ON THE FRACTAL PROPERTIES OF GENERALIZED CANTOR SETS AND DEVIL'S STAIRCASE FUNCTIONS

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Abstract. We present a generalization of Cantor's ternary set and ternary function through the Cantor expansion of real numbers. Our analysis demonstrates that the Hausdorff dimension of these generalized sets and the Hölder regularity of the corresponding functions are influenced by the specific sequence used to define the expansion. Additionally, we explore the measure-theoretic properties of these sets through their Hausdorff h-measure, highlighting the connections between numeral systems and fractal geometry.

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1. Introduction

The intricate interplay between discrete mathematics, number theory, and the notion of fractals has long been recognized [7, 14, 20, 16, 2, 28]. This relationship persists in the investigation of fractal regularity in functions [14, 15, 16, 17], a topic that remains underexplored in the existing literature. Notably, substituting the decimal numeral system with more exotic ones often induces profound changes in a function's fractal characteristics [16, 24]. This work exemplifies such phenomena by generalizing Cantor's ternary set and ternary function. The sets analyzed herein are part of the class of Moran sets [10], a subject of extensive generalization in prior studies (e.g., [10, 8, 3, 22, 19, 27, 1, 18]). However, to the best of our knowledge, no prior exploration has directly employed generalized numeral systems.

Cantor's ternary set, $\mathfrak{C}$, was introduced as a quintessential example of a perfect set that is nowhere dense [11]. Its Hausdorff dimension (see Definition 2.1) is well-known to be $\alpha_0 = \log 2/\log 3$. Similarly, the Devil's staircase (also called the Cantor function), $\mathfrak{D}$ - a paradigmatic singular function [26] - is continuous and non-constant, with a derivative that vanishes outside $\mathfrak{C}$, almost everywhere. The Hölder regularity (see Definition 3.1) of $\mathfrak{D}$ on $\mathfrak{C}$ matches α_0, as the Hölder exponent at any point in $\mathfrak{C}$ is precisely α_0. Intuitively, this result can be explained from the fact that, on $\mathfrak{C}$, $\mathfrak{D}$ associates a number in base 2 to a number in base 3 in a very simple way. This study extends both $\mathfrak{C}$ and $\mathfrak{D}$ to explore the Hausdorff dimension of the resulting sets and the Hölder exponents of the corresponding functions. Under our framework, these quantities can achieve values as high as 1, relying on the Cantor expansion of real numbers.

Let $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ be a sequence of natural numbers greater than 1. A representation of the form

$$\sum_{j=1}^{\infty} \frac{x_j}{q_1 q_2 \cdots q_j}, \tag{1}$$

where $x_j \in 0, 1, \ldots, q_j - 1$ for all j, is referred to as a Cantor series [4, 5]. By setting $q_j = b > 1$ for all j, this framework recovers the familiar representation in base b. For $x \in [0, 1]$, we denote its representation in base $\mathbf{q}$ as $x = (x_1, x_2, \ldots)_{\mathbf{q}}$. Notably, this representation admits proper and improper forms, depending on the values of x_j.

To two sequences $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ and $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ of natural numbers, we associate a new sequence $\mathbf{q}' = (q'_j)_{j\in\mathbb{N}}$ defined as

$$q'_j = n_j(q_j - 1) + 1. \tag{2}$$

In the sequel, we will always assume that $q_j > 1$ and $n_j > 1$ for all $j \in \mathbb{N}$. We then define the set

$$C_{\mathbf{q},\mathbf{n}} = \{(x_1, x_2, \ldots)_{\mathbf{q}'} : \; x_j \in n_j\mathbb{N}, \, \forall j\},$$

which generalizes the Cantor ternary set $\mathfrak{C}$ when $q_j = n_j = 2$ for all j. Like $\mathfrak{C}$, $C_{\mathbf{q},\mathbf{n}}$ is totally disconnected and can be constructed iteratively, as noted in Remark 2.2.

A generalization of the Devil's staircase, $D_{\mathbf{q},\mathbf{n}}$, is similarly defined:

$$D_{\mathbf{q},\mathbf{n}} : C_{\mathbf{q},\mathbf{n}} \to [0, 1] \quad (n_1p_1, n_2p_2, \ldots)_{\mathbf{q}'} \mapsto (p_1, p_2, \ldots)_{\mathbf{q}},$$

This function extends continuously to $[0, 1]$ via

$$D_{\mathbf{q},\mathbf{n}}(x) = \sup\{D_{\mathbf{q},\mathbf{n}}(y) : y \leq x, \; y \in C_{\mathbf{q},\mathbf{n}}\}.$$

For $q_j = 2$ and $n_j = 2$, $D_{\mathbf{q},\mathbf{n}}$ reduces to $\mathfrak{D}$. Following techniques employed for $\mathfrak{D}$ [6], it can be shown that $D_{\mathbf{q},\mathbf{n}}$ is nowhere differentiable on $C_{\mathbf{q},\mathbf{n}}$. Figure 1 illustrates one such example, with $D_{\mathbf{q},\mathbf{n}}$ exhibiting a structure reminiscent of $\mathfrak{D}$, as anticipated from Remark 2.2.

