &= \min_{i=1,\cdots,n}\left\{(n+1-i)\left(1-\sum_{k=i}^{n-1}\zeta_{i,k}^{(n)}u_k\right)\right\}. \end{aligned} \tag{3.23}$$

Integration of the joint PDF of the independent increments on D_n yields the survival probability S_n. The resulting recurrence for S_n is finally

$$S_n = S_{n-1} + 2\int\!\cdots\!\int_{D_n} \phi\,(u_1)\,\phi\,(u_2)\,\mathrm{d}u_2\mathrm{d}u_1 \tag{3.24}$$

where $D_n = \{(u_1, \cdots, u_n) \mid (u_1, \cdots, u_{n-1}) \in D_{n-1}, u_n \in [\sup_n; +\infty]\}$. Since the joint PDF of the independent increments u_i factorizes into its marginals, this recurrence provides a simple numerical approach to the estimation of survival probabilities. And, since the survival probability corresponds to the CDF of the stack height at collapse, the PDF of the stack height at collapse is simply obtained through finite differentiation of S_n with respect to n.

4 Regularization of the model

It is tempting to analyze a continuous version of the same problem, obtained by considering that the stack is composed of a continuous medium, for which a tower with finite height can be materialized as an infinite collection of infinitely thin sheets. The solutions of this regularized version of the problem matches those of the discrete model as $n \to +\infty$, i.e. for high stacks.

In this case, Equations (2.1), (2.3) and (2.5) defining the random sequences x_i, $y_i^{(n)}$ and $z_i^{(n)}$, respectively, become

$$x(\xi) = b \int_0^{\xi} \mathrm{d}B \quad ; \quad y(\xi,\tau) = \frac{1}{\tau-\xi} \int_{\xi}^{\tau} x(\bar{\xi})\,\mathrm{d}\bar{\xi} \quad ; \quad z(\xi,\tau) = \frac{1}{b}\left[y(\xi,\tau) - x(\xi)\right] \tag{4.1}$$

where $\tau \geq \xi$, and $\mathrm{d}B(t)$ represents increments of the Brownian motion ($x(\xi)$ is therefore a Wiener process). With this formulation, the time-space nature of the problem is explicit in the overturning condition $z(\xi,\tau)$. Differentiation of the definition of $y(\xi,\tau)$ with respect to the space coordinate ξ yields

$$\partial_{\xi} y(\xi,\tau) = \frac{-1}{(\tau-\xi)^2} \int_{\xi}^{\tau} x(\bar{\xi})\,\mathrm{d}\bar{\xi} - \frac{x(\xi)}{\tau-\xi} = \frac{y(\xi,\tau) - x(\xi)}{\tau-\xi} = \frac{b\,z(\xi,\tau)}{\tau-\xi}. \tag{4.2}$$

This expression is used in the derivative of $z(\xi,\tau)$ to yield

$$\partial_{\xi} z(\xi,\tau) = \frac{z(\xi,\tau)}{\tau-\xi} - W(\xi) \tag{4.3}$$

where $W(\xi) = \mathrm{d}B/\mathrm{d}\xi$ is the zero mean delta correlated white Gaussian process, with intensity σ_u^2, i.e. $\mathbb{E}\left[W(\xi)\,W(\xi+\zeta)\right] = \sigma_u^2 \delta(\zeta)$.

The general solution of (4.3) satisfying the boundary condition $z(\tau,\tau) = 0$, $\forall \tau$, $\forall \xi \leq \tau < +\infty$ takes the form of a stochastic integral,

$$z(\xi,\tau) = \frac{1}{\tau-\xi} \int_{\xi}^{\tau} \left(\tau - \bar{\xi}\right) W\left(\bar{\xi}\right) \mathrm{d}\bar{\xi} \tag{4.4}$$

from which it is possible to derive for instance $\mathbb{E}\left[z(\xi,\tau)\right] = 0$, and

$$\begin{aligned}\mathbb{E}\left[z^2(\xi,\tau)\right] &= \frac{1}{(\tau-\xi)^2} \iint_{\xi}^{\tau} \left(\tau-\bar{\xi}_1\right)\left(\tau-\bar{\xi}_2\right) \mathbb{E}\left[W\left(\bar{\xi}_1\right) W\left(\bar{\xi}_2\right)\right] \mathrm{d}\bar{\xi}_1 \mathrm{d}\bar{\xi}_2 \\ &= \frac{1}{(\tau-\xi)^2} \int_{\xi}^{\tau} \left(\tau-\bar{\xi}_1\right)^2 \sigma_u^2 \mathrm{d}\bar{\xi}_1 = \sigma_u^2 \frac{\tau-\xi}{3}.\end{aligned} \tag{4.5}$$

In particular, the standard deviation of the overturning condition at the base $z(0,\tau)$ is $\sigma_u \sqrt{\tau/3}$. This result is consistent with the solution (3.11) obtained previously in the discrete case as $n \to +\infty$. The first passage problem associated with the determination of the collapsing height can also be translated in the continuous version

$$h_r = \arg\min_{\tau} \left\{ \tau > 0 \,|\, \left(\forall \tau_0 < \tau, \forall \xi_0 \in [0,\tau_0], |z(\xi_0,\tau_0)| \leq 1\right) \cap \left(\exists \xi \in [1,\tau] \mid |z(\xi,\tau)| > 1\right) \right\}.$$

Its solution does not appear simpler than in the discrete case and is not further discussed.

