Derivative-Free Arclength Control-Based Continuation for Secondary Resonances Identification

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ABSTRACT
Mechanical structures exhibiting nonlinear dynamical behaviors possess co-existing stable or unstable periodic solutions at given excitation parameters, with possible jumps between solutions when the parameters are varied. In addition to their fundamental resonances, they can resonate at any multiple or fraction of the excitation frequency, a phenomenon known as superharmonic or subharmonic resonance, respectively, which has received less attention so far in the technical literature.

Control-based approaches have been developed for more systematic and reliable testing of nonlinear structures. This work exploits a novel derivative-free method for experimental continuation, termed arclength control-based continuation (ACBC). An extended version of ACBC (x-ACBC) relying on a double-control strategy is proposed herein. We show that this approach can successfully identify superharmonic and subharmonic resonances. Furthermore, x-ACBC can transition effectively to isolated branches of periodic solutions by leveraging a temporarily invasive control. Numerical and experimental demonstrations of x-ACBC carried out on a Duffing oscillator serve to validate the proposed developments.

Keywords: Control-based continuation, Secondary resonances, Derivative-free method, Isolated responses

INTRODUCTION
Despite significant progress in the theoretical understanding and numerical prediction of nonlinear dynamics, designing mechanical systems with nonlinear behavior remains a challenge. Developing accurate models requires experimental data, and analyzing this data is often complex and time-consuming.

Traditional open-loop experimental testing methods are insufficient for characterizing nonlinear structures. To address this, control-based methods have been introduced to simplify the identification of nonlinear responses, requiring no prior knowledge of the system. Experimental continuation methods combine non-invasive feedback control and continuation to construct complete bifurcation diagrams experimentally, stabilizing any open-loop response.

Control-based nonlinear vibration testing has developed into two primary approaches: phase-locked loop (PLL) testing [1] and control-based continuation (CBC) [2]. The key distinction between these methods lies in the controlled experimental parameter. In PLL, the phase lag is controlled, while in CBC, it is the amplitude of the response's first harmonic. Establishing a one-to-one relationship between the control parameter and the bifurcation parameter is crucial for both methods.

PLL testing and CBC are widely used for characterizing primary resonances, and until recently, experimental tests focused mainly on this resonance. Higher harmonics can also offer valuable insights into a system's behavior. Nonlinear modes can resonate when a structure is excited at multiples or fractions of the primary resonance frequency, often producing harmonics with larger amplitudes than the fundamental harmonic. Characterizing secondary resonances using traditional CBC is challenging due to the presence of bifurcation parameter folds. Using phase resonance nonlinear modes [3], PLL testing can be applied to identify backbone curves and nonlinear frequency response curves (FRCs) around superharmonic and subharmonic resonances [4, 5]. However, the full isola cannot be captured because it cannot be uniquely parametrized by the phase lag.

Both PLL and CBC methods exploit the control of a specific bifurcation parameter to unfold FRCs, although this is not always guaranteed. Identifying folded FRCs directly would require numerical continuation methods that involve derivative computations [6], which are difficult to perform experimentally due to unavoidable noise. A new approach, called arclength control-based continuation (ACBC), was proposed, which is conceptually simple and derivative-free [5]. Initial experimental results demonstrated its effectiveness in identifying the fundamental resonance of a geometrically nonlinear beam. Besides managing bifurcations and unstable branches, the algorithm successfully identified an isolated branch of periodic solutions containing both stable and unstable segments. Limitations were noted in detecting superharmonic resonances in an electronic Duffing oscillator [7].

This work introduces an extended version of ACBC, referred to as x-ACBC, which enables the identification of response curves near secondary resonances in an experimental Duffing oscillator, as outlined in [8]. (Ultra-)subharmonic resonances may manifest as attached or isolated branches of periodic responses. The x-ACBC method facilitates transitions between the main response branch and these isolated branches.

BACKGROUND

A simplified version of the original CBC is considered here [2]. CBC stabilizes any orbit of the uncontrolled system by indirectly enforcing the response amplitude. Due to the presence of nonlinearities in the experiment, the system exhibits a multi-harmonic response. As illustrated in Figure 1, the measured response, x, is compared to a reference response, x^*, to generate an error signal, $x^* - x$, which is then fed into a PID controller. In contrast to traditional feedback control, the objective of the control-based experiment is not to ensure that x matches the reference x^*. Rather, it involves adjusting the reference x^* itself until the controller produces an output that effectively excites the structure as intended.