In this work, after observing that $C_{\mathbf{q},\mathbf{n}}$ belongs to the class of Moran, whose Hausfordd dimension is consequently well-established [9, 10], we proceed to compute the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ for a broad class of sequences $\mathbf{q}$ and $\mathbf{n}$. Specifically, when $\mathbf{q}$ and $\mathbf{n}$ are constant sequences, the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ and the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point in $C_{\mathbf{q},\mathbf{n}}$ are equal to (6), with the standard case corresponding to $q_j = 2$ and $n_j = 2$ for all j. We also explore cases where $\mathbf{q}$ deviates significantly from a constant sequence. For instance, we consider scenarios where q_j is the $j + 1$-th prime number (see Example 3.12). More generally, we establish a straightforward growth condition on the sequence $\mathbf{q}$ such that the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ attains the maximum value of 1 (albeit with an associated measure that vanishes), and the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at $x \in C_{\mathbf{q},\mathbf{n}}$ also equals 1.

For $s \in [0, 1]$, an s-set is defined as a set whose s-dimensional Hausdorff measure is finite and positive (see Definition 2.1). We provide examples of sets $C_{\mathbf{q},\mathbf{n}}$ that fail to qualify as s-sets (see Corollary 3.11). To further analyze such sets, we leverage the Hausdorff h-measure, which extends the classical Hausdorff measure framework (see Definition 4.1) [13, 25]. Building on the methods introduced in [3], we compute a Hausdorff h-measure of $C_{\mathbf{q},\mathbf{n}}$, revealing how the growth of the sequence $\mathbf{q}$ impacts both the dimensional and measure-theoretic properties of these sets.

This study presents a framework that highlights novel connections between numeral systems and the geometric intricacies of such sets.

2. The construction of $C_{\mathbf{q},\mathbf{n}}$

It is interesting to note that $C_{\mathbf{q},\mathbf{n}}$ inherits the fractal properties of $\mathfrak{C}$. More precisely, the usual approach for computing the Hausdorff measure of $\mathfrak{C}$ can be extended to $C_{\mathbf{q},\mathbf{n}}$.

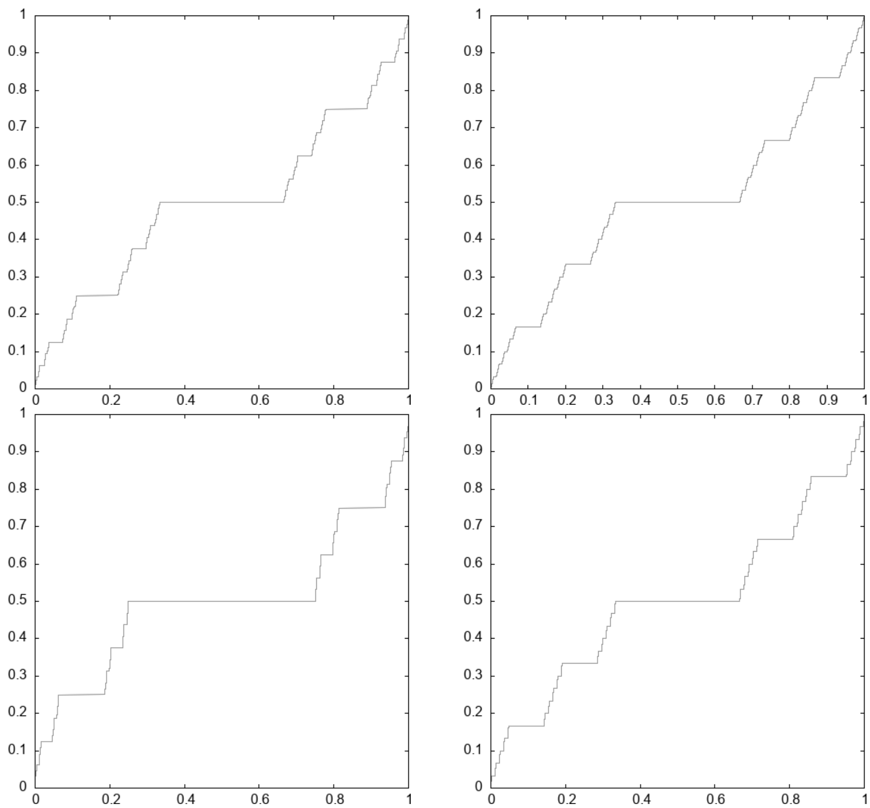

FIGURE 1. In the upper left, the so-called Devil's staircase $\mathfrak{D}$ is depicted In the upper right, the function $D_{\mathbf{q},\mathbf{n}}$ is shown, where q_j represents the $(j+1)$-th prime number and $n_j = 2$ for all j. The lower left illustrates the function $D_{\mathbf{q},\mathbf{n}}$ with $q_j = 2$ and $n_j = 3$ for all j, while the lower right presents the function $D_{\mathbf{q},\mathbf{n}}$ where q_j is the $(j+1)$-th prime number and $n_j = q_j$ for all j.