5 Description of the Monte Carlo Simulation

Algorithm (1) is a pseudo-code for the generation of a random stack. Blocks are positioned on top of each other, one at a time, until collapse occurs. As input, the code accepts the block half-width (b), the standard deviation of the offset (*sigu*), the number of repetitions of simulation *nbSim* (Monte Carlo runs), and a maximum number N of independent increments.

Algorithm 1: Pseudo-code representation

```
Input: b, sigu, nbSim, N
Output: maxHeight, brkHeight, jointPDF
for k ← 1 to nbSim do
    Generate N random numbers from a standard normal distribution and multiply them
      by sigu, store the result in U;
    do_add ← true;
    n ← 1;
    while do_add do
        Take the first n elements of U and store them in u, u ← U(1 : n);
        Calculate the cumulative sum of u and append a zero at the beginning, store it in x;
        Calculate the cumulative sum of x in reverse order, excluding the first element, and
          divide each element by its corresponding index, store it in y;
        Calculate the overturn vector as the difference between y and the first n elements of
          x divided by b, store it in z;
        n ← n + 1;
        if max|z| > 1 then
            do_add ← false;
            exitflag ← 1;
        if n = N then
            do_add ← false;
            exitflag ← −1;
    maxHeight(k) ← n − 1;
    brkHeight(k) ← index of max |z|;
    Increment the value of jointPDF at position (brkHeight(k), maxHeight(k)) by 1;
```

The main loop of the algorithm represents the iterative process of simulating the random stacking of blocks. . As indicated in the pseudo-code, a set of N samples can advantageously be generated all at once to avoid the repeated generation of individual samples. Realizations of x_i, $y_i^{(n)}$ and $z_i^{(n)}$ are then obtained, and blocks are piled up until one of the $z_i^{(n)}$ is larger than 1 in absolute value.

The final stacking height is stored in an array *maxHeight*, to track the maximum height reached in each trial, as well as the height at which collapse occurred and stores this value in another array, *brkHeight*. Additionally, the algorithm updates the counts to determine an estimate of the joint probability *jointPDF* between stack height h and breaking height ℓ when collapse occurs.

The execution time of the algorithm remains relatively short, typically a few seconds, even for a substantial number of trials (e.g., *nbSim*$\sim 10^5$). However, it is important to note that the execution time increases as the standard deviation of the random numbers (*sigu*) decreases. This trend is expected, as lower values of *sigu* lead to a greater number of stacked blocks at failure..

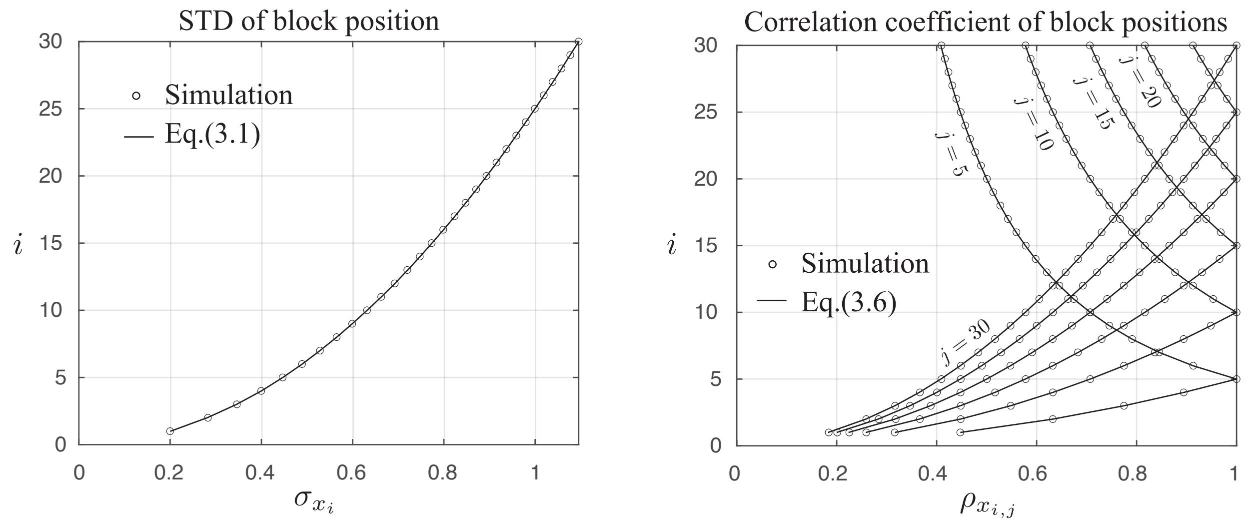

Fig. 6.1: Comparison of analytical solutions and numerical results for the statistics of block positions ($n = 30$). Left: standard deviation of transverse block position σ_{x_i}; Right: correlation coefficient of transverse block positions for some values of j, as a function of elevation i of the block in the stack. Obtained for $\sigma_u = 0.2$.