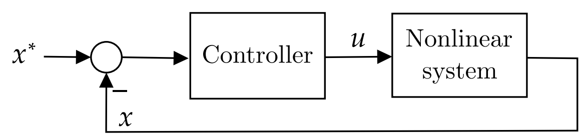


Figure 1: Block diagram of control-based experiment [8].

If the controller is non-invasive, meaning its output is mono-harmonic, the stabilized orbit aligns with the desired open-loop orbit. Since the PID controller is linear and time-invariant, its output remains mono-harmonic as long as the error, $x^* - x$, is also mono-harmonic. Non-invasiveness is guaranteed when the non-fundamental harmonics of the reference match those of the actual response, i.e., $x^*_{\mathrm{nf}} = x_{\mathrm{nf}}$. This condition can be satisfied either through offline Picard iterations or online adaptive filters.
Once this condition is met, the first harmonic of x^* controls the excitation level. The fundamental amplitude of x does not equal the one of x^*, but it converges to it as the controller gain is increased. With sufficiently high gain, the fundamental amplitude becomes a monotonic function of the excitation amplitude, allowing for sequential continuation to trace the S-curve at a specific frequency. A response surface can be constructed by measuring multiple S-curves across different frequencies. This surface can then be sliced at a constant force level to identify an FRC.
The main limitation of this CBC version is that it explores a large portion of the response surface, whereas only a specific segment is typically needed.

Arclength control-based continuation (ACBC) is a novel method designed to directly identify FRCs, eliminating folding issues and jumps. The structure of the control-based experiment is similar to CBC, with the key difference being the choice of the reference signal. As in CBC, the non-fundamental harmonics of the reference x^* are determined using adaptive filters to meet non-invasiveness criteria. The continuation process involves combining the excitation frequency with the fundamental harmonic of the response.
Figure 2 illustrates the basic idea of the arclength continuation. Assuming distinct solutions rarely share the same fundamental amplitude at a given frequency, FRCs are essentially one-dimensional in the plane of fundamental amplitude versus excitation frequency. A small ellipse, centered on a point of the branch, will intersect the curve at two points. At these intersections, the force from the controller u matches the target excitation amplitude u^*. Other points on the ellipse correspond to FRCs at higher

or lower forcing levels.

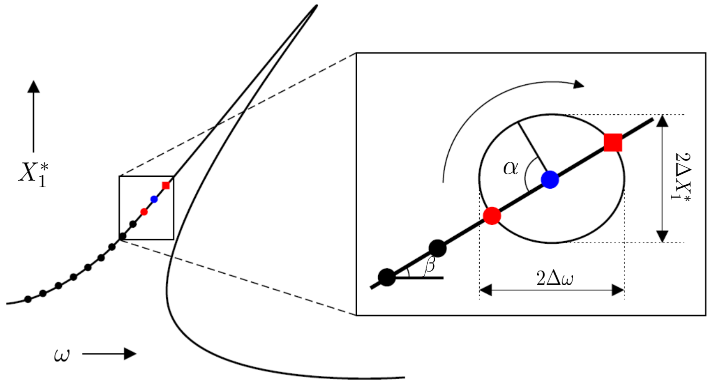


Figure 2: Illustration of the arclength continuation method introduced in [5]. Circles indicate previous responses, the square marks the next response, the blue point is the ellipse's center, and the red markers are intersections along the arc.

The continuation procedure is as follows: once two points on the FRC are identified, a small ellipse is drawn, with the most recent point at the center and the previous one on the ellipse. Starting from this intersection, the ellipse is swept by varying the eccentric anomaly α, until reaching the next intersection. This provides the next FRC point, which becomes the center of the subsequent ellipse.
Different strategies can be applied to adjust the sweeping rate. While a higher sweeping rate enables faster continuation, it is constrained by the system's transient response and the speed of the adaptive filters to ensure non-invasive control. A continuous integral law, $\dot{\alpha} = k_i(u - u^*)$, controls the angle α. This ensures the algorithm converges toward intersections and prevents retracing previously identified points.