Let us first briefly introduce the notions of Hausdorff measure and Hausdorff dimension (for more details, see e.g. [13, 7, 25]).

Definition 2.1. Given a set E of $\mathbb{R}^n$ and $s > 0$, the quantity

$$\mathcal{H}^s(E) = \sup_{\epsilon>0} \inf\{\sum_{j=1}^{\infty} |E_j|^s : E \subset \bigcup_{j=1}^{\infty} E_j,\ |E_j| < \epsilon\} \tag{3}$$

is called the s-dimensional Hausdorff (outer-)measure of E. The Hausdorff dimension of a non empty set E is the real number

$$\sup\{s : \mathcal{H}^s(E) > 0\}.$$

Since the diameter of a set is the same as the diameter of its convex hull, we may assume that the E_j in the previous definition are convex sets. If $s > 0$ is the Hausdorff dimension of E, we have $\mathcal{H}^{s+\epsilon}(E) = 0$ and $\mathcal{H}^{s-\epsilon}(E) = \infty$ for any $\epsilon > 0$ such that $s - \epsilon > 0$.

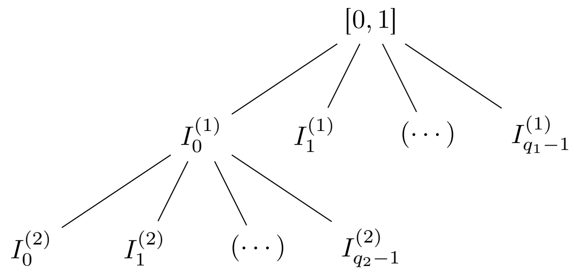

FIGURE 2. Illustration of the construction of the intervals $I_k^{(\ell)}$ from Remark 2.2.

It is easy to show that $C_{\mathbf{q},\mathbf{n}}$ can be defined as a countable intersection of compact sets: $\mathfrak{C}$ can be obtained by iteratively deleting the open middle third from a set of intervals and this construction can be generalized for $C_{\mathbf{q},\mathbf{n}}$.

Remark 2.2. To build $C_{\mathbf{q},\mathbf{n}}$, the first step consists in removing open intervals from $K_0 = [0, 1]$ to obtain the set $K_1 = \cup_{j=0}^{q_1-1} I_j^{(1)}$, with

$$I_j^{(1)} = [(n_1 j, 0, \dots)_{\mathbf{q}'}, (n_1 j + 1, 0, \dots)_{\mathbf{q}'}],$$

if $j < q_1 - 1$ and

$$I_{q_1-1}^{(1)} = [(n_1(q_1 - 1), 0, \dots)_{\mathbf{q}'}, 1].$$

Next, one removes open intervals from each $I_j^{(1)}$ to obtain sets of the form $\cup_{k=0}^{q_2-1} I_k^{(2)}$ with

$$I_k^{(2)} = [(p, n_2 k, 0, \dots)_{\mathbf{q}'}, (p, n_2 k + 1, 0, \dots)_{\mathbf{q}'}],$$

for some $p \in n_2\mathbb{N}_0$. The union of all these intervals is denoted K_2. If K_ℓ is a finite union of closed intervals $I_j^{(\ell)}$, one removes open intervals from each $I_j^{(\ell)}$ so that the subset $K_{\ell+1}$ of K_ℓ is the finite union of intervals of the form

$$I_k^{(\ell+1)} = [(*, n_{\ell+1} k, 0, \dots)_{\mathbf{q}'}, (*, n_{\ell+1} k + 1, 0, \dots)_{\mathbf{q}'}], \tag{4}$$

with $k \in \{0, \dots, q_{\ell+1} - 1\}$ (with an obvious modification if $k = q_{\ell+1} - 1$), where $*$ is a prefix of length ℓ, so that the numbers $n_{\ell+1}k$ and $n_{\ell+1}k + 1$ are at the $\ell + 1$-th position in the endpoints of the interval. By construction, we have $C_{\mathbf{q},\mathbf{n}} = \cap_\ell K_\ell$.

Remark 2.3. At iteration ℓ, the distance between two consecutive intervals $I_k^{(\ell)}$ is $n_\ell - 1$ times the length of each $I_k^{(\ell)}$, while the ratio between the lengths of $I_k^{(\ell)}$ to $I_j^{(\ell+1)}$ is equal to $q'_{\ell+1}$. An interval $I_k^{(\ell)}$ generates $q_{\ell+1}$ intervals of the form $I_j^{(\ell+1)}$ at iteration $\ell + 1$.

Given sequence $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ of natural numbers, let us set $r_j = q_1 q_2 \cdots q_j$ and $r'_j = q'_1 q'_2 \cdots q'_j$, where q'_j is specifieded as in (2). The previous remark shows that $C_{\mathbf{q},\mathbf{n}}$ can be covered with r_ℓ intervals of length $1/r'_\ell$ ($\ell \in \mathbb{N}$). As a consequence $C_{\mathbf{q},\mathbf{n}}$ is a nonempty compact set that contains no interval.