6 Verification of the Analytical Solutions

Figures 6.1 and 6.2 show comparisons of statistics of the main results of the problem obtained with the Monte Carlo simulations, and the analytical solutions developed in Section 3. They correspond to a positioning uncertainty $\sigma_u = 0.2$, and the block half-width is set to $b = 1$. To validate the statistics before the first passage problem is considered, towers with $n = 30$ are generated, irrespective of wether they would collapse or not. Simulations have been repeated $nbSim = 5 \cdot 10^5$ times. The statistics are represented as a function of the block elevation i.

More specifically, Figure 6.1 shows on the left the standard deviation of the block position. The numerical solution validates the diffusion nature of the sequence x_i with a standard deviation proportional to the square root of the block elevation i. With such a large number of simulations, the agreement is virtually perfect. On the right, some correlation coefficients between the transverse block positions at various heights are represented. We represented only the correlations with block positions at levels 5, 10, $\cdots 30$. In each case, the correlation coefficient is equal to 1 when $i = j$ (as expected), and decreases as $|i - j|$ increases, as shown in (3.4). Again, the analytical solution matches perfectly with the results of the Monte Carlo simulations.

On the left, Figure 6.2 shows the standard deviation of the transverse positions $\sigma_{y_i^{(n)}}$ of the centroid of upper parts of the stock, for various values of n and $i \in [1, n]$. Analytical solutions are shown in continuous and dashed lines, while the results of the Monte Carlo simulations are shown for $n = 30$ only, to not overload the graphical representation. The upper envelope of these lines shows $\sigma_{y_n^{(n)}}$ for various values of n and corresponds to σ_{x_i}, i.e. the solution shown in Figure 6.1. This is because the transverse position of one block at the top precisely corresponds to the position of that block. These representations indicate that $\sigma_{y_i^{(n)}}$ could be idealized as evolving linearly with the block elevation i.

Last but not least, the right of Figure 6.2 shows a comparison of the standard deviation of the overturning condition $\sigma_{z_i^{(n)}}$ for successive stack height, i.e. for n running from 1 to 30. At the top

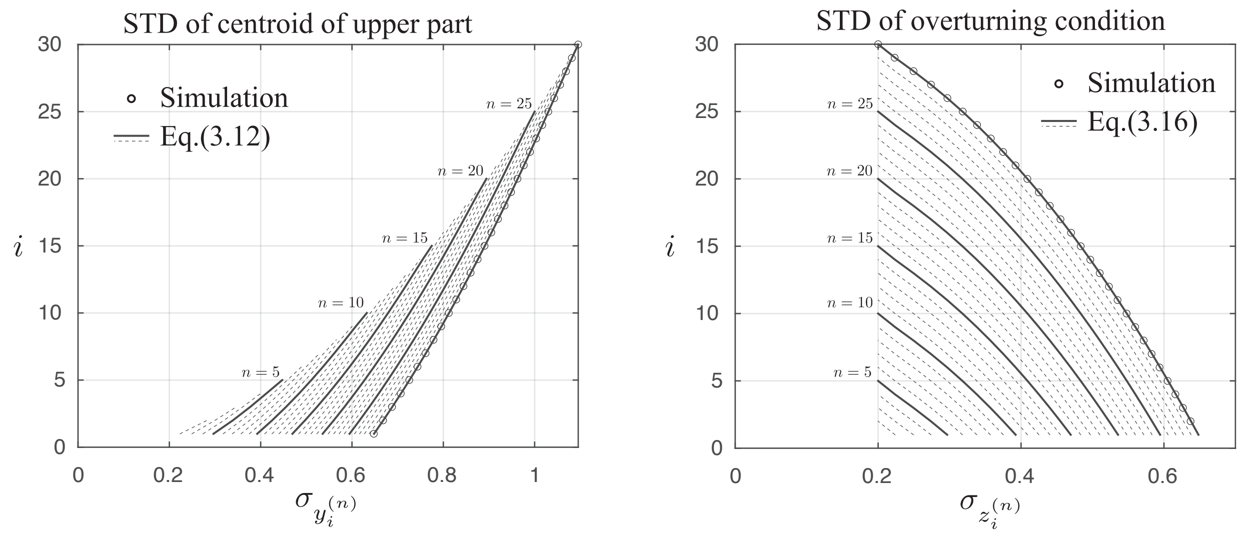

Fig. 6.2: Comparison of analytical solutions and numerical results for the statistics of $y_i^{(n)}$ and $z_i^{(n)}$ $(n = 1, \cdots, 30)$. Left: standard deviation of transverse position of centroid of top part of the tower, above level i; Right: standard deviation of the overturning condition $z_i^{(n)}$, as a function of elevation i of the block in the stack. Obtained for $\sigma_u = 0.2$.