Both CBC and ACBC assume that distinct FRC solutions do not share the same fundamental amplitude at a specific excitation frequency. While this holds for fundamental resonances, it breaks down near secondary resonances. In these regions, multiple system responses at different forcing levels may exhibit the same fundamental amplitude at a given excitation frequency, although their non-fundamental components differ [5, 7].
x-ACBC is an enhanced version of ACBC that addresses this limitation by introducing control over the non-fundamental resonant harmonic, enabling the identification of secondary resonances. This allows it to distinguish between distinct responses that would otherwise overlap in the (A)CBC working plane. An example can be considered when the l^{th} harmonic is resonating. In Figure 3 (left plot), each solution in the region with three solutions is characterized by a unique pair of Fourier coefficients $(s_{x,l}, c_{x,l})$. An ellipse with a zero-frequency semi-major axis is considered for the arclength sweep, forming a vertical line to simplify the subsequent explanations.

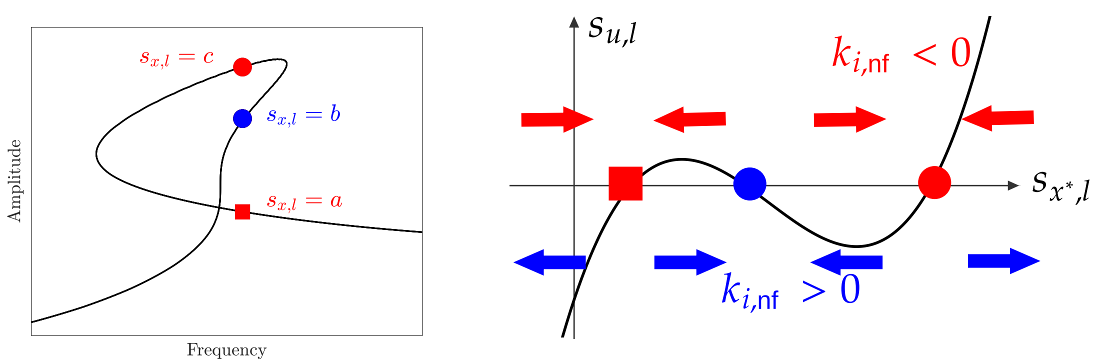


Figure 3: Left: FRC around a secondary resonance, with colored points indicating solutions on different branches of this resonance. Right: Variation of a non-fundamental resonant harmonic Fourier coefficient of the control action, $s_{u,l}$, with respect to the controlled non-fundamental resonant harmonic Fourier coefficient of the reference displacement, $s_{x^*,l}$.

Control is applied to one of the l^{th} harmonic Fourier coefficients of the reference, e.g., $s_{x^*,l}$. This non-fundamental control aims to match the controlled non-fundamental Fourier coefficient $s_{x^*,l}$ with one of the three non-fundamental Fourier coefficients $s_{x,l}$. The error $s_{x^*,l} - s_{x,l}$ is the input to the non-fundamental control. An integral controller ensures smooth convergence to the sought solution, defined by the control law $\dot{s}_{x^*,l} = k_{i,\text{nf}}\ s_{u,l} = k_{i,\text{nf}}\ (s_{x^*,l} - s_{x,l})$. As shown in Figure 3 (right plot), when multiple solutions are possible, the sign of the controller gain $k_{i,\text{nf}}$ and the initial value of the controlled Fourier coefficient $s_{x^*,l}$ determine which solution will be stabilized.

Non-invasiveness is not guaranteed throughout the arclength sweep. It is only achieved when the controlled non-fundamental component of the reference $s_{x^*,l}$ matches that of the response $s_{x,l}$, i.e., $s_{u,l} = 0$. An FRC solution is found when the arclength sweep intersects with the desired FRC and the non-fundamental control ceases to be invasive.

Control of the resonating non-fundamental harmonic is performed simultaneously with the fundamental sweep, which is assumed to be faster than the non-fundamental sweep. This allows the non-fundamental reference to remain approximately constant while the fundamental component is adjusted.

By changing the sign of the controller gain $k_{i,\text{nf}}$ and the initial value of the controlled Fourier coefficient $s_{x^*,l}$, distinct solutions can be identified at a specific excitation frequency, including isolated branches of response. This approach is valid as long as the arc is sufficiently large in the fundamental amplitude versus the excitation frequency plane.