Example 2.4. Setting $q_1 = n_1 = 2$ and $q_2 = n_2 = 3$, we obtain $q'_1 = 1/3$ and $q'_2 = 7$. The interval $[0, 1]$ yields two intervals $I_0^{(1)} = [0, 1/3]$ and $I_1^{(1)} = [2/3, 1]$. These two intervals define the following intervals: $I_0^{(2)} = [0, 1/21]$, $I_1^{(2)} = [1/7, 4/21]$, $I_2^{(2)} = [2/7, 1/3]$, $I_3^{(2)} = [2/3, 5/7]$, $I_4^{(2)} = [17/21, 6/7]$ and $I_5^{(2)} = [20/21, 1]$.

FIGURE 3. Construction of the initial intervals defining $C_{\mathbf{q},\mathbf{n}}$ for $q_1 = 2$, $q_2 = 3$, $n_1 = 2$ et $n_2 = 3$.

For $\ell \in \mathbb{N}$, let us set

$$s_\ell = \frac{\log r_\ell}{\log r'_\ell}.$$

It has been demonstrated in [9] that the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ is given by

$$\alpha = \liminf_{\ell} s_\ell. \tag{5}$$

Corollary 2.5. *Given two natural numbers $b > 1$ and $n > 1$, if $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ and $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ are defined by $q_j = b$ $n_j = n$, respectively, then the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ is*

$$\alpha = \frac{\log b}{\log(n(b-1)+1)} \tag{6}$$

and this set has full measure: $\mathcal{H}^\alpha(C_{\mathbf{q},\mathbf{n}}) = 1$.

Proof. This follows as a special case of Theorem 1.15 in [7]. □

Nevertheless, Corollary 3.11 reveals that this result fails to hold universally for all sequences $\mathbf{q}$.

3. Hölder regularity of Cantor functions

In this section, we delve into the Hölder regularity of the functions $D_{\mathbf{q},\mathbf{n}}$. The results presented here can be viewed as a natural extension of the classical case.

To analyze the regularity of $D_{\mathbf{q},\mathbf{n}}$, we introduce some necessary notations. Consider a sequence $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ of natural numbers strictly greater than 1, and define $r_j = q_1 q_2 \cdots q_j$. For a point $x = (x_1, x_2, \dots)_{\mathbf{q}}$ such that $x_j = 0$ for $j \le \ell$ and $x_{\ell+1} > 0$ ($\ell \in \mathbb{N}$), we clearly have $x \ge x_{\ell+1}/r_{\ell+1} \ge 1/r_{\ell+1}$. Furthermore, since $q_j \ge 2$ for all $j \in \mathbb{N}$, it follows that $r_j \ge 2^{j-k} r_k$ for all $j, k \in \mathbb{N}$ with $j \ge k$. Consequently, we can write

$$x = \sum_{j=\ell+1}^{\infty} \frac{x_j}{r_j} < \sum_{j=\ell+1}^{\infty} \frac{q_j}{r_j} \le \sum_{j=\ell+1}^{\infty} \frac{1}{r_{j-1}} \le \frac{2}{r_\ell}.$$

Now, let $x, y \in [0, 1]$, and denote by $(z_1, z_2, \dots)_{\mathbf{q}}$ the proper representation of $|x - y|$ in base $\mathbf{q}$, Define:

$$\gamma_{\mathbf{q}}(x, y) = \inf\{j \in \mathbb{N} : z_j \ne 0\} - 1.$$

In other words, $\gamma_{\mathbf{q}}(x, y)$ represents the length of the longest prefix of zeros in $(z_1, \dots)_{\mathbf{q}}$. It is straightforward to verify that if $x > y$ and $\gamma_{\mathbf{q}}(x, y) = \ell$, then we have either

$$x = \sum_{j=1}^{\ell} \frac{x_j}{r_j} + \sum_{j>\ell} \frac{x_j}{r_j} \quad \text{and} \quad y = \sum_{j=1}^{\ell} \frac{x_j}{r_j} + \sum_{j>\ell} \frac{y_j}{r_j}, \tag{7}$$

where $\sum_{j>\ell} y_j/r_j < \sum_{j>\ell} x_j/r_j$, or

$$x = \sum_{j=1}^{k} \frac{x_j}{r_j} + \sum_{j>\ell} \frac{x_j}{r_j} \quad \text{and} \quad y = \sum_{j=1}^{k-1} \frac{x_j}{r_j} + \frac{x_k - 1}{r_k} + \sum_{j=k+1}^{\ell} \frac{q_j - 1}{r_j} + \sum_{j>\ell} \frac{y_j}{r_j} \tag{8}$$

for some $k \leq \ell$, with $\sum_{j>\ell} y_j/r_j > \sum_{j>\ell} x_j/r_j$.

Next, we introduce the notion of Hölder exponent, defined from the Hölder spaces. We present here the pointwise version for non-differentiable functions (we restrict our interest to exponents in $[0,1]$).

