Resolution of elasto-plastic finite-elements problems by quantum annealing

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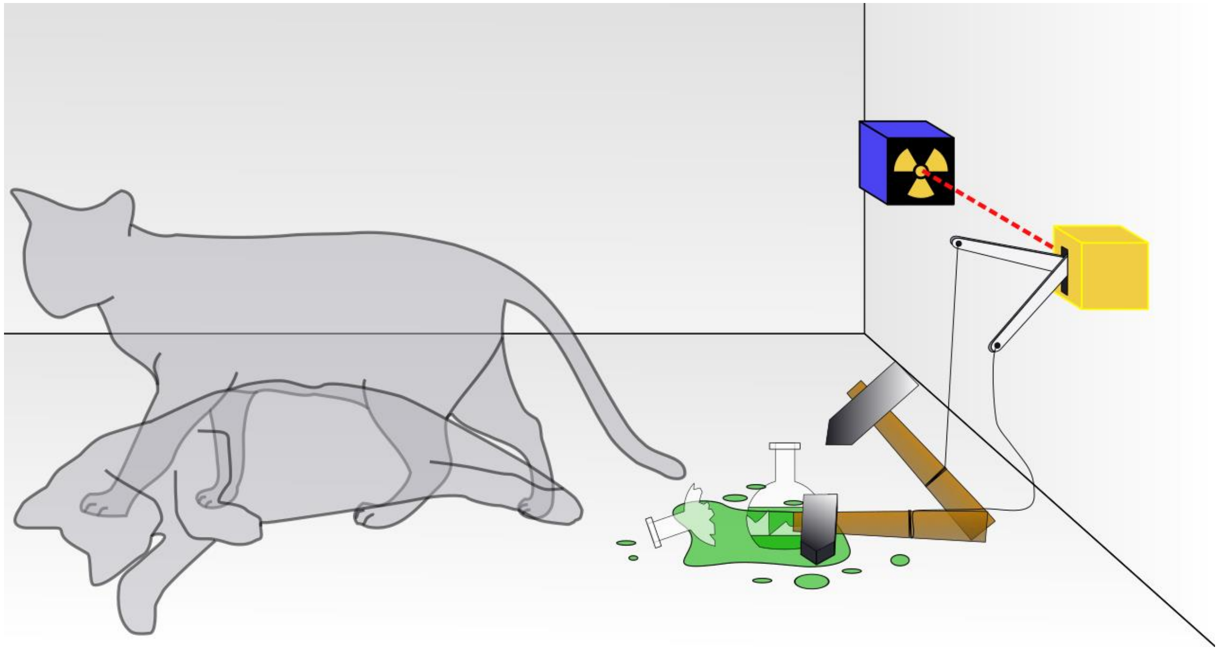

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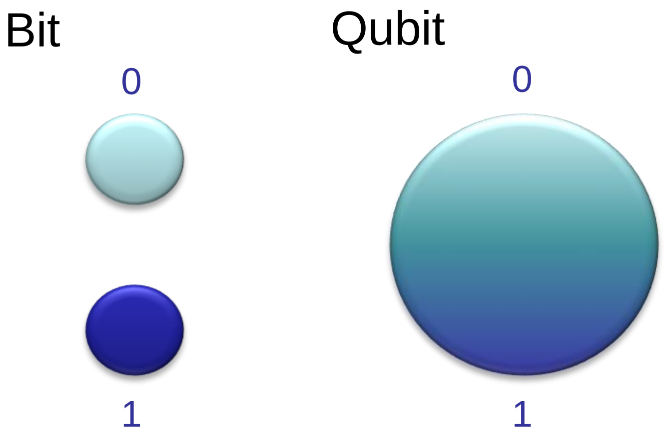
Introduction to Quantum Computing

- Bits vs. Qubits:
  - Superposition of states:
    - A quantum bit can be 0 or 1 at the same time

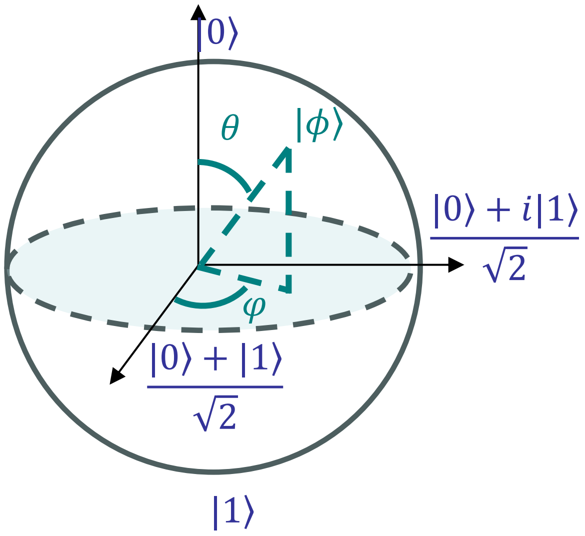


  - State vector of a qubit
    - Computational basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
    - Notations: $\begin{cases} |\phi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle \\ \langle\phi| = (\alpha^* \quad \beta^*) \end{cases}$ $|\alpha|^2 + |\beta|^2 = 1$
    - Qubit represented on the surface of the Bloch Sphere

$$|\phi\rangle = e^{i\delta}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right)$$

    - Global phase $\mathrm{e}^{i\delta}$ has no observable consequence

      (NB relative phase has consequence)



  - At measurement (in the computational basis)
    - Either $|0\rangle$ or $|1\rangle$ with respective probability $|\alpha|^2$ and $|\beta|^2$

Introduction to Quantum Computing

- Multiple (connected) qubits:
  - Product state of 2 1-qubit states: $\begin{cases} |\phi_0\rangle = \alpha_0|0\rangle + \beta_0|1\rangle \\ |\phi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \end{cases}$

$$|\boldsymbol{\phi}\rangle = |\phi_0\rangle \otimes |\phi_1\rangle = \boxed{\alpha_0\alpha_1}|00\rangle + \boxed{\alpha_0\beta_1}|01\rangle + \boxed{\alpha_1\beta_0}|10\rangle + \boxed{\beta_0\beta_1}|11\rangle$$

  - Most general 2-qubit state

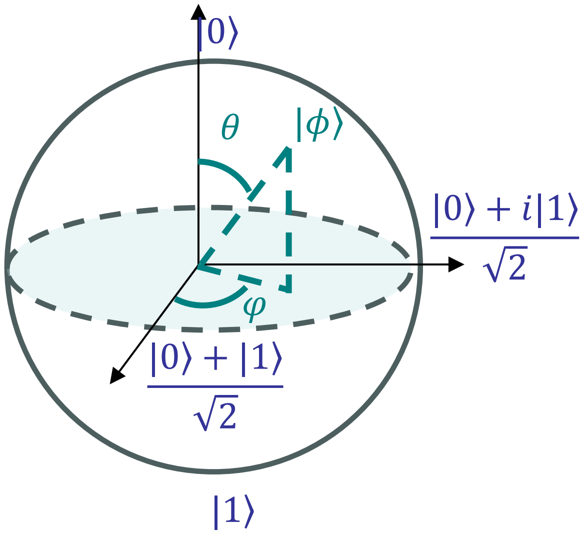
$$|\boldsymbol{\phi}\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Because of entanglement, a K-qubit state is more general

(it cannot always be written as the product of K 1-qubit states)

There is not always K equivalent 1-qubit states to a K-qubit state, e.g.

$$|\boldsymbol{\phi}\rangle = \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + 0|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$



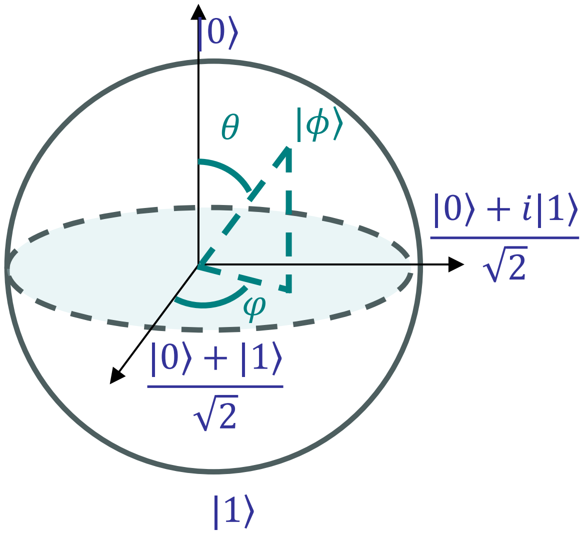
  - A system of K coupled qubits
    - Is a 2^K -state quantum-mechanical system
    - Whose state can be represented by any normalised linear combination of 2^K basis states:

$$|\boldsymbol{\phi}\rangle = \phi_0|0\rangle \otimes |0\rangle \dots \otimes |0\rangle + \phi_1|0\rangle \otimes |0\rangle \dots \otimes |1\rangle + \cdots + \phi_{2^K-1}|1\rangle \dots \otimes |1\rangle \otimes |1\rangle$$

with $\sum_{i=0}^{2^K-1} |\phi_i|^2 = 1$

Because of superposition, potentially, a quantum computer with K qubits can take 2^K bitstrings of size K in parallel at the same time. A classical computer can only take 1 bitstring of size K

Introduction to Quantum Computing



- Quantum computers:
  - Trapped-ion quantum computer
    - Suspended ions in electromagnetic field
    - Ground state and excited states
    - Interactions controlled by laser
    - E.g. IonQ & Quantinuum
  - Photonic quantum computers
    - State corresponds to direction of photon travel
  - Superconducting quantum computers
    - Superconducting qubits as artificial atoms (ground state and excited state)
    - Superconducting capacitors and inductors are used to produce a resonant circuit
    - Operate at temperature of 10 mK
    - Qubit state controlled by external microwave signals
    - IBM, D-Wave
  - Different platforms (2 different resolution methodologies)
    - IBM
      - 2022: 433-qubit Osprey’
      - 2023: 1121-qubit Condor
    - D-Wave
      - 5000+-qubit Advantage (35000 couplers)
      - Each qubit is only connected to a reduced number of other qubits