of a stack of any height ($\forall n$), $\sigma_{z_n^{(n)}} = \sigma_u$ (chosen a 0.2 in this illustration) for the same reasons as above. The standard deviation of the overturning condition increases from top to bottom, which translates a more pronounced susceptibility to overturning around a block which is positioned at lower floors in the stack.

The survival probability computed, recursively, with (3.24) is represented in Figure 6.3. The corresponding numerical values are given in Table 1. It is shown for various values of σ_u to anticipate the detailed discussion on the influence of the misalignment error σ_u on the dynamic behavior of the stacking process. The probability that the stack is still stable after having superimposed 5 blocks, $\mathcal{S}_5$ varies from 1.00 to 0.004 as σ_u varies from 0.1 to 2. For a given number n of stacked blocks, the survival probability is therefore significantly decreasing as σ_u increases.

Overall, the agreement between numerical and analytical solutions is very good for all considered statistics, so that the analytical solutions and the numerical simulations can be considered as cross-validated.

7 Detailed Discussion

More extensive numerical simulations have been performed for $\sigma_u \in \{0.1, 0.15, 0.2, 0.3, 0.4, \ 0.6, 0.8, 1, 2, 4\}$, and repeated $nbSim = 50 \cdot 10^6$ times, which is sufficiently large to provide neat empirical distributions. For each simulation, the stack is piled up until collapse, and the height at collapse h as well as the breaking level ℓ are memorized. The joint histogram of these two random quantities is represented in Figure 7.1 for $\sigma_u = 0.6$ (top), $\sigma_u = 0.4$ (middle) and $\sigma_u = 0.2$ (bottom). It is defined over the triangular area $\ell \leq h$, since the breaking level is smaller that the tower height at collapse. The representations on the right show slice cuts in the joint PDF, i.e. (unnormalized) conditional distributions of the breaking level for given tower height at collapse. They are represented for equally spaced values of h, and the solid lines are also represented by dashed lines on the contour plots of the joint distribution. As a first observation, the centroid of the joint distribution

σ_u	0.1	0.15	0.2	0.3	0.4	0.6	0.8	1.	2.
S_1	1.0000	1.0000	1.0000	0.9991	0.9876	0.9044	0.7886	0.6826	0.3831
S_2	1.0000	1.0000	1.0000	0.9960	0.9604	0.7743	0.5727	0.4222	0.1295
S_3	1.0000	1.0000	0.9999	0.9886	0.9180	0.6389	0.3978	0.2489	0.0416
S_4	1.0000	1.0000	0.9997	0.9752	0.8639	0.5143	0.2686	0.1425	0.0130
S_5	1.0000	1.0000	0.9990	0.9556	0.8024	0.4066	0.1780	0.0801	0.0040
S_6	1.0000	1.0000	0.9976	0.9304	0.7374	0.3172	0.1164	0.0443	0.0012
$S_{\grave{e}}$	1.0000	0.9999	0.9951	0.9005	0.6718	0.2449	0.0753	0.0243	0.0004
S_8	1.0000	0.9997	0.9914	0.8669	0.6076	0.1875	0.0483	0.0132	0.0001
S_9	1.0000	0.9994	0.9862	0.8307	0.5463	0.1426	0.0307	0.0071	0.0000
S_{10}	1.0000	0.9989	0.9795	0.7927	0.4887	0.1079	0.0195	0.0038	0.0000
S_{20}	0.9997	0.9712	0.8450	0.4287	0.1355	0.0055	0.0002	0.0000	0.0000
S_{30}	0.9954	0.8967	0.6585	0.2011	0.0322	0.0002	0.0000	0.0000	0.0000
S_{40}	0.9819	0.7943	0.4845	0.0880	0.0071	0.0000	0.0000	0.0000	0.0000
S_{50}	0.9573	0.6833	0.3438	0.0370	0.0015	0.0000	0.0000	0.0000	0.0000
S_{100}	0.7314	0.2566	0.0479	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000

Tab. 1: Survival probability S_n for $n = 1, \cdots, 10$ and above. These values are obtained with the recurrence (3.24) but they also match the results obtained with the Monte Carlo simulation, as shown in Figure 6.3.