Definition 3.1. A locally bounded function f belongs to the Hölder space $\Lambda^\alpha(x_0)$ with $\alpha \in [0,1]$ if there exists a constant C such that, for any x in a neighborhood of x_0,

$$|f(x_0) - f(x)| \leq C|x_0 - x|^\alpha.$$

The Hölder exponent of f at x_0 is then defined as

$$\sup\{\alpha : f \in \Lambda^\alpha(x_0)\}.$$

The Hölder exponent provides a measure of the regularity of a function: the larger the exponent, the smoother the function [21]. In particular, if this exponent is strictly less than 1 at x_0, then the function is non-differentiable at x_0.

Let $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ and $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ be sequences of natural numbers greater than 1, and let $\mathbf{q}' = (q'_j)_{j\in\mathbb{N}}$ denote the sequence defined by $q'_j = n_j(q_j - 1) + 1$. As before, we define $r'_j = q'_1 q'_2 \cdots q'_j$. It is already established that the Hölder regularity of $D_{\mathbf{q},\mathbf{n}}$ at any point of $[0,1] \setminus C_{\mathbf{q},\mathbf{n}}$ is equal infinite, as the function is piecewise linear on this set.

Lemma 3.2. *The quantity*

$$\alpha_1 = \liminf_{\ell} \frac{\log r_\ell}{\log r'_{\ell+1}} \tag{9}$$

is a lower bound for the Hölder exponent of $D_{\mathbf{n},\mathbf{q}}$ at any point of $C_{\mathbf{n},\mathbf{q}}$.

Proof. Let $x \in C_{\mathbf{q},\mathbf{n}}$, and choose $y \in [0,1]$ such that $\gamma_{\mathbf{q}'}(x,y) = \ell$. If (7) holds, the expansions in base $\mathbf{q}$ of $D_{\mathbf{q},\mathbf{n}}(x)$ and $D_{\mathbf{q},\mathbf{n}}(y)$ both begin with $\sum_{j=1}^{\ell} p_j/r_j$, implying that $\gamma_{\mathbf{q}}(D_{\mathbf{q},\mathbf{n}}(x), D_{\mathbf{q},\mathbf{n}}(y)) \geq \ell$. If (8) holds, then $y \notin C_{\mathbf{q},\mathbf{n}}$ because $x_k - 1$ cannot be divisible by n_k. Consequently, $D_{\mathbf{q},\mathbf{n}}(y) = \sum_{j=1}^{k} p_j/r'_j$, which again ensures that $\gamma_{\mathbf{q}}(D_{\mathbf{q},\mathbf{n}}(x), D_{\mathbf{q},\mathbf{n}}(y)) \geq \ell$.

Thus, we have

$$\begin{aligned} \log|D_{\mathbf{q},\mathbf{n}}(x) - D_{\mathbf{q},\mathbf{n}}(y)| \leq \log 2 - \log r_{\gamma'_{\mathbf{q}}(x,y)} &= \frac{\log 2 - \log r_\ell}{\log r'_{\ell+1}} \log r'_{\ell+1} \\ &\leq \frac{\log r_\ell - \log 2}{\log r'_{\ell+1}} \log|x - y|, \end{aligned}$$

which concludes the proof. ∎

Lemma 3.3. *The quantity*

$$\beta_1 = \limsup_\ell \frac{\log r_{\ell+1}}{\log r'_\ell}$$

is an upper bound for the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$.

Proof. We can assume that the upper limit

$$\limsup_\ell \frac{\log r_{\ell+1}}{\log r'_\ell - \log 2} \tag{10}$$

is strictly less than 1. Given $x \in C_{\mathbf{q},\mathbf{n}}$, one can always find $y \in C_{\mathbf{q},\mathbf{n}}$ such that (7) holds, which implies that $\gamma_{\mathbf{q}'}(x,y) = \gamma_{\mathbf{q}}((D_{\mathbf{q},\mathbf{n}}(x), D_{\mathbf{q},\mathbf{n}}(y))$. Therefore,

$$\log |D_{\mathbf{q},\mathbf{n}}(x) - D_{\mathbf{q},\mathbf{n}}(y)| \geq \frac{\log r_{\ell+1}}{\log r'_\ell - \log 2} \log |x-y|.$$

Consequently, for any sufficiently small $\epsilon > 0$, there exists $L > 0$ such that for all $\ell \geq L$, one can find y satisfying $\gamma_{\mathbf{q}'}(x,y) = \ell$ and

$$|D_{\mathbf{q},\mathbf{n}}(x) - D_{\mathbf{q},\mathbf{n}}(y)| \geq |x-y|^{\beta+\epsilon}.$$