- Universal gate
  - Gate on 1 qubit
    - E.g. Hadamard $\mathbf{H^d} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ $\quad |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$|0\rangle \text{—}\boxed{\mathbf{H^d}}\text{—} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |1\rangle \text{—}\boxed{\mathbf{H^d}}\text{—} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

  - Gate on 2 qubits
    - NB: $|01\rangle$: $\begin{pmatrix}1 & \cdot\begin{pmatrix}0\\1\end{pmatrix}\\ 0 & \cdot\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ & $|10\rangle$: $\begin{pmatrix}0 & \cdot\begin{pmatrix}1\\0\end{pmatrix}\\ 1 & \cdot\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ & ……
    - E.g. controlled-not: $\mathbf{C}_{10} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$

$$\begin{array}{l} |x\rangle\text{—}\\ |y\rangle\text{—}\end{array}\boxed{\mathbf{C}_{10}}\begin{array}{l}\text{—}|x\rangle\\ \text{—}|y \oplus x\rangle\end{array}$$

$x, y \in \{0,1\}$, the second qubit is flipped if and only if first is 1

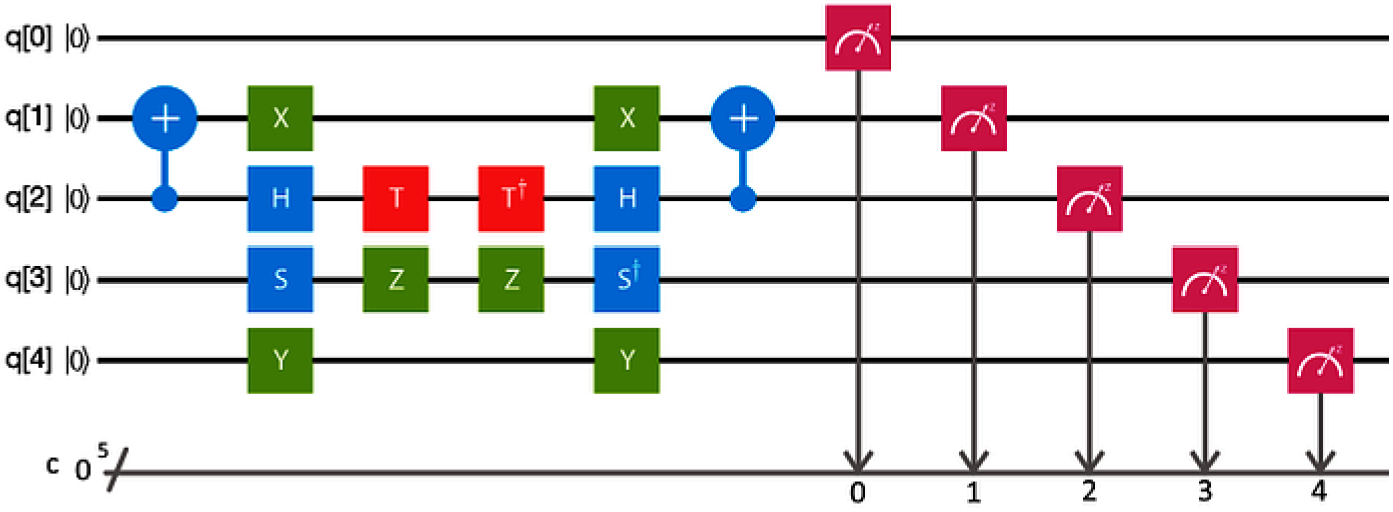
E.g. for $x = 1$ $y = 0$: $|\boldsymbol{\phi}\rangle = 0\underset{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}{|00\rangle} + 0\underset{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}{|01\rangle} + 1\underset{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}{|10\rangle} + 0\underset{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}{|11\rangle}$

$$|\boldsymbol{\phi}'\rangle = \mathbf{C}_{10}|\boldsymbol{\phi}\rangle = 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle$$

  - Gate on n qubits …

Introduction to Quantum Computing

- Universal gate
  - Circuit, e.g. on 5-qubits



  - Gate-based QC
    - Universal approach (like classical computers operations are performed on qubits)
    - Highly sensitive to noise ⇒ difficulty in controlling error
    - Error controlled by using control qubits

Introduction to Quantum Computing

- Quantum annealer
  - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$

$$|\boldsymbol{\phi}_0\rangle = \arg\min_{|\boldsymbol{\phi}\rangle} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle$$

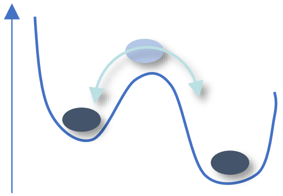
  - Based on quantum adiabatic theorem:
    - Considering a time-varying Hamiltonian $\mathbf{H}_{\mathbf{QA}}(t)$ initially at ground state, if its time evolution is slow enough, it is likely to remain at the ground state
  - Adiabatic quantum computing:
    - Starts from the ground state of an easy to prepare Hamiltonian $\mathbf{H}_i$
    - Evolves to the ground state of the Hamiltonian $\mathbf{H}$ which encodes the sought solution

$$\mathbf{H}_{\mathbf{QA}}(t) = \frac{(t_a - t)}{t_a}\,\mathbf{H}_i + \frac{t}{t_a}\,\mathbf{H}$$

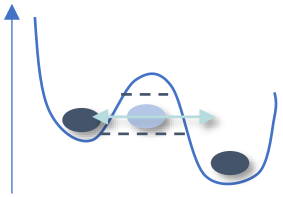
  - Quantum annealing
    - Exploits quantum effect such as quantum tunneling
    - Less sensitive to noise than Gate-based QC

      ⇒ We still need to design error-contained algorithm !!!
    - Less versatile than Gate-based QC

      ⇒ But minimizing energy in mechanics is natural !!!



Thermal annealing



Quantum annealing

Ising Hamiltonian-based quantum annealing

- Ising Hamiltonian
  - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$

$$|\boldsymbol{\phi}_0\rangle = \arg\min_{|\boldsymbol{\phi}\rangle} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle$$

  - Some definitions
    - Set of K qubits $V = \{0, \dots K-1\}$
    - Set of interactions between 2 qubits $E \subset \{(i,j) \mid i \in V, j \in V, i < j)\}$
    - Pauli- Z operator $\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and identity $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
    - Pauli- Z operator applied on qubit i: $\mathbf{Z}_i = \underbrace{\mathbf{I}}_{0} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{i} \otimes \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{I}}_{K-1}$
    - Pauli- Z operator applied on qubits i and j:

$$\mathbf{Z}_{ij} = \underbrace{\mathbf{I}}_{0} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{i} \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{j} \otimes \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{I}}_{K-1}$$

  - Ising Hamiltonian represented by an undirected graph (V, E):
    - $\mathbf{H} = \sum_{i \in V} h_i \mathbf{Z}_i + \sum_{(i,j) \in E} J_{ij} \mathbf{Z}_{ij}$
    - Is a $2^K \times 2^K$ diagonal operator in the computational basis

Ising Hamiltonian-based quantum annealing

- Quadratic Unconstrained Binary Optimization (QUBO)
  - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$

$$|\boldsymbol{\phi}_0\rangle = \arg\min_{|\boldsymbol{\phi}\rangle} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle \qquad \text{with} \quad \mathbf{H} = \sum_{i\in V} h_i \mathbf{Z}_i + \sum_{(i,j)\in E} J_{ij}\mathbf{Z}_{ij}$$

  - In terms of spin variables
    - Computational basis of $\mathbf{H}$ $\quad |\boldsymbol{\phi}\rangle = |b_0\ b_1\ \dots b_{K-1}\rangle$ $\quad$ with $\quad b_i \in \{0,1\}$
    - We have successively

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Z}|b_i\rangle = (-1)^{b_i}|b_i\rangle \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{Z}_i = \underbrace{\mathbf{I}}_{0} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{i} \otimes \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{I}}_{K-1} \quad \Rightarrow \quad \mathbf{Z}_i|\boldsymbol{\phi}\rangle = (-1)^{b_i}|\boldsymbol{\phi}\rangle$$

$$\mathbf{Z}_{ij} = \underbrace{\mathbf{I}}_{0} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{i} \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{Z}}_{j} \otimes \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{I}}_{K-1} \quad \Rightarrow \quad \mathbf{Z}_{ij}|\boldsymbol{\phi}\rangle = (-1)^{b_i}(-1)^{b_j}|\boldsymbol{\phi}\rangle$$

    - Defining the vector of spin variables: $\mathbf{s} = [(-1)^{b_i}\ \ \forall i \in V]$

⇒ The eigenvalue of $\mathbf{H}$ reads $\quad \mathcal{F}_{\text{Ising}} = \sum_{i\in V} h_i s_i + \sum_{(i,j)\in E} J_{ij} s_i s_j = \mathbf{s}^T\mathbf{h} + \mathbf{s}^T\mathbf{J}\mathbf{s}$

with $\quad \mathbf{h} = [h_i\ \ \forall i \in V]$ $\quad$ & $\quad \mathbf{J} = \left[J_{ij}\ \ \forall (i,j) \in E\right]$

$$\Rightarrow \quad |\boldsymbol{\phi}_0\rangle = \arg\min_{\boldsymbol{\phi}} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle \quad \Leftrightarrow \quad \mathbf{s} = \arg\min_{\mathbf{s}'} \mathcal{F}_{\text{Ising}}(\mathbf{s}'; \boxed{\mathbf{h},\mathbf{J}})$$

User programmable parameters

Ising Hamiltonian-based quantum annealing

- Quadratic Unconstrained Binary Optimization (QUBO)
  - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$

$$|\boldsymbol{\phi}_0\rangle = \arg\min_{|\boldsymbol{\phi}\rangle} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle \qquad \text{with} \quad \mathbf{H} = \sum_{i\in V} h_i \mathbf{Z}_i + \sum_{(i,j)\in E} J_{ij}\mathbf{Z}_{ij}$$

  - In terms of spin variables
    - Computational basis of $\mathbf{H}$ $\quad |\boldsymbol{\phi}\rangle = |b_0\, b_1\, \dots b_{K-1}\rangle \quad$ with $\quad b_i \in \{0,1\}$
    - Vector of spin variables: $\mathbf{s} = [(-1)^{b_i} \;\; \forall i \in V]$