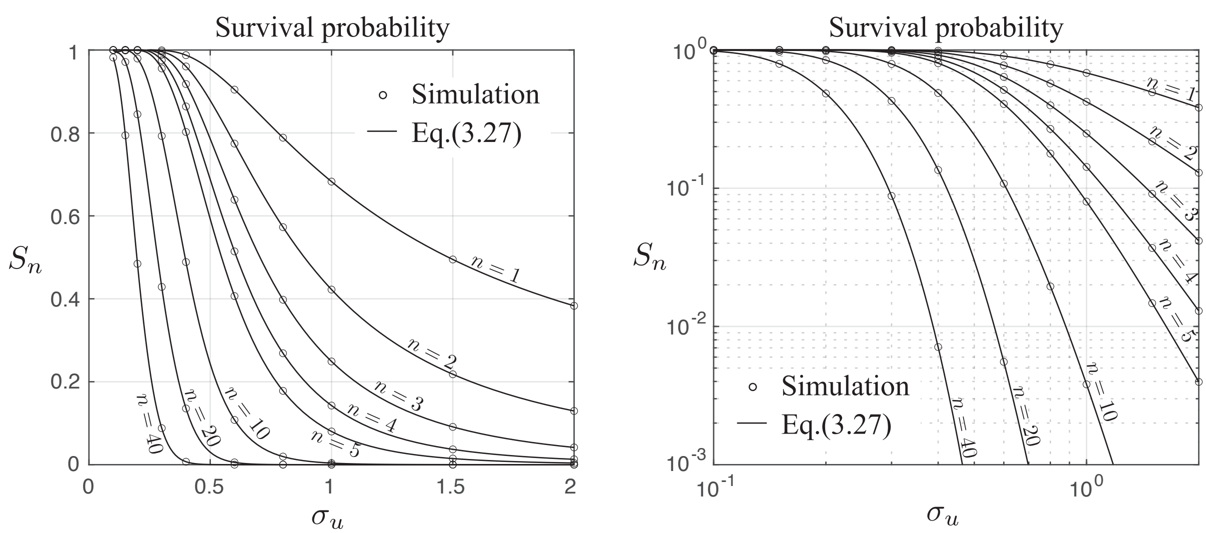

Fig. 6.3: Comparison of the semi-analytical solution 3.24 and numerical results for the survival probability $\mathcal{S}_n$ ($n = 1, \cdots, 5$). Shown in linear scales (left) and log scales (right), and for various values of σ_u.

is located closer to the origin (lower stack heights) for larger uncertainty σ_u, as expected. The marginal distributions are discussed later.

Figure 7.1 illustrates the transition between two limiting cases. Indeed, for large uncertainty (top), $\sigma_u = 0.6$, the distribution of the conditional breaking height is monotonic and increasing almost everywhere. The most probable weakest interface is $\ell = n$ so that the most probable collapse for a given stack at collapse is at the top level. We underline that the experimental distributions are normalized in the sense of a PDF to ease comparison for increasing values of n. In the very rare cases where the stack height can reach up to $h = 20$ and higher, the conditional PDF of the breaking level appears to be steady: as $h \to +\infty$, it tends to be expressed as a function of $h - \ell$. For small positioning uncertainty (bottom), e.g. $\sigma_u = 0.2$, the conditional distribution of the breaking level for large h also appears to be stationary, as $h \to +\infty$. However, it features now a zone located about 10 blocks below the top of the stack where failure is the most probable. This actually results from thin boundary layers in the joint PDF between the stack height at collapse and the breaking level, as indicated in the bottom left of Figure 7.1.

The first specificity of the problem, related to the stationarity of the conditional PDF of the breaking level, can be explained by the nature of the governing equations. Specifically, in the triangular set of equations (3.18), the format of the last few elements in each line is the same. For instance, $z_1^{(2)}$ and $z_2^{(2)}$ are expressed as a function of u_1 and u_2 with the same coefficients as $z_2^{(3)}$ and $z_3^{(3)}$ as a function of u_2 and u_3, which comes from the format of the combination coefficients $\zeta_{i,k}^{(n)}$ being in fact expressed as a function of the two distances $n - k$ and $n - i$ from the top of the stack.

The second specificity consists in the existence of thin boundary layers in the joint PDF of h and ℓ. They develop for small σ, and as $n, h \to +\infty$, a configuration where the problem can be regularized, as described in Section 4. It is difficult to explain the boundary layers in the discrete case, but we can argue that such boundary layers exists in similar first passage problems, in continuous time-space models [23, 24]. They results from the existence of small parameters in the diffusion equation [25], and appear, for instance, as small coefficients in the Pontryagin equation governing the moments of first passage times. An accurate description of the similarity with the boundary layer observed here goes beyond the scope of this paper.