This establishes the result. ☐

Remark 3.4. It is natural to conjecture that the Hölder exponent of $D_{\mathbf{n},\mathbf{q}}$ is given by the formula $\liminf_\ell \log r_\ell / \log r'_\ell$. While this conjecture is highly intuitive, it is far from trivial to establish rigorously. One of the reviewers proposed the following heuristic argument. Let ℓ be the smallest integer such that both x and y belong to a basic interval $I^{(\ell)}$ of the set K_ℓ, as defined in Remark 2.2. If k denotes the number of full basic intervals between x and y, then the inequality

$$\frac{\log |D_{\mathbf{q},\mathbf{n}}(x) - D_{\mathbf{q},\mathbf{n}}(y)|}{\log |x-y|} \leq \frac{\log(k+2) - \log r_\ell}{\log((k+1)n_\ell - 1) - \log r'_\ell}.$$

holds. However, we assert that the right-hand side of this inequality does not, in general, behave asymptotically like $\log r_\ell / \log r'_\ell$. To illustrate this, consider the sequences $\mathbf{q}$ and $\mathbf{n}$ defined by $q_j = n_j = {}^j 2$, where the notation ${}^j x$ denotes the tetration operation [12]. It is always possible to choose points $x, y \in C_{\mathbf{q},\mathbf{n}}$ such that $|x-y|$ is arbitrarily small and $k = 0$; thus, it suffices to analyze the case $k = 0$. On the one hand, we have

$$\begin{aligned}\lim_\ell \frac{\log 2 - \log r_\ell}{\log(n_\ell - 1) - \log r'_\ell} &= \lim_\ell \frac{\log r_\ell}{\log r'_\ell - \log n_\ell} \\ &= \lim_\ell \frac{\log q_\ell + \log r_{\ell-1}}{2\log r_{\ell-1} + \log q_\ell} = 1.\end{aligned}$$

On the other hand, we obtain

$$\lim_\ell \frac{\log r_\ell}{\log r'_\ell} = \lim_\ell \frac{\log r_\ell}{2\log r_\ell} = 1/2.$$

Knowing that $D_{\mathbf{q},\mathbf{n}}$ is nowhere differentiable on $C_{\mathbf{q},\mathbf{n}}$, the following proposition is a straightforward consequence of Lemmata 3.2 and 3.3.

Proposition 3.5. *If $\alpha_1 = \beta_1$ or $\alpha_1 = 1$ (α_1 and β_1 being defined by* (9) *and* (10)*, respectively), then the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$ is equal to α_1.*

In particular, if there exists $n, b \in \mathbb{N}$, both greater than 1, such that $q_j = b$ and $n_j = n$ for any j, then the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$ is equal to $\log b / \log(n(b-1)+1)$.

Let us provide a simple condition to construct a Cantor set whose Hausdorff dimension equals 1, while its associate measure vanishes. Using the same construction, we can obtain Cantor functions with Hölder exponent equal to 1 on the corresponding Cantor set. Let

$$\gamma = \limsup_{\ell} \frac{\log q_{\ell+1}}{\log r_\ell}$$

Definition 3.6. A function $\phi : \mathbb{N} \to \mathbb{N}$ is called sup-linear if $\phi(j) \geq j$ for any $j \in \mathbb{N}$.

Corollary 3.7. *Let $\mathbf{q} = (q_j)_{j\in\mathbb{N}}$ be a sequence of the form $q_j = \phi(j)$, where ϕ is a sup-linear function and let $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ be a bounded sequence of natural numbers greater than 1.*

- *If $\gamma = 0$ then the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$ is equal to 1.*
- *If $\gamma \in (0, +\infty)$ then the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$ is greater than $1/(1+\gamma)$.*

Proof. Let $C > 0$ be such that $n_j \leq C$ for all $j \in \mathbb{N}$. For any $\ell \in \mathbb{N}$, we have

$$\begin{aligned} 1 \geq \frac{\log r_\ell}{\log r'_{\ell+1}} &= \frac{\log \prod_{j=1}^{\ell} \phi(j)}{\log \prod_{j=1}^{\ell+1}(n_j(\phi(j)-1)+1)} = \frac{\log \prod_{j=1}^{\ell} n_j\phi(j) - \log \prod_{j=1}^{\ell} n_j}{\log \prod_{j=1}^{\ell+1}(n_j(\phi(j)-1)+1)} \\ &\geq \frac{\log \prod_{j=1}^{\ell} n_j\phi(j) - \log \prod_{j=1}^{\ell} n_j}{\log \prod_{j=1}^{\ell+1} n_j\phi(j)} \geq \frac{\log \prod_{j=1}^{\ell} n_j\phi(j)}{\log \prod_{j=1}^{\ell+1} n_j\phi(j)} - \frac{\log \prod_{j=1}^{\ell} n_j}{\log(C^\ell \ell!)}. \end{aligned}$$

It is evident that the second term vanishes as $\ell \to \infty$. Furthermore, we observe

$$\liminf_{\ell} \frac{\log \prod_{j=1}^{l} n_j\phi(j)}{\log \prod_{j=1}^{\ell+1} n_j\phi(j)} \geq \liminf_{\ell} \frac{1}{1 + \dfrac{\log n_{\ell+1} + \log q_{\ell+1}}{\log r_\ell + \ell \log \inf_j n_j}} = \frac{1}{1+\gamma},$$

which completes the proof. □

Remark 3.8. Using reasoning analogous to the proof of Corollary 3.7, we have

$$\limsup_{\ell} \frac{\log r_{\ell+1}}{\log r'_\ell} \geq \limsup_{\ell} \frac{\log \prod_{j=1}^{\ell} n_j\phi(j)}{\log \prod_{j=1}^{\ell} n_j\phi(j)} + \frac{\log(n_{\ell+1}q_{\ell+1})}{\log r_\ell + \ell \log \inf_j n_j} = 1 + \gamma.$$

We now present an example where the exact Hölder exponent can be determined.