The eigenvalue of $\mathbf{H}$ reads $\mathcal{F}_{\text{Ising}} = \sum_{i\in V} h_i s_i + \sum_{(i,j)\in E} J_{ij} s_i s_j = \mathbf{s}^T\mathbf{h} + \mathbf{s}^T\mathbf{J}\mathbf{s}$

with $\quad \mathbf{h} = [h_i \;\; \forall i \in V] \qquad$ & $\mathbf{J} = \left[J_{ij} \;\; \forall (i,j) \in E\right]$

User programmable parameters

⇒ $|\boldsymbol{\phi}_0\rangle = \arg\min_{\boldsymbol{\phi}} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle$ ⇔ $\mathbf{s} = \arg\min_{\mathbf{s}'} \mathcal{F}_{\text{Ising}}(\mathbf{s}'; \mathbf{h}, \mathbf{J})$

  - In terms of binary variables
    - Vector of binary variables $\mathbf{b} = [b_i \; \forall i \in V]$
    - Spin-binary variable transformation $s_i = 2b_i - 1 : \{0,1\} \to \{-1,1\}$ & property $b_i^2 = b_i$

⇒ $\mathcal{F}_{\text{Ising}} = \sum_{i\in V} h_i s_i + \sum_{(i,j)\in E} J_{ij} s_i s_j$ ⇒ $\mathcal{F}_{\text{QUBO}} = \sum_{(i,j)\in E\cup\{(i,i)\forall i\in V\}} A_{ij} b_i b_j = \mathbf{b}^T\mathbf{A}\mathbf{b}$

⇒ $|\boldsymbol{\phi}_0\rangle = \arg\min_{\boldsymbol{\phi}} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle$ ⇔ $\mathbf{b} = \arg\min_{\mathbf{b}'} \mathcal{F}_{\text{QUBO}}(\mathbf{b}'; \mathbf{A})$

User programmable parameters

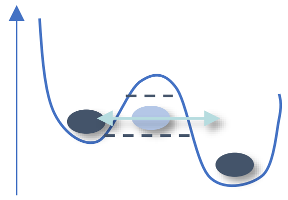
Ising Hamiltonian-based quantum annealing

- Summary
  - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$

$$|\boldsymbol{\phi}_0\rangle = \arg\min_{|\boldsymbol{\phi}\rangle} \langle\boldsymbol{\phi}|\mathbf{H}\,|\boldsymbol{\phi}\rangle \qquad \text{with} \quad \mathbf{H} = \sum_{i\in V} h_i \mathbf{Z}_i + \sum_{(i,j)\in E} J_{ij}\mathbf{Z}_{ij}$$

  - Adiabatic annealing
    - Starts from the ground state of an easy to prepare $\mathbf{H}_i$
    - Evolves to the ground state of the Hamiltonian $\mathbf{H}$

$$\mathbf{H}_{\mathbf{QA}}(t) = \frac{(t_a - t)}{t_a}\,\mathbf{H}_i + \frac{t}{t_a}\,\mathbf{H}$$



Quantum annealing

  - Problem reformulated in terms of binary variables
    - $\mathbf{b} = [b_i \; \forall i \in V]$ with $b_i \in \{0,1\}$
    - Eigenvalue $\mathcal{F}_{\text{QUBO}} = \mathbf{b}^T\mathbf{A}\mathbf{b}$
    - QUBO optimization $\mathbf{b} = \arg\min_{\mathbf{b}'} \mathcal{F}_{\text{QUBO}}(\mathbf{b}';\boxed{\mathbf{A}})$

User programmable parameters

  - In practice
    - Provide the QUBO matrix **A**
    - Set the annealing time t_a (typically 20 µs)
    - One annealing returns a sample of $\mathbf{b}$
    - A single run may not provide the global minimum due to environmental noises, hardware imperfections, pre- and post-processing errors → requires several reads

Quantum computing & finite elements

- Set of PDEs to be solved
  - Strong form $\Rightarrow$ Weak form:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{b}_{\mathbf{0}}(\boldsymbol{x}) = \mathbf{0} \quad \Rightarrow \quad \int_V \boldsymbol{\sigma}(\boldsymbol{x}) : \nabla \otimes^s \boldsymbol{\delta u}(\boldsymbol{x}) dV = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV + \int_{\partial_N V} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\delta u} d\partial V$$

  - Constitutive model:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}\big(\boldsymbol{\nabla} \otimes^s \boldsymbol{u}(\boldsymbol{x}, t); \mathbf{q}(\boldsymbol{x}, t)\big) \quad \text{with evolution law} \quad \boldsymbol{Q}\big(\boldsymbol{\sigma}(\boldsymbol{x}, t), \mathbf{q}(\boldsymbol{\nabla} \otimes^s \boldsymbol{u}(\boldsymbol{x}, \tau) : \tau \leq t)\big) = \mathbf{0}$$

- Finite element formulation
  - Displacement field at quadrature point Ξ from nodal displacements vector $\mathbf{U}$

$$\boldsymbol{u}(\Xi) = N_a(\Xi)\boldsymbol{U}_a \quad \Rightarrow \quad \boldsymbol{\varepsilon}(\Xi) = \boldsymbol{\nabla} \otimes^s \boldsymbol{u}(\Xi) = \mathbf{B}_a(\Xi)\boldsymbol{U}_a$$

  - Resulting non-linear system of equations on time interval $[t_n \; t_{n+1}]$

$$\int_V \boldsymbol{\sigma}(\boldsymbol{x}) : \nabla \otimes^s \boldsymbol{\delta u}(\boldsymbol{x}, t) dV = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV + \int_{\partial_N V} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\delta u} d\partial V$$

$$\Rightarrow \quad \boldsymbol{\delta U}_{\boldsymbol{b}}^{\mathrm{T}} \cdot \sum_{\Xi} \mathbf{B}_b^{\mathrm{T}}(\Xi)\, \boldsymbol{\sigma}\big((\Xi)\big)\omega^{\Xi} = \boldsymbol{\delta U}_{\boldsymbol{b}}^{\mathrm{T}} \cdot \sum_{\Xi} N_b(\Xi)\, \boldsymbol{b}_0(\Xi)\omega^{\Xi}$$

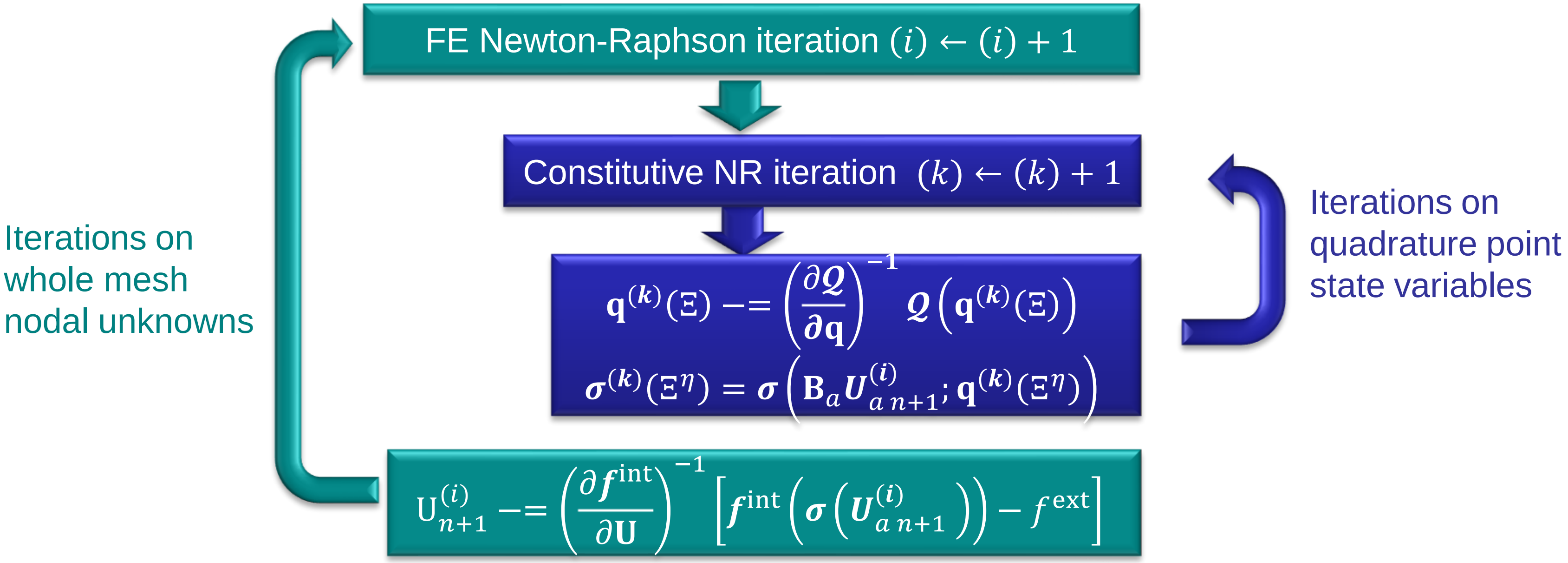
Omitting surface tractions

$$\Rightarrow \quad \boldsymbol{f}_b^{\mathrm{int}} = \sum_{\Xi} \mathbf{B}_b^{\mathrm{T}}(\Xi)\, \boldsymbol{\sigma}(\Xi)\omega^{\Xi} = \sum_{\Xi} N_b(\Xi)\, \boldsymbol{b}_0(\Xi)\omega^{\Xi} = \boldsymbol{f}_b^{\mathrm{ext}}$$

$$\text{with} \quad \begin{cases} \boldsymbol{\sigma}(\Xi, t_{n+1}) = \boldsymbol{\sigma}\big(\mathbf{B}_a(\Xi)\boldsymbol{U}_{a\,n+1}; \mathbf{q}(\Xi, \boldsymbol{t_{n+1}})\big) \\ \boldsymbol{Q}\big(\boldsymbol{\sigma}(\Xi, t_{n+1}), \mathbf{q}(\Xi, t_{n+1}), \mathbf{q}(\Xi, t_n)\big) = \mathbf{0} \end{cases}$$

Quantum computing & finite elements

- Consider classical finite element resolution on Quantum Computers?