Beside these curiosities, more tangible information as a practical solution of this problem consists of the marginal distributions of h and ℓ. They are shown in Figure 7.2 and obtained by marginalizing the joint PDFs of Figure 7.1. The plots are given with a linear scale on the left and log scale on the right. For $\sigma_u = \sigma_{u,\mathrm{cr}} \simeq 0.8$, close to yellow solid line, the PDF of the height at collapse has a horizontal tangent at the origin, meaning that the tower tumbles most ofter after having placed only 1 block ($h = 1$). The exact value of this threshold, $h = 0.818$, can be determined from the survival probabilities S_1 and S_2, solving $S_2 = 2S_1$ for σ_u. For smaller values of σ_u, corresponding to applications in which alignment errors are better controlled thus limited, the most probable stack height is much larger, for instance $h = 103$ for $\sigma_u = 0.1$ and $h = 27$ for $\sigma = 0.2$. The distributions are heavily skewed to the right, with non negligible probabilities of reaching stack heights much larger than the mode. For instance, 1% of the stacks collapse for a height larger than 502 blocks when $\sigma_u = 0.1$.

Although the conditional collapse level is bimodal as $n \to +\infty$, as illustrated in Figure 7.1, the marginal distribution of the collapse level is monotonically decreasing. Overall, this indicates that the most probable way for the stack to collapse is by overturning around a block at the bottom of the stack. However, on average it is observed that the position ℓ of the weak interface increases as $\sigma_u \to 0$.

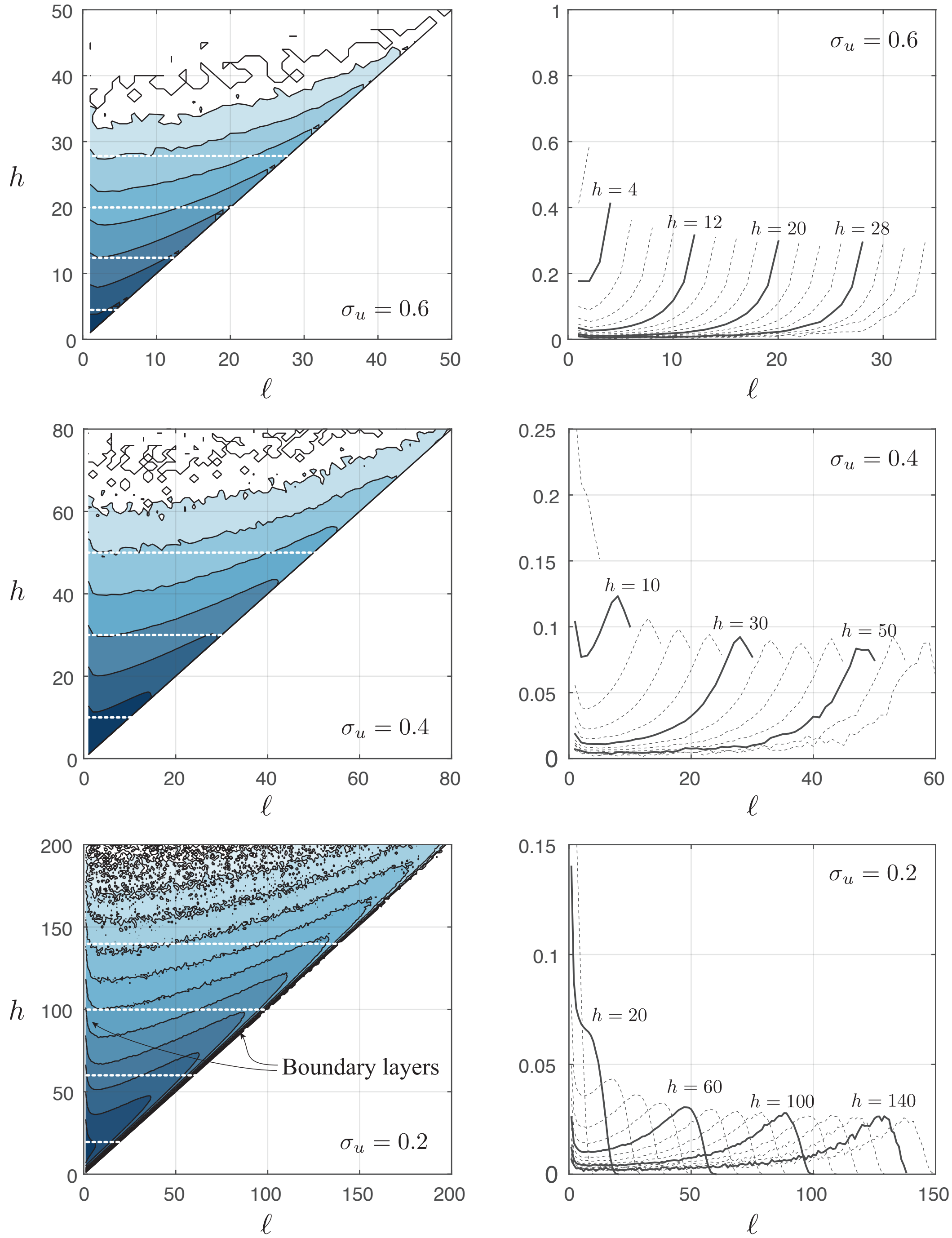

Fig. 7.1: (a) Joint PDF of the stack height at collapse h and the breaking level ℓ, (b) conditional PDF of the breaking level for various values of h. (Shown for $\sigma_u = 0.6$ (top), $\sigma_u = 0.4$ (middle) and $\sigma_u = 0.2$ (bottom).