Corollary 3.9. *Under the same conditions as Corollary 3.7, if ϕ is a sup-linear function satisfying $\phi(j) \leq a^j$ for all $j \in \mathbb{N}$ $(a > 1)$, then $\gamma = 0$.*

In particular, the Hölder exponent of $D_{\mathbf{q},\mathbf{n}}$ at any point of $C_{\mathbf{q},\mathbf{n}}$ is equal to 1.

Proof. We immediately have

$$\gamma \leq \limsup_{\ell} \frac{\log a^{\ell+1}}{\log r_\ell} = \limsup_{\ell} \frac{\log a^{\ell+1}}{\log \prod_{j=1}^{\ell} \phi(j)} \leq \limsup_{\ell} \frac{(\ell+1)\log a}{\log \ell!} = 0.$$

□

However, the above result does not hold for a function ϕ that grows too quickly, as demonstrated in the following example.

Example 3.10. Let $q_j = 2^{2^j}$ for all $j \in \mathbb{N}$. In this case, we directly find $\gamma = 1$.

To conclude this section, we return to the discussion of the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$.

Corollary 3.11. *If $\mathbf{q} = (q_j)j \in \mathbb{N}$ is defined by $q_j = \phi(j)$, where ϕ is a sup-linear function, and $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ is a bounded sequence of natural numbers greater than 1, then the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ is equal to 1, while its Lebesgue measure vanishes: $\mathcal{H}^1(C_{\mathbf{q},\mathbf{n}}) = 0$.*

Proof. The argument for the Hausdorff dimension follows similarly from Corollary 3.7. For the Lebesgue measure, we find

$$\mathcal{H}^1(C_{\mathbf{q},\mathbf{n}}) = \lim_{\ell} \mathcal{H}^1(K_\ell) = \lim_{\ell} \frac{\prod_{j=1}^{\ell} \phi(j)}{\prod_{j=1}^{\ell}(n_j(\phi(j)-1)+1)} = 0.$$

□

Example 3.12. Let $\mathbf{q} = (q_j)j \in \mathbb{N}$ denote the sequence of prime numbers, where q_j represents the $(j+1)$-th prime. The prime number theorem, combined with Corollary 3.9, establishes that $\gamma = 0$. Moreover, using Corollary 3.11, we deduce that if $\mathbf{n}$ is a bounded sequence, then the Hausdorff dimension of $C_{\mathbf{q},\mathbf{n}}$ is equal to 1, while its Lebesgue measure vanishes.

4. The Hausdorff h-measure of $C_{\mathbf{q},\mathbf{n}}$

Let h be a nondecreasing, right-continuous function that is zero at the origin. The Hausdorff h-measure, denoted by $\mathcal{H}^h$, generalizes the standard Hausdorff measure by replacing the function x^s in (3) with $h(x)$ [13, 25].

Definition 4.1. Let h be a right-continuous nondecreasing function defined on $[0,\infty)$ such that $h(0) = 0$. The Hausdorff h-measure of a set E is given by

$$\mathcal{H}^h(E) = \sup_{\epsilon>0} \inf\{\sum_{j=1}^{\infty} h(|E_j|) : E \subset \bigcup_{j=1}^{\infty} E_j,\ |E_j| < \epsilon\}.$$

The function h is called a dimension function for E if $0 < \mathcal{H}^h(E) < \infty$.

It is worth noting that if h is a dimension function, then rh (where $r > 0$) is also a dimension function.

Extend $D_{\mathbf{q},\mathbf{n}}$ to the whole real line by defining $D_{\mathbf{q},\mathbf{n}}(x) = 0$ for $x < 0$ and $D_{\mathbf{q},\mathbf{n}}(x) = 1$ for $x > 1$. We now demonstrate that the function $h = D_{\mathbf{q},\mathbf{n}}$ serves as a valid dimension function for $C_{\mathbf{q},\mathbf{n}}$.

Theorem 4.2. *Given a set $C_{\mathbf{q},\mathbf{n}}$, the function $h = D_{\mathbf{q},\mathbf{n}}$ is a dimension function for $C_{\mathbf{q},\mathbf{n}}$ and*

$$\mathcal{H}^h(C_{\mathbf{q},\mathbf{n}}) = 1.$$

Proof. Using the notations from Remark 2.2, observe that the length of each $I_j^{(\ell)}$ tends to zero as $\ell \to \infty$. Furthermore, we have

$$\sum_{j=0}^{r_\ell - 1} h(|I_j^{(\ell)}|) = r_\ell h(\frac{1}{r'_\ell}) = 1,$$

which implies $\mathcal{H}^h(C_{\mathbf{q},\mathbf{n}}) \leq 1$.