Iterations on whole mesh nodal unknowns

FE Newton-Raphson iteration $(i) \leftarrow (i) + 1$

Constitutive NR iteration $(k) \leftarrow (k) + 1$

Iterations on quadrature point state variables

$$\mathbf{q}^{(k)}(\Xi) -= \left(\frac{\partial \boldsymbol{Q}}{\partial \mathbf{q}}\right)^{-1} \boldsymbol{Q}\left(\mathbf{q}^{(k)}(\Xi)\right)$$

$$\boldsymbol{\sigma}^{(k)}(\Xi^{\eta}) = \boldsymbol{\sigma}\left(\mathbf{B}_a \boldsymbol{U}_{a\,n+1}^{(i)}; \mathbf{q}^{(k)}(\Xi^{\eta})\right)$$

$$\mathrm{U}_{n+1}^{(i)} -= \left(\frac{\partial \boldsymbol{f}^{\mathrm{int}}}{\partial \mathbf{U}}\right)^{-1} \left[\boldsymbol{f}^{\mathrm{int}}\left(\boldsymbol{\sigma}\left(\boldsymbol{U}_{a\,n+1}^{(i)}\right)\right) - f^{\mathrm{ext}}\right]$$

  - What can be solved on a Quantum Computer?
    - Optimization problems can be solved (Actually Quantum Annealers look for a ground state)
    - Some operations can be achieved efficiently on classical computers like assembly
  - Do we need the same resolution structure?
    - Do we need intricated NR loops?
    - Do we even need to use the discretized form of the weak form?

$$\int_V \boldsymbol{\sigma}(\boldsymbol{x}) : \nabla \otimes^s \boldsymbol{\delta u}(\boldsymbol{x}) dV = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV \Rightarrow \boldsymbol{f}_b^{\mathrm{int}} = \sum_{\Xi} \mathbf{B}_b^{\mathrm{T}}(\Xi)\, \boldsymbol{\sigma}(\Xi) \omega^{\Xi} = \boldsymbol{f}_b^{\mathrm{ext}}$$

Quantum computing & finite elements

- Non-linear finite element resolution on Quantum Computers?
  - Weak form: $\int_V \boldsymbol{\sigma}(\boldsymbol{x}) : \nabla \otimes^s \boldsymbol{\delta u}(\boldsymbol{x}) dV = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV$
  - Assuming non-linear elasticity
    - Existence of a free energy $\Psi\big(\boldsymbol{\varepsilon}(x)\big)$ with $\boldsymbol{\varepsilon}(x) = \nabla \otimes^s \boldsymbol{u}(\boldsymbol{x})$
    - Stress results from $\boldsymbol{\sigma}(\boldsymbol{x}) = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}$

    The weak form becomes $\int_V \dfrac{\partial \Psi(\boldsymbol{x})}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\delta \varepsilon}(\boldsymbol{x}) dV = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV$
  - Introduction of a functional
    - $\Phi\big(\boldsymbol{u}(V)\big) = \int_V \Psi\big(\nabla \otimes^s \boldsymbol{u}(\boldsymbol{x})\big) dV - W^{\text{ext}}\big(\boldsymbol{u}(V)\big)$ & $W^{\text{ext}} = \int_V \boldsymbol{b}_0 \cdot \boldsymbol{u}(\boldsymbol{x}) dV$
    - The weak form results from nulling the Gâteaux derivative

    $$\Phi'\left(\boldsymbol{u}(V); \delta \boldsymbol{u}\big((V)\big)\right) = \int_V \boldsymbol{\sigma}(\boldsymbol{x}) : \nabla \otimes^s \boldsymbol{\delta u}(\boldsymbol{x}) dV - \int_V \boldsymbol{b}_0 \cdot \boldsymbol{\delta u} dV = \boldsymbol{0}$$

    The solution of the weak form minimizes the energy: $\boldsymbol{u}(V) = \arg\min_{\boldsymbol{u'}(V)} \Phi\big(\boldsymbol{u}'(V)\big)$
  - We are looking for the solution of a minimization problem
    - The potential is convex
    - But it is not quadratic
    - Quid inelastic materials?

Quantum computing & finite elements

- Non-linear finite element resolution on Quantum Computers?
  - Inelastic materials
    - Existence of a Helmholtz free energy $\Psi(\boldsymbol{\varepsilon}(x), \mathbf{q}(x))$ with $\begin{cases} \text{internal variables } \mathbf{q}(x) \\ \boldsymbol{\varepsilon}(x) = \nabla \otimes^s \boldsymbol{u}(\boldsymbol{x}) \end{cases}$
  - Dissipation $\mathcal{D}$ and Clausius-Duhem inequality
    - $\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geq 0$ with $\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}}$
    - Equality holds in case of a reversible transformation

⇒ $\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}$ ⇒ for an irreversible process: $\mathcal{D} = \mathbf{Y} \cdot \dot{\mathbf{q}} \geq 0$ with $\mathbf{Y} = -\frac{\partial \Psi}{\partial \mathbf{q}}$

  - Postulate the existence of a pseudo-potential $\Theta(\dot{\mathbf{q}})$ and its convex dual $\Theta^*(\mathbf{Y})$
    - $\Theta(\dot{\mathbf{q}}) = \max_{\mathbf{Y}} [\mathbf{Y} \cdot \dot{\mathbf{q}} - \Theta^*(\mathbf{Y})]$ ⇒ $\dot{\mathbf{q}} = \frac{\partial \Theta^*(\mathbf{Y})}{\partial \mathbf{Y}}$ & $\mathbf{Y} = \frac{\partial \Theta(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}$
  - Power functional $\mathcal{E}$
    - New independent variables $(\dot{\boldsymbol{\varepsilon}}, \dot{\mathbf{q}})$
    - $\mathcal{E}(\dot{\boldsymbol{\varepsilon}}, \dot{\mathbf{q}}) = \dot{\Psi} + \Theta(\dot{\mathbf{q}}) = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \mathbf{Y} \cdot \dot{\mathbf{q}} + \Theta(\dot{\mathbf{q}})$

⇒ $\frac{\partial \mathcal{E}}{\partial \dot{\mathbf{q}}} = -\mathbf{Y} + \frac{\partial \Theta(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} = \mathbf{0}$ ⇒ $\mathcal{E}$ has to be minimized with respect to internal state

    - Effective power functional* $\mathcal{E}^{\text{eff}}(\dot{\boldsymbol{\varepsilon}}) = \min_{\dot{\mathbf{q}}} \mathcal{E}(\dot{\boldsymbol{\varepsilon}}, \dot{\mathbf{q}})$ with $\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \mathcal{E}^{\text{eff}}}{\partial \dot{\boldsymbol{\varepsilon}}}$
  - The constitutive model is also a minimization problem

*Radovitzky, R. Ortiz M, CMAME 1999
Ortiz, M., Stainier, L., CMAME 1999

Quantum computing & finite elements

- Non-linear finite element resolution on Quantum Computers?
  - In elasticity we had
    - $\boldsymbol{u}(V) = \arg\min_{\boldsymbol{u}'(V)} \Phi\big(\boldsymbol{u}'(V)\big)$ with $\Phi\big(\boldsymbol{u}(V)\big) = \int_V \Psi\big(\nabla \otimes^s \boldsymbol{u}(\boldsymbol{x})\big) dV - W^{\text{ext}}\big(\boldsymbol{u}(\boldsymbol{x})\big)$
  - Double minimization problem in inelasticity
    - Power functional $\mathcal{E}$

$$\mathcal{E}(\dot{\boldsymbol{\varepsilon}}, \dot{\mathbf{q}}) = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \mathbf{Y} \cdot \dot{\mathbf{q}} + \Theta(\dot{\mathbf{q}}) \quad \& \quad \mathcal{E}^{\text{eff}}(\dot{\boldsymbol{\varepsilon}}) = \min_{\dot{\mathbf{q}}} \mathcal{E}(\dot{\boldsymbol{\varepsilon}}, \dot{\mathbf{q}}) \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial \mathcal{E}^{\text{eff}}}{\partial \dot{\boldsymbol{\varepsilon}}}$$

    - Volume power functional

$$\Phi\big(\dot{\boldsymbol{u}}(V), \dot{\mathbf{q}}(V)\big) = \int_V \mathcal{E}(\nabla \otimes^s \dot{\boldsymbol{u}}, \dot{\mathbf{q}}) - \dot{W}^{\text{ext}}\big(\dot{\boldsymbol{u}}(V)\big)$$

    - Incremental volume energy functional on time interval $[t_n \; t_{n+1}]$*

$$\Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}) = \int_V \Delta\mathcal{E}(\nabla \otimes^s \boldsymbol{u}_{n+1}, q_{n+1}) - \Delta W^{\text{ext}}(\boldsymbol{u}_{n+1})$$

with $\Delta\mathcal{E}(\nabla \otimes^s \boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}) = \int_{t_n}^{t_{n+1}} \mathcal{E}(\nabla \otimes^s \dot{\boldsymbol{u}}, \dot{\mathbf{q}})$ & $\Delta\mathcal{E}^{\text{eff}}(\boldsymbol{\varepsilon}) = \min_{\mathbf{q}} \Delta\mathcal{E}(\boldsymbol{\varepsilon}, \mathbf{q})$, $\boldsymbol{\sigma} = \dfrac{\partial \Delta\mathcal{E}^{\text{eff}}}{\partial \boldsymbol{\varepsilon}}$