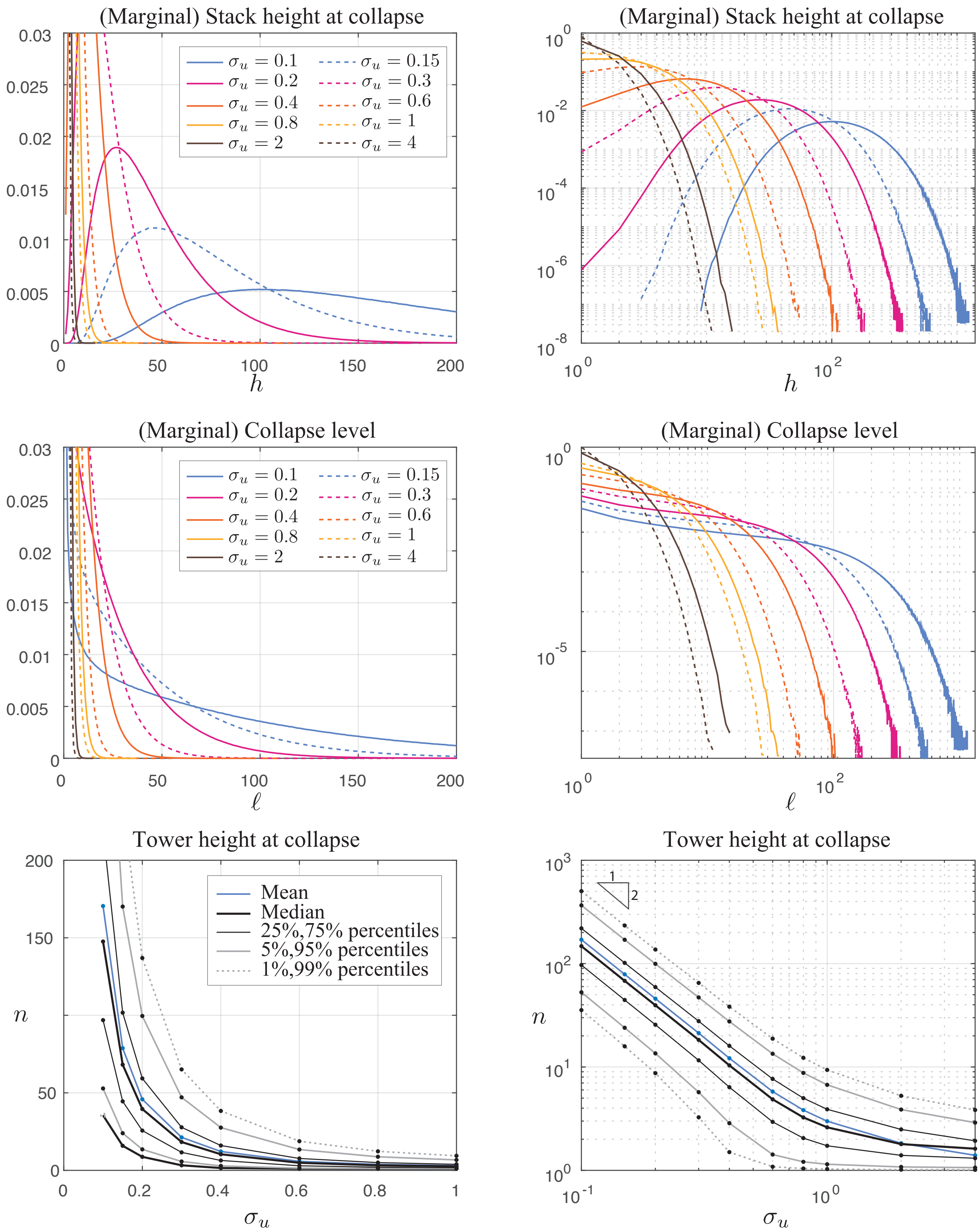

Fig. 7.2: (a) Probability distribution of the stack height at collapse h, (b) (a) Probability distribution of the breaking level ℓ , (c) statistics of the stack height at collapse h. Graphs in the left and right are the same, but shown with linear and log scales (the color palette is colorblind-friendly; colored version online)