To establish the reverse inequality, let $\mu_{\mathbf{q},\mathbf{n}}$ be the unique Lebesgue-Stieltjes measure associated to $D_{\mathbf{q},\mathbf{n}}$ (see [23]), such that

$$D_{\mathbf{q},\mathbf{n}}(x) = \int_{-\infty}^{x} d\mu_{\mathbf{q},\mathbf{n}}.$$

This defines a Borel probability measure satisfying

$$\mu_{\mathbf{q},\mathbf{n}}(I_j^{(\ell)}) = \frac{1}{r_\ell},$$

for all $\ell \in \mathbb{N}$ and $j \in \{0, \dots, r_\ell - 1\}$, with $r_0 = 1$.

Now, consider an open interval U from some cover of $C_{\mathbf{q},\mathbf{n}}$. We claim that

$$h(|U|) \geq \mu_{\mathbf{q},\mathbf{n}}(U) \tag{11}$$

First assume that $\inf U$ coincides with the left endpoint of a component interval $I_l^{(\ell)}$ of the set K_ℓ (from Remark 2.2), and that $\sup U$ coincides with the right endpoint of a component interval $I_r^{(\ell')}$ of $K_{\ell'}$. By choosing sufficiently large ℓ and ℓ', we may assume that $I_l^{(\ell)}$ and $I_r^{(\ell')}$ are both contained in $\overline{U}$ and that $\ell = \ell'$. Let J and J' denote the component intervals of $K_{\ell-1}$ containing $I_l^{(\ell)}$ and $I_r^{(\ell)}$ respectively. Define $U_* = (0, |U|)$ and observe that proving $\mu_{\mathbf{q},\mathbf{n}}(U) \leq \mu_{\mathbf{q},\mathbf{n}}(U_*)$ will suffice to obtain the inequality $\mu_{\mathbf{q},\mathbf{n}}(U) \leq h(|U|)$.

We begin by observing that the ratio n_ℓ/r'_ℓ represents the distance between the left endpoints of two consecutive fundamental intervals at level ℓ. If $U \subset J$, then by the homogeneity of $C_{\mathbf{q},\mathbf{n}}$, we have $\mu_{\mathbf{q},\mathbf{n}}(U) = \mu_{\mathbf{q},\mathbf{n}}(U_*)$. Otherwise, since J and J' are disjoint with positive separation, we consider the translation $U_\ell = U - n_\ell/r'_\ell$, yielding $\mu_{\mathbf{q},\mathbf{n}}(U) \leq \mu_{\mathbf{q},\mathbf{n}}(U_\ell)$. Repeating this translation recursively, we may assume that the left endpoint of the translated set U_ℓ aligns with the left endpoint of J. Then, translating by $n_{\ell-1}/r'_{\ell-1}$, we obtain a further translated set $U_{\ell-1}$ whose left endpoint aligns with a left end point of a component interval $I^{(\ell-2)}$ of $K_{\ell-2}$, and which satisfies $\mu_{\mathbf{q},\mathbf{n}}(U_{\ell-1}) \geq \mu_{\mathbf{q},\mathbf{n}}(U_\ell)$. If $U_{\ell-1} \subset I^{(\ell-2)}$, then again $\mu_{\mathbf{q},\mathbf{n}}(U_{\ell-1}) = \mu_{\mathbf{q},\mathbf{n}}(U_*)$. Otherwise, the process continues. This recursive translation must terminate, ultimately leading to $\mu_{\mathbf{q},\mathbf{n}}(U) \leq \mu_{\mathbf{q},\mathbf{n}}(U_*)$.

To address the general case, when $\inf U$ or $\sup U$ are not endpoints of component intervals, we may assume they lie in $C_{\mathbf{q},\mathbf{n}}$. Let $(x_j)_{j\in\mathbb{N}}$ be an increasing sequence of left endpoints of component intervals converging to $\inf U$, and $(y_j)_{j\in\mathbb{N}}$ a decreasing sequence of right endpoints converging to $\sup U$. By considering the intervals (x_j, y_j) and invoking the continuity of both h and $\mu_{\mathbf{q},\mathbf{n}}$, we conclude that inequality (11) extends to the general case.

Finally, for any covering $(C_j)_{j\in\mathbb{N}}$ of $C_{\mathbf{q},\mathbf{n}}$ consisting of intervals, we obtain

$$\sum_{j\in\mathbb{N}} h(|C_j|) \geq \sum_{j\in\mathbb{N}} \mu_{\mathbf{q},\mathbf{n}}(C_j) \geq \mu_{\mathbf{q},\mathbf{n}}(C_{\mathbf{q},\mathbf{n}}) = 1,$$

which completes the proof. □

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