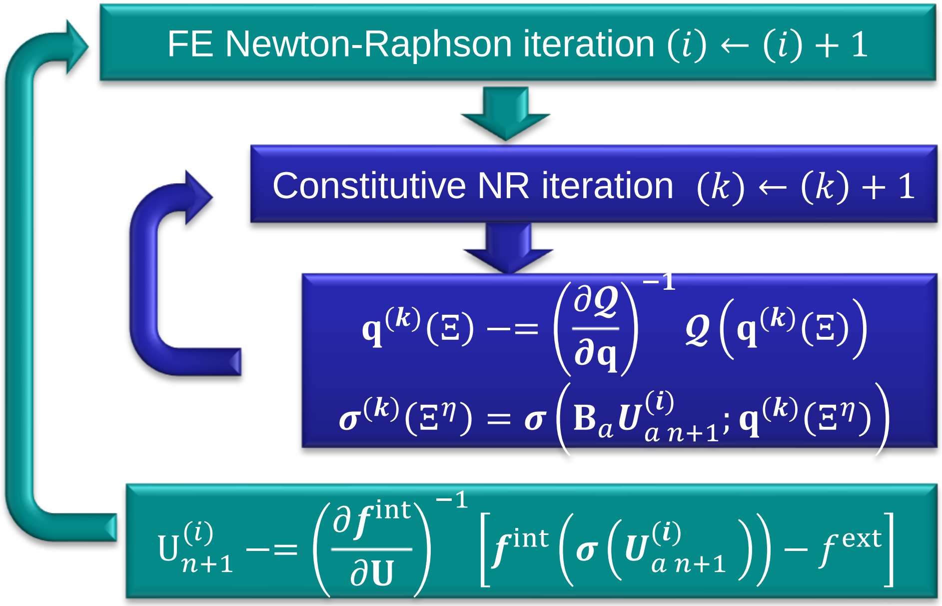
    - The problem solution reads

$$\left\{\begin{aligned} &\mathbf{q}_{n+1} = \arg\min_{\mathbf{q}'} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}') \\ &\Delta\Phi^{\text{eff}}(\boldsymbol{u}_{n+1}) = \min_{\mathbf{q}'} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}') = \int_V \Delta\mathcal{E}^{\text{eff}}(\nabla \otimes^s \boldsymbol{u}_{n+1}) - \Delta W^{\text{ext}} \\ &\boldsymbol{u}_{n+1} = \arg \min_{\mathbf{u}' \text{admissible}} \Delta\Phi^{\text{eff}}(\boldsymbol{u}') \end{aligned}\right.$$

*Ortiz, M., Stainier, L., CMAME 1999

Quantum computing & finite elements

- Classical finite element resolution



FE Newton-Raphson iteration $(i) \leftarrow (i) + 1$

Constitutive NR iteration $(k) \leftarrow (k) + 1$

$$\mathbf{q}^{(k)}(\Xi) -= \left(\frac{\partial \mathcal{Q}}{\partial \mathbf{q}}\right)^{-1} \mathcal{Q}\left(\mathbf{q}^{(k)}(\Xi)\right)$$

$$\boldsymbol{\sigma}^{(k)}(\Xi^{\eta}) = \boldsymbol{\sigma}\left(\mathbf{B}_a \boldsymbol{U}_{a\,n+1}^{(i)}; \mathbf{q}^{(k)}(\Xi^{\eta})\right)$$

$$\mathrm{U}_{n+1}^{(i)} -= \left(\frac{\partial \boldsymbol{f}^{\mathrm{int}}}{\partial \mathbf{U}}\right)^{-1} \left[\boldsymbol{f}^{\mathrm{int}}\left(\boldsymbol{\sigma}\left(\boldsymbol{U}_{a\,n+1}^{(i)}\right)\right) - f^{\mathrm{ext}}\right]$$

- Finite element as a double-minimization problem

Loop until convergence

$$\mathbf{q}_{n+1} = \arg \min_{\mathbf{q}'\text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')\,;$$

$$\Delta\Phi^{\mathrm{eff}} = \min_{\mathbf{q}'\text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')$$

$$\boldsymbol{u}_{n+1} = \arg \min_{\mathbf{u}'\text{admissible}} \Delta\Phi^{\mathrm{eff}}(\mathbf{u}')$$

Internal variables can be constrained (e.g. $\mathbf{N}:\mathbf{N} = \frac{3}{2}, \Delta\gamma \geq 0$)

  - Quantum annealers: ground state of an Ising-Hamiltonian
    - No need for Jacobians
    - No problem of convergence (but needs to be noise-contained)
  - But how to make the optimisation problem solvable by quantum annealing?

Double-minimization process solved by Quantum annealing

- Finite element as a double-minimization problem
  - Finite element problem

Loop until convergence

$$\mathbf{q}_{n+1} = \arg \min_{\mathbf{q}' \text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')\,;$$

$$\Delta\Phi^{\text{eff}} = \min_{\mathbf{q}' \text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')$$

$$\boldsymbol{u}_{n+1} = \arg \min_{\mathbf{u}' \text{admissible}} \Delta\Phi^{\text{eff}}(\mathbf{u}')$$

  - Ising Hamiltonian for Quantum annealing
    - Goal: finding the ground state of a Hamiltonian $\mathbf{H}$: $\mathbf{H} = \sum_{i \in V} h_i \mathbf{Z}_i + \sum_{(i,j) \in E} J_{ij} \mathbf{Z}_{ij}$

$$|\boldsymbol{\phi}_0\rangle = \arg \min_{|\boldsymbol{\phi}\rangle} \langle \boldsymbol{\phi} | \mathbf{H} | \boldsymbol{\phi} \rangle$$

    - Problem reformulated in terms of binary variables $\mathbf{b} = [b_i \; \forall i \in V]$ with $b_i \in \{0, 1\}$
    - QUBO optimisation problem $\mathcal{F}_{\text{QUBO}} = \sum_{(i,j) \in E \cup \{(i,i) \forall i \in V\}} A_{ij} b_i b_j = \mathbf{b}^T \mathbf{A} \mathbf{b}$

$$\mathbf{b} = \arg \min_{\mathbf{b}'} \mathcal{F}_{\text{QUBO}}(\mathbf{b}'; \mathbf{A})$$

User programmable parameters

  - Steps to follow
    - Transform the constrained minimization problem into an unconstrained one
    - Transform the general unconstrained optimization problem into a series of quadratic ones
    - Transform each continuous quadratic optimization problem into a binarized one
    - Apply the double-minimization framework

Double-minimization process solved by Quantum annealing

- Transform the constrained minimization problem into an unconstrained one
  - Constrained multivariate minimization problem
    - $\min_{\mathbf{w}} f(\mathbf{w})$ with $\mathbf{w}^{\min} \leq \mathbf{w} \leq \mathbf{w}^{\max}$
    - Under constraints $h(\mathbf{w}) = 0$ & $l(\mathbf{w}) \leq 0$
  - Augmented minimization problem
    - $f_{\text{aug}}(\mathbf{v}) = f_{\text{aug}}(\mathbf{w}, \lambda) = f(\mathbf{w}) + c^h\left(h(\mathbf{w})\right)^2 + c^l(l(\mathbf{w}) + \lambda)^2$ with $\mathbf{v} = \{\boldsymbol{w}, \lambda \geq 0\}$
  - Unconstrained minimization problem
    - $\min_{\mathbf{v}} f_{\text{aug}}(\mathbf{v})$ with $\mathbf{v}^{\min} \leq \mathbf{v} \leq \mathbf{v}^{\max}$
    - Bounds will be enforced during the binarization process
  - Definition of the double-unconstrained minimization problem

Loop until convergence

$$\mathbf{q}_{n+1} = \arg \min_{\mathbf{q}' \text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')\,;$$

$$\Delta\Phi^{\text{eff}} = \min_{\mathbf{q}' \text{constrained}} \Delta\Phi(\boldsymbol{u}_{n+1}, \mathbf{q}')$$

$$\boldsymbol{u}_{n+1} = \arg \min_{\mathbf{u}' \text{admissible}} \Delta\Phi^{\text{eff}}(\mathbf{u}')$$

Loop until convergence

$$\mathbf{q}_{n+1}, \lambda = \arg \min_{\{\mathbf{q}', \lambda'\}} \Delta\Phi_{\text{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}', \lambda')\,;$$

$$\Delta\Phi^{\text{eff}} = \min_{\{\mathbf{q}', \lambda'\}} \Delta\Phi_{\text{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}', \lambda')$$

$$\boldsymbol{u}_{n+1} = \arg \min_{\mathbf{u}' \text{admissible}} \Delta\Phi^{\text{eff}}(\mathbf{u}')$$

Double-minimization process solved by Quantum annealing

- Transform the optimization problem into a series of quadratic ones
  - Unconstrained optimization problem
    - $\min_{\mathbf{v}} f_{\text{aug}}(\mathbf{v})$ with $\mathbf{v}^{\min} \leq \mathbf{v} \leq \mathbf{v}^{\max}$
  - Taylor's expansion
    - $f_{\text{aug}}(\mathbf{v}+\mathbf{z}) \simeq f_{\text{aug}}(\mathbf{v}) + \underbrace{\mathbf{z}^{\mathrm{T}} f_{\text{aug},\mathbf{v}} + \frac{1}{2}\mathbf{z}^{\mathrm{T}} f_{\text{aug},\mathbf{vv}}\,\mathbf{z}}_{\mathrm{QF}(\mathbf{z};\, f_{\text{aug},\mathbf{v}}\,,\, f_{\text{aug},\mathbf{vv}}\,)}$
  - New series of optimization problems
    - Iterate on $\mathbf{z}$ with: $\mathbf{z} = \arg\min_{\mathbf{z}'} \mathrm{QF}(\mathbf{z}';\, f_{\text{aug},\mathbf{v}}\,,\, f_{\text{aug},\mathbf{vv}}\,)$
  - Application to the double minimisation problem

$$\left\{\begin{array}{l} f_{\text{aug},\mathbf{v}\ i} = \left.\dfrac{\partial f_{\text{aug}}}{\partial v_i}\right|_{\mathbf{v}} \\ f_{\text{aug},\mathbf{vv}\ ij} = \left.\dfrac{\partial^2 f_{\text{aug}}}{\partial v_i \partial v_j}\right|_{\mathbf{v}} \end{array}\right.$$

Loop until convergence

$$\mathbf{q}_{n+1}, \lambda = \arg\min_{\{\mathbf{q}',\lambda'\}} \Delta\Phi_{\text{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}', \lambda')\,;$$

$$\Delta\Phi^{\text{eff}} = \min_{\{\mathbf{q}',\lambda'\}} \Delta\Phi_{\text{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}', \lambda')$$

$$\boldsymbol{u}_{n+1} = \arg\min_{\mathbf{u}'\text{admissible}} \Delta\Phi^{\text{eff}}(\mathbf{u}')$$