ANALYSIS

The x-ACBC algorithm was tested experimentally on an electronic Duffing oscillator subjected to harmonic forcing. The setup is depicted in Figure 4. The electronic Duffing oscillator [7] is an electronic circuit designed to implement a weakly dissipative oscillator with very strong nonlinearity and relatively small resonance frequencies. This setup offers a significant advantage, allowing x-ACBC to face traditional experimental challenges without involving shaker-structure interactions. The electronic system can be associated with equivalent coefficients similar to those in the Duffing equation, which is expressed as follows

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = p\sin\omega t \tag{1}$$

The equivalent coefficients are $m = 10^{-4}$ [s^2], $c = 4.9\,\text{e}-4$ [s], $k = 1.68$ [-], and $k_3 = 0.99$ [V^{-2}]. The forcing level p is 2 [V]. FRCs are identified with the ACBC and x-ACBC.

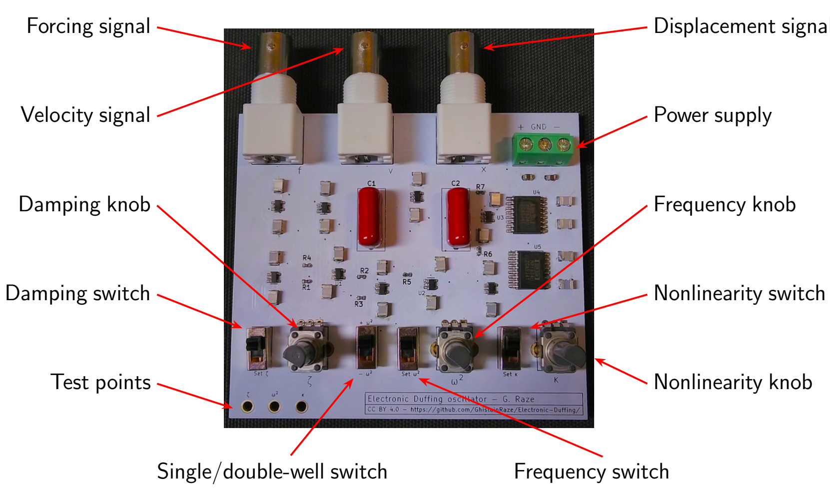


Figure 4: Setup of the electronic Duffing system [7].

Figure 5 represents the identified bifurcation diagram of the electronic Duffing. Assessing stability in CBC presents a significant challenge in experimental research. The objective of this study is to identify both stable and unstable open-loop responses. The subsequent figures will not distinguish between stable and unstable solutions.

The fundamental resonance was identified using ACBC. Similar to a hardening Duffing oscillator, the fundamental resonance of the electronic oscillator shifts to higher frequencies with an increase in the forcing level. Behaviors such as multistability and amplitude-frequency dependence are observed. The complete identification of secondary resonances is only possible using the x-ACBC algorithm. Three secondary resonances are identified: two superharmonic resonances (2 : 1 and 3 : 1), and one subharmonic resonance (1 : 3). Secondary resonances occur at values greater than one-third, one-half, or three times the natural

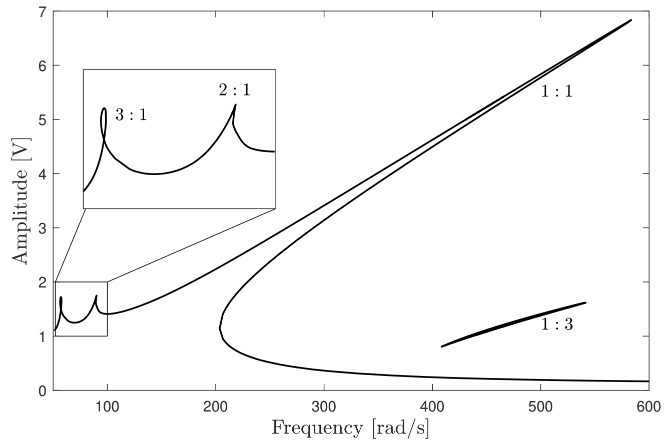
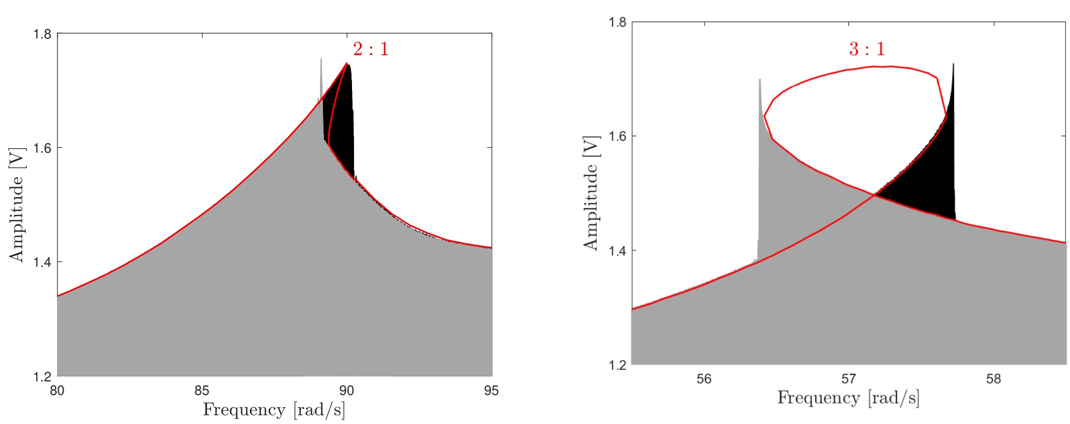