σ_u	0.1	0.15	0.2	0.3	0.4	0.6	0.8	1.	2.	4.
$1/\sigma_u^2$	100.	44.4	25.	11.1	6.25	2.78	1.56	1.	0.24	0.06
mode	103	46	27	12	7	3	2	1	1	1
median	147.59	68.15	39.56	18.28	10.37	4.87	3.25	2.61	1.79	1.62
mean	170.48	78.82	45.83	21.28	12.16	5.79	3.83	2.99	1.83	1.40
P-1%	35.51	15.85	8.72	3.26	1.50	1.08	1.04	1.03	1.02	1.01
P-5%	52.82	23.96	13.54	5.71	2.86	1.42	1.21	1.15	1.08	1.06
P-25%	96.89	44.51	25.65	11.60	6.38	2.94	2.05	1.73	1.40	1.31
P-75%	219.57	101.70	59.28	27.76	16.01	7.66	5.00	3.90	2.49	1.93
P-95%	366.27	170.13	99.55	47.09	27.58	13.42	8.77	6.72	3.87	2.90
P-99%	502.73	233.81	136.96	65.10	38.33	18.80	12.28	9.37	5.26	3.84

Tab. 2: Statistics of the stack height at collapse h. Mean, mode, median and percentiles 1%, 5%, 25%, 75%, 95%, 99%.

The mode, median and mean tower height at collapse, as well as some percentiles, are represented in the bottom part of Figure 7.2. The corresponding numerical values are given in Table 2. The two limiting behaviors are also clearly illustrated with these statistics. For large positioning uncertainty, all statistics of the tower height at collapse tend to $h = 1$. On the other end, it is observed that they all grow proportionally to σ_u^2 as $\sigma_u \to 0$. With $\sigma_u = 0.1$, it appears we have reached the asymptote and it is deemed reasonable to avoid the much longer simulations corresponding to smaller values of σ_u. Instead, extrapolation with the 1-to-2 slope in the log scale seems possible. In particular, for $\sigma_u < 0.1$, the mode and the 25$^{\text{th}}$ percentile ($P = 25\%$) can be approached by $1/\sigma_u^2$.

8 Conclusions

Based on the analysis presented in this paper, several key insights into the stochastic stability of randomly stacked blocks have been uncovered. The study systematically investigated the statistical properties of block stacking, considering the uncertainties introduced by the random misalignments of each block. The study found that the stack height at collapse scales asymptotically with the square of the positioning error, and the most probable stack height is approximately $1/\sigma_u^2$, which aligns closely with the 25th percentile.

For a given stack height at collapse h, the conditional distribution of the breaking level has revealed two probable scenarios: either a weak interface at the base of the stack or at some position slightly below but near the top of the stack. This phenomenon was shown to be related to a boundary layer in the joint probability density function (PDF) of the tower height at collapse and the breaking level.

In essence, this seemingly whimsical problem has unveiled rich mathematical concepts related to first-passage problems in stochastic fields, including non-homogeneous non-standard Markovian chains. Possible applications find relevance in diverse scientific and engineering domains. From a theoretical standpoint, further works could study extensions of this basic problem by considering, for example, that the blocks themselves have random geometry or random density, could be stacked in 3-D, or focusing on the continuous version of the problem as the stack height grows.

9 Appendix

A useful expression for the computation of some integrals

$$\int_{\mathbb{R}} x \exp\left[-\alpha^2\left((x-x_0)^2+\beta^2\right)\right] dx = \frac{\sqrt{\pi}e^{-\alpha^2\beta^2}x_0}{\alpha} \tag{9.1}$$

Details of the derivation of (3.7)

$$\begin{aligned}
\sigma_{y_i}^2 &= \frac{b^2}{(n+1-i)^2}\sum_{j_1=i}^{n}\sum_{j_2=i}^{n}\min(j_1,j_2)\sigma_u^2 \\
&= \frac{b^2\sigma_u^2}{(n+1-i)^2}\sum_{j_1=i}^{n}\left(\sum_{j_2=i}^{j_1} j_2 + \sum_{j_2=j_1+1}^{n} j_1\right) \\
&= \frac{b^2\sigma_u^2}{(n+1-i)^2}\sum_{j_1=i}^{n}\left(\frac{(j_1+i)(j_1-i+1)}{2} + j_1(n-j_1)\right) \\
&= \frac{b^2\sigma_u^2}{6(n+1-i)^2}\sum_{j_1=i}^{n}\frac{i-i^2+j_1(2n+1)-j_1^2}{2} \\
&= \frac{b^2\sigma_u^2}{6}\frac{2n^2+2in+n-4i^2+5i}{n+1-i}.
\end{aligned}$$

Details of the derivation of (3.9)

$$\begin{aligned}
\operatorname{cov}\left(y_i^{(n)}, x_{i-1}\right) &= \mathbb{E}\left[\frac{x_{i-1}}{n+1-i}\sum_{j=i}^{n} x_j\right] = \frac{1}{n+1-i}\sum_{j=i}^{n}\operatorname{cov}(x_{i-1},x_j) \\
&= \frac{1}{n+1-i}\sum_{j=i}^{n}(i-1)\sigma_u^2 = (i-1)\sigma_u^2.
\end{aligned}$$

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