Loop until convergence

Loop on $\boldsymbol{u}_{n+1} \leftarrow \boldsymbol{u}_{n+1} + \boldsymbol{\Delta}\mathbf{u}$

$$\boldsymbol{\Delta u} = \arg\min_{\boldsymbol{\Delta}\mathbf{u}'\text{admissible}} \boldsymbol{\Delta}\mathbf{u}'^{\mathrm{T}} \Delta\Phi^{\text{eff}}_{,\mathbf{u}} + \frac{1}{2}\boldsymbol{\Delta}\mathbf{u}'^{\mathrm{T}} \Delta\Phi^{\text{eff}}_{,\mathbf{uu}} \ \boldsymbol{\Delta}\mathbf{u}'$$

Loop on $\boldsymbol{q}_{n+1} \leftarrow \boldsymbol{q}_{n+1} + \boldsymbol{\Delta}\mathbf{q},\ \boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \Delta\lambda$

$$\boldsymbol{\Delta}\mathbf{q}, \Delta\lambda = \arg\min_{\{\boldsymbol{\Delta}\mathbf{q}',\Delta\lambda'\}} \left[\boldsymbol{\Delta}\mathbf{q}'^{\mathrm{T}} \Delta\lambda'\right] \Delta\Phi_{\text{aug},\{\mathrm{q}\ \lambda\}} + \tfrac{1}{2}\left[\boldsymbol{\Delta}\mathbf{q}'^{\mathrm{T}} \Delta\lambda'\right] \Delta\Phi_{\text{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}} \left[\boldsymbol{\Delta}\mathbf{q}'^{\mathrm{T}} \Delta\lambda'\right]^{\mathrm{T}}$$

$$\Delta\Phi^{\text{eff}} = \Delta\Phi_{\text{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}, \lambda)$$

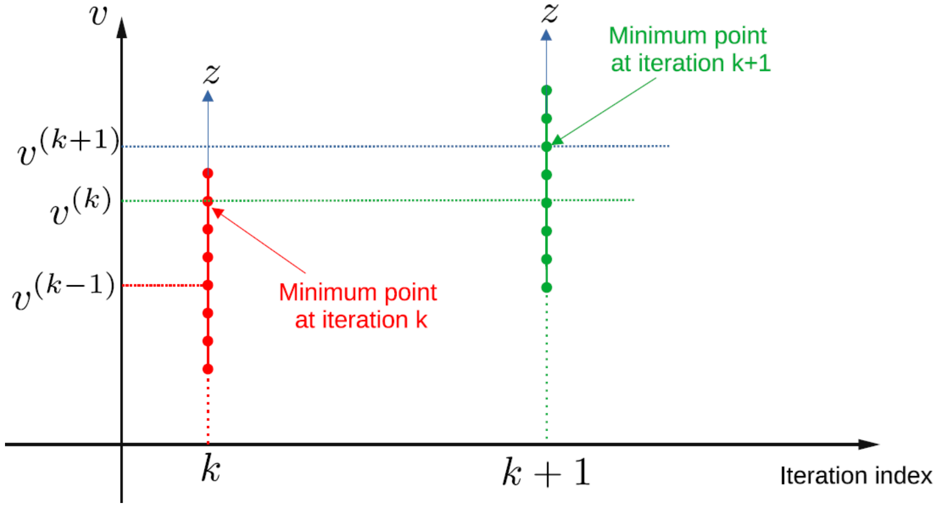
Allow to contain the noise !!!

Double-minimization process solved by Quantum annealing

- Transform each continuous quadratic optimization problem into a binarized one
  - Optimization problems to be solved
    - $\mathbf{z} = \arg\min_{\mathbf{z}'} \mathrm{QF}(\mathbf{z}', f_{\text{aug},\mathbf{v}}, f_{\text{aug},\mathbf{vv}})$ & $\mathrm{QF}(\mathbf{z}; f_{\text{aug},\mathbf{v}}, f_{\text{aug},\mathbf{vv}}) = \mathbf{z}^{\mathrm{T}} f_{\text{aug},\mathbf{v}} + \frac{\mathbf{1}}{\mathbf{2}} \mathbf{z}^{\mathrm{T}} f_{\text{aug},\mathbf{vv}}\, \mathbf{z}$
    - With bounds: $\mathbf{v}_{\min} \le \mathbf{v} + \mathbf{z} \le \mathbf{v}_{\max}$
  - Binarization of $z \in \mathbb{R}^N$ into $N \times L$ qubits
    - $b_{L-1} \dots b_0 \equiv \sum_{j=0}^{L-1} b_j\, 2^j = \boldsymbol{\beta}^{\mathrm{T}} \mathbf{b}_i$ ⇨ $z_1 = z_i^{\min} + \epsilon_i \boldsymbol{\beta}^T \mathbf{b}_i$
    - $\mathbf{z} = \mathbf{a} + \mathbf{D}(\boldsymbol{\epsilon})\mathbf{b}$ with the bounds defining $\mathbf{a} = \mathbf{z}^{\min}$ & the scale $\boldsymbol{\epsilon} = \dfrac{\mathbf{z}^{\max} - \mathbf{z}^{\min}}{2^L - 1}$

⇨ $$\mathrm{QF}\big(\mathbf{z}; f_{\text{aug},\mathbf{v}}, f_{\text{aug},\mathbf{vv}}\big) = \underbrace{\frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}} f_{\text{aug},\mathbf{vv}}\, \mathbf{D}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\big(f_{\text{aug},\mathbf{v}} + f_{\text{aug},\mathbf{vv}}\, \mathbf{a}\big)}_{\mathcal{F}_{\text{QUBO}}(\mathbf{b};\,\mathbf{A})} + \frac{\mathbf{1}}{\mathbf{2}} \mathbf{a}^{\mathrm{T}}\big(f_{\text{aug},\mathbf{vv}}\, \mathbf{a} + f_{\text{aug},\mathbf{v}}\big)$$

  - Minimization
    - Bound $\mathbf{a} = \mathbf{z}^{\min}$ and
    - Scale $\boldsymbol{\epsilon} = \dfrac{\mathbf{z}^{\max} - \mathbf{z}^{\min}}{2^L - 1}$
    - Updated when building the QUBO



Double-minimization process solved by Quantum annealing

- Application to the double-minimization problem

```
Loop until convergence
    Loop on $\boldsymbol{u}_{n+1} \leftarrow \boldsymbol{u}_{n+1} + \mathbf{\Delta u}$
        $\boldsymbol{\Delta u} = \arg \min_{\mathbf{\Delta u}' \text{admissible}} \mathbf{\Delta u}'^{\mathrm{T}} \Delta\Phi^{\mathrm{eff}}_{,\mathbf{u}} + \frac{1}{2} \mathbf{\Delta u}'^{\mathrm{T}} \Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}} \quad \mathbf{\Delta u}'$
    Loop on $\boldsymbol{q}_{n+1} \leftarrow \boldsymbol{q}_{n+1} + \mathbf{\Delta q},\ \boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \Delta\lambda$
        $\mathbf{\Delta q}, \Delta\lambda = \arg \min_{\{\mathbf{\Delta q}', \mathbf{\Delta}\lambda'\}} [\mathbf{\Delta q}'^{\mathrm{T}} \Delta\lambda'] \, \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}} + \frac{1}{2} [\mathbf{\Delta q}'^{\mathrm{T}} \Delta\lambda'] \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}\{\mathrm{q}\,\lambda\}} \, [\mathbf{\Delta q}'^{\mathrm{T}} \Delta\lambda']^{\mathrm{T}}$
    $\Delta\Phi^{\mathrm{eff}} = \Delta\Phi_{\mathrm{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}, \lambda)$
```

```
Loop until convergence
    Loop on $\boldsymbol{u}_{n+1} \leftarrow \boldsymbol{u}_{n+1} + \mathbf{\Delta u}$
        $f(\mathbf{\Delta u}) = \mathbf{\Delta u}^{\mathrm{T}} \Delta\Phi^{\mathrm{eff}}_{,\mathbf{u}} + \frac{1}{2} \mathbf{\Delta u}^{\mathrm{T}} \Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{\Delta u}$
        Update $\mathbf{a}_{\mathrm{u}}, \mathbf{D}(\boldsymbol{\epsilon}_{\mathbf{u}})$
        $\mathbf{b}_{\mathbf{u}} = \arg \min_{\mathbf{b}'_{\mathbf{u}}} \left( \frac{1}{2} \mathbf{b}'^{\mathrm{T}}_{\mathbf{u}} \mathbf{D}^{\mathrm{T}} \Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{D} \mathbf{b}'_{\mathbf{u}} + \mathbf{b}'^{\mathrm{T}}_{\mathbf{u}} \mathbf{D}^{\mathrm{T}} \left( \Delta\Phi^{\mathrm{eff}}_{,\mathbf{u}} + \Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{a}_{\mathbf{u}} \right) \right)$    <- Quantum annealing
    Loop on $\boldsymbol{q}_{n+1} \leftarrow \boldsymbol{q}_{n+1} + \mathbf{\Delta q},\ \boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \Delta\lambda$
        $f(\mathbf{\Delta q}, \Delta\lambda) = [\mathbf{\Delta q}^{\mathrm{T}} \Delta\lambda] \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}} + \frac{1}{2} [\mathbf{\Delta q}^{\mathrm{T}} \Delta\lambda] \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}\{\mathrm{q}\,\lambda\}} [\mathbf{\Delta q}^{\mathrm{T}} \Delta\lambda]^{\mathrm{T}}$
        Update $\mathbf{a}_{\mathbf{q}}, \mathbf{D}(\boldsymbol{\epsilon}_{\mathbf{q}})$
        $\mathbf{b}_{\mathbf{q}} = \arg \min_{\mathbf{b}'_{\mathbf{q}}} \left( \frac{1}{2} \mathbf{b}'^{\mathrm{T}}_{\mathbf{q}} \mathbf{D}^{\mathrm{T}} \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}\{\mathrm{q}\,\lambda\}} \mathbf{D} \mathbf{b}'_{\mathbf{q}} + \mathbf{b}'^{\mathrm{T}}_{\mathbf{q}} \mathbf{D}^{\mathrm{T}} \left( \Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}} + \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\,\lambda\}\{\mathrm{q}\,\lambda\}} \mathbf{a}_{\mathbf{q}} \right) \right)$    <- Quantum annealing
    $\Delta\Phi^{\mathrm{eff}} = \Delta\Phi_{\mathrm{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}, \lambda)$
```

Double-minimization process solved by Quantum annealing

- Application to the double-minimization problem

```
Loop until convergence
    Loop on $\boldsymbol{u}_{n+1} \leftarrow \boldsymbol{u}_{n+1} + \mathbf{\Delta u}$
        $f(\mathbf{\Delta u}) = \mathbf{\Delta u}^{\mathrm{T}} \Delta \Phi^{\mathrm{eff}}_{,\mathbf{u}} + \frac{1}{2} \mathbf{\Delta u}^{\mathrm{T}} \Delta \Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{\Delta u}$
        Update $\mathbf{a}_{\mathrm{u}}, \mathrm{D}(\boldsymbol{\epsilon}_{\mathbf{u}})$
        $\mathbf{b}_{\mathbf{u}} = \arg\min_{\mathbf{b}'_{\mathbf{u}}} \left( \frac{1}{2} \mathbf{b}'_{\mathbf{u}}{}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \Delta \Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{D} \mathbf{b}'_{\mathbf{u}} + \mathbf{b}'_{\mathbf{u}}{}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \left( \Delta \Phi^{\mathrm{eff}}_{,\mathbf{u}} + \Delta \Phi^{\mathrm{eff}}_{,\mathbf{uu}} \mathbf{a}_{\mathbf{u}} \right) \right)$
    Loop on $\boldsymbol{q}_{n+1} \leftarrow \boldsymbol{q}_{n+1} + \mathbf{\Delta q}, \boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \Delta \lambda$
        $f(\mathbf{\Delta q}, \Delta \lambda) = [\mathbf{\Delta q}^{\mathrm{T}} \Delta \lambda] \Delta \Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}} + \frac{1}{2} [\mathbf{\Delta q}^{\mathrm{T}} \Delta \lambda] \Delta \Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}} [\mathbf{\Delta q}^{\mathrm{T}} \Delta \lambda]^{\mathrm{T}}$
        Update $\mathbf{a}_{\mathbf{q}}, \mathrm{D}(\boldsymbol{\epsilon}_{\mathbf{q}})$
        $\mathbf{b}_{\mathbf{q}} = \arg\min_{\mathbf{b}'_{\mathbf{q}}} \left( \frac{1}{2} \mathbf{b}'_{\mathbf{q}}{}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \Delta \Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}} \mathbf{D} \mathbf{b}'_{\mathbf{q}} + \mathbf{b}'_{\mathbf{q}}{}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \left( \Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}} + \Delta \Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}} \mathbf{a}_{\mathbf{q}} \right) \right)$
    $\Delta \Phi^{\mathrm{eff}} = \Delta \Phi_{\mathrm{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}, \lambda)$
```

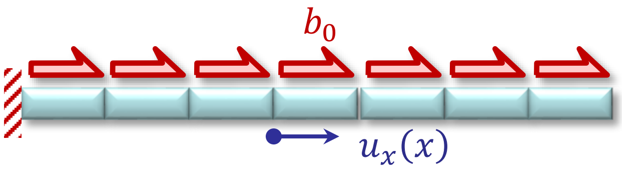
Quantum annealing

Quantum annealing

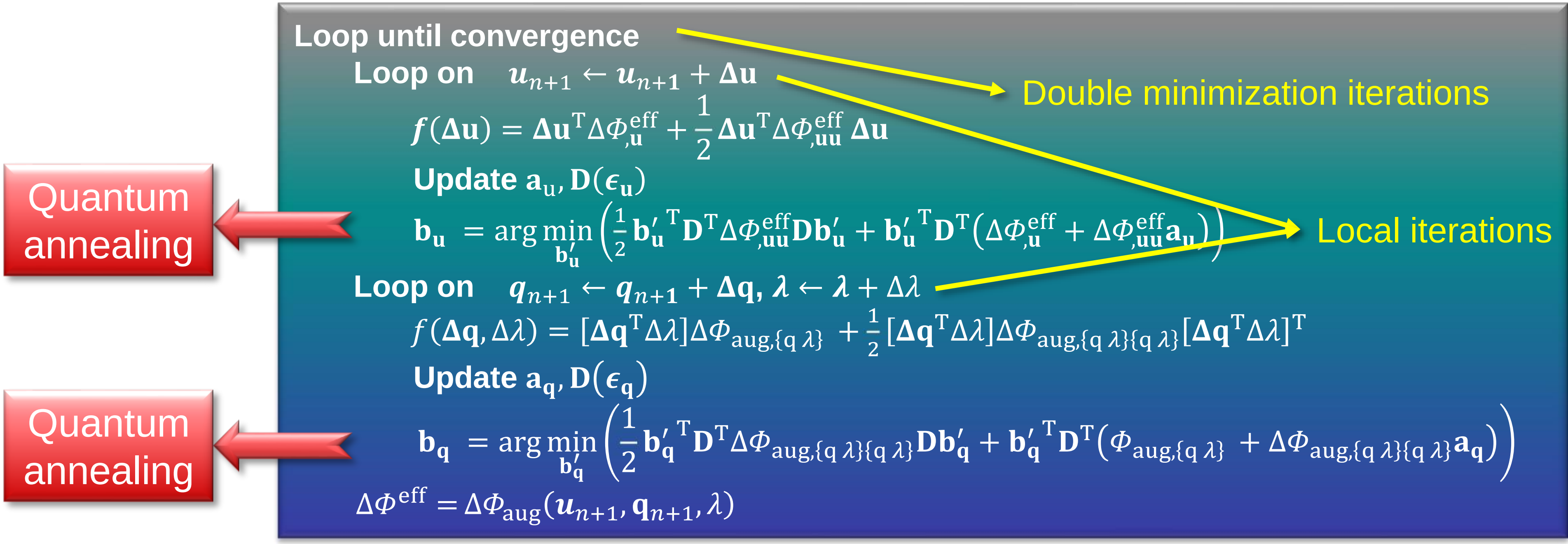
```
from dwave.system import DWaveSampler, EmbeddingComposite
sampler = EmbeddingComposite(DWaveSampler())
sampleset = sampler.sample_qubo(A, num_reads=100, annealing_time=20)
b = sampleset.first.sample
```

Application on 1D problems

- Uniaxial-strain test



- Elasto-plastic case
  - Double minimization
    - Binarizations L of each nodal displacement and internal variable: $b_{L-1}\ \dots b_0 \equiv \sum_{j=0}^{L-1} b_j\ 2^j = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}_{\boldsymbol{i}}$
  - Resolution by quantum annealing on DWave Advantage QPU



```
Loop until convergence
    Loop on  $\boldsymbol{u}_{n+1} \leftarrow \boldsymbol{u}_{n+1} + \boldsymbol{\Delta}\mathbf{u}$
        $f(\boldsymbol{\Delta}\mathbf{u}) = \boldsymbol{\Delta}\mathbf{u}^{\mathrm{T}}\Delta\Phi^{\mathrm{eff}}_{,\mathbf{u}} + \frac{1}{2}\boldsymbol{\Delta}\mathbf{u}^{\mathrm{T}}\Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}}\ \boldsymbol{\Delta}\mathbf{u}$
        Update $\mathbf{a}_{\mathrm{u}}, \mathbf{D}(\boldsymbol{\epsilon}_{\mathbf{u}})$
        $\mathbf{b}_{\mathbf{u}}\ = \arg\min_{\mathbf{b}'_{\mathbf{u}}}\left(\frac{1}{2}\mathbf{b}'_{\mathbf{u}}{}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}}\mathbf{D}\mathbf{b}'_{\mathbf{u}} + \mathbf{b}'_{\mathbf{u}}{}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\left(\Delta\Phi^{\mathrm{eff}}_{,\mathbf{u}} + \Delta\Phi^{\mathrm{eff}}_{,\mathbf{uu}}\mathbf{a}_{\mathbf{u}}\right)\right)$
    Loop on  $\boldsymbol{q}_{n+1} \leftarrow \boldsymbol{q}_{n+1} + \boldsymbol{\Delta}\mathbf{q}, \boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \Delta\lambda$
        $f(\boldsymbol{\Delta}\mathbf{q}, \Delta\lambda) = [\boldsymbol{\Delta}\mathbf{q}^{\mathrm{T}}\Delta\lambda]\Delta\Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}} + \frac{1}{2}[\boldsymbol{\Delta}\mathbf{q}^{\mathrm{T}}\Delta\lambda]\Delta\Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}}[\boldsymbol{\Delta}\mathbf{q}^{\mathrm{T}}\Delta\lambda]^{\mathrm{T}}$
        Update $\mathbf{a}_{\mathbf{q}}, \mathbf{D}(\boldsymbol{\epsilon}_{\mathbf{q}})$
        $\mathbf{b}_{\mathbf{q}}\ = \arg\min_{\mathbf{b}'_{\mathbf{q}}}\left(\frac{1}{2}\mathbf{b}'_{\mathbf{q}}{}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\Delta\Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}}\mathbf{D}\mathbf{b}'_{\mathbf{q}} + \mathbf{b}'_{\mathbf{q}}{}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\left(\Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}} + \Delta\Phi_{\mathrm{aug},\{\mathrm{q}\ \lambda\}\{\mathrm{q}\ \lambda\}}\mathbf{a}_{\mathbf{q}}\right)\right)$
    $\Delta\Phi^{\mathrm{eff}} = \Delta\Phi_{\mathrm{aug}}(\boldsymbol{u}_{n+1}, \mathbf{q}_{n+1}, \lambda)$
```

Double minimization iterations

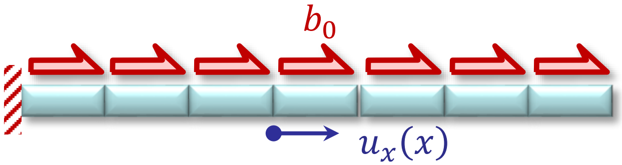
Local iterations

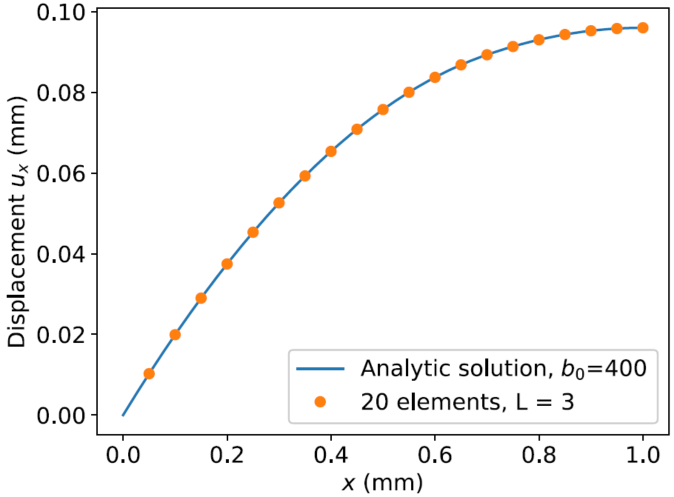
Quantum annealing

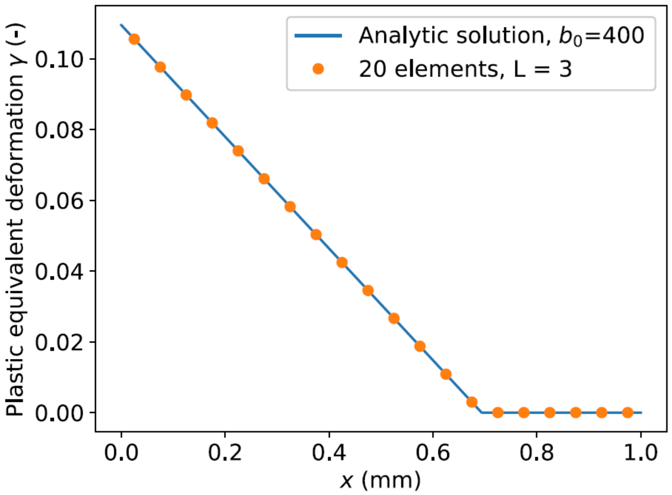
Quantum annealing

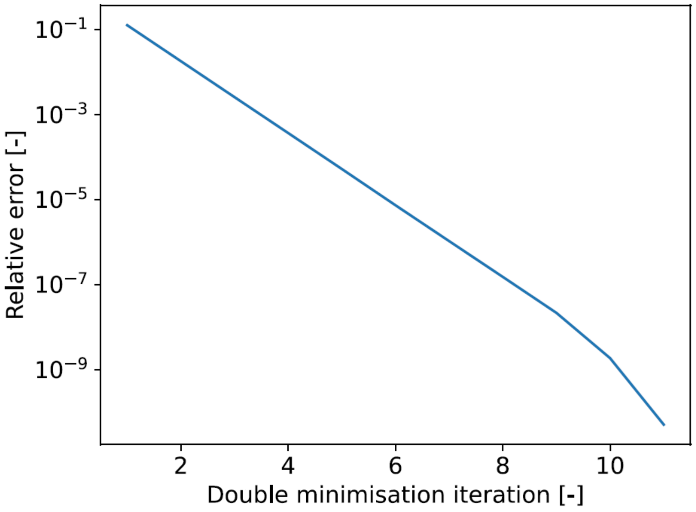
Application on 1D problems

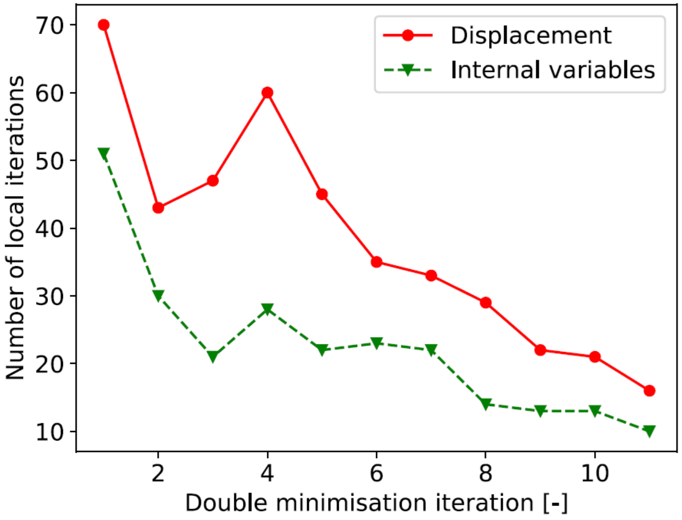
- Uniaxial-strain test
- Elasto-plastic case
  - Effect of double-minimization & local iterations







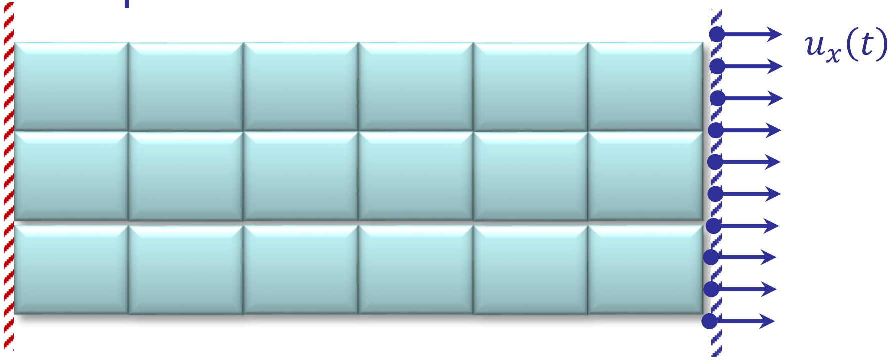


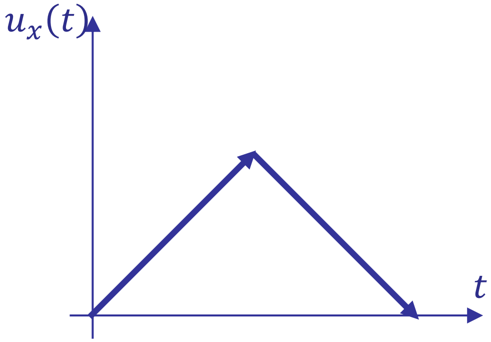


The number of local iterations decreases as the double minimisation iterations proceed

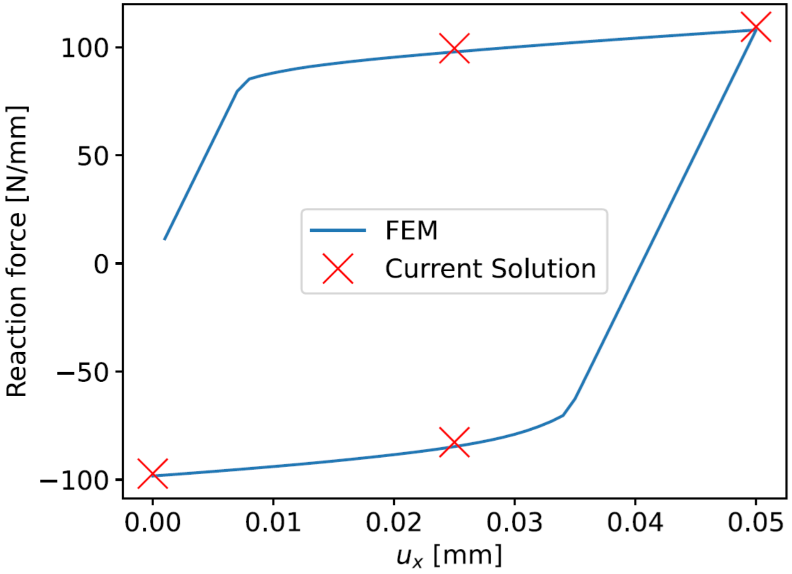
Application on 2D problems

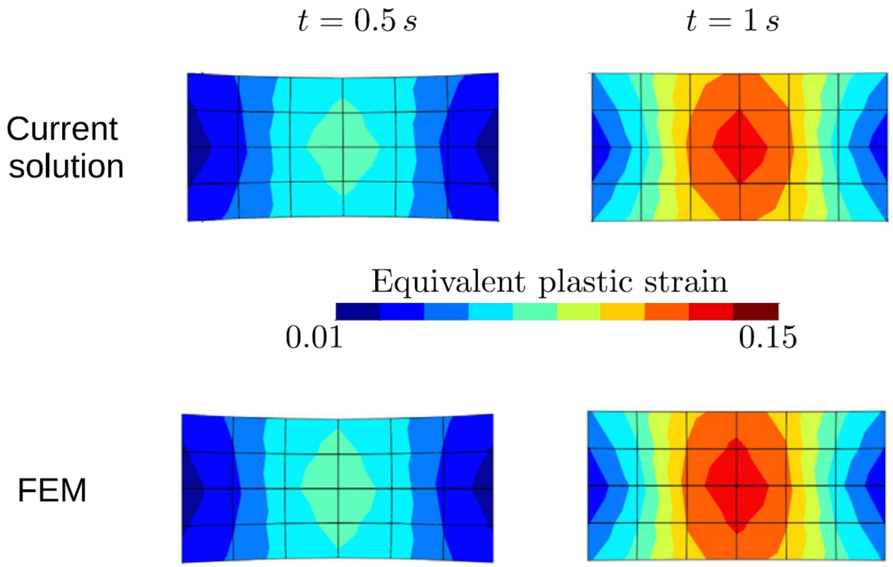
- 2D-elasto-plastic case





  - Double minimization
    - Binarizations L of each nodal displacement and internal variable: $b_{L-1}\dots b_0 \equiv \sum_{j=0}^{L-1} b_j\ 2^j = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}_{\boldsymbol{i}}$
  - Resolution by quantum annealing on DWave Advantage QPU





Application on 2D problems

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