Figure 5: Bifurcation diagram of the Duffing oscillator with a forcing amplitude of 2 [V]. The FRC was identified using ACBC and x-ACBC. Stable and unstable open-loop solutions are represented in the same way.

frequency (130 rad/s). The system exhibits strong nonlinear behavior due to its high forcing level and cubic stiffness. The identification of other secondary resonances was not pursued in this study.
Unlike in the numerical Duffing oscillator, the pair-secondary resonances (here the 2 : 1 resonance) appear attached to the main branch due to the asymmetry present in the electronic system. These asymmetries come from unavoidable offsets.

Isolated branches of solution require specific identification methods as they do not lie in the direct continuation of the frequency response branch. Using the x-ACBC algorithm, the 1 : 3 subharmonic resonance detection begins from a point on the main branch at a frequency near three times the natural frequency. The isolated subharmonic resonance was detected and fully identified experimentally, a remarkable result. A complete isolated subharmonic resonance had already been identified in [9] using an ad hoc method to detect the isola. The x-ACBC algorithm offers a more robust detection approach.

Figure 6 compares the FRC identified with the (x-)ACBC methods with swept sine tests for a 2 [V] forcing amplitude. The (x-)ACBC method matches swept-up and -down sine results in stable solutions and identifies unstable solutions or complex resonance shapes, which open-loop tests cannot. Nonlinear hysteresis is observed by reversing the swept sine test, revealing distinct branches in regions with multiple solutions. The jump phenomenon manifests when the system transitions from multiple possible solutions to a single solution.
Close to fold bifurcations, identification becomes more challenging because the Fourier coefficients of the merging branches are closer in value, suggesting that a reduced integral gain may be needed for the non-fundamental control.



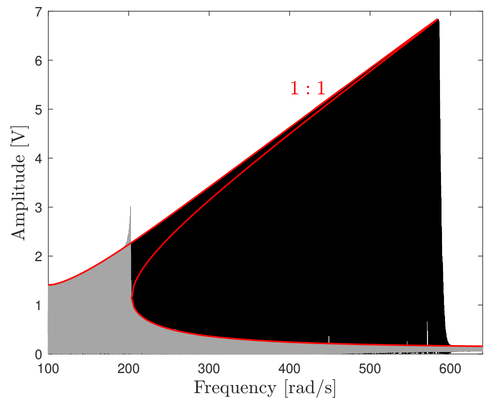


Figure 6: FRC around the 2 : 1, 3 : 1, and 1 : 1 resonances of the electronic Duffing oscillator at $p = 2$ [V]. The (x-)ACBC results (—) are compared with swept-up (—) and swept-down (—) sine tests.

CONCLUSION

This work introduced an extended version of the ACBC method, a novel online, derivative-free method for experimental continuation. By controlling the non-fundamental resonant harmonic, x-ACBC can detect and identify complex superharmonic and subharmonic resonance branches. The method was demonstrated experimentally using an electronic Duffing system. Beside the fundamental resonance, three secondary resonances, namely 3:1, 2:1 and 1:3 resonances, could be completely characterized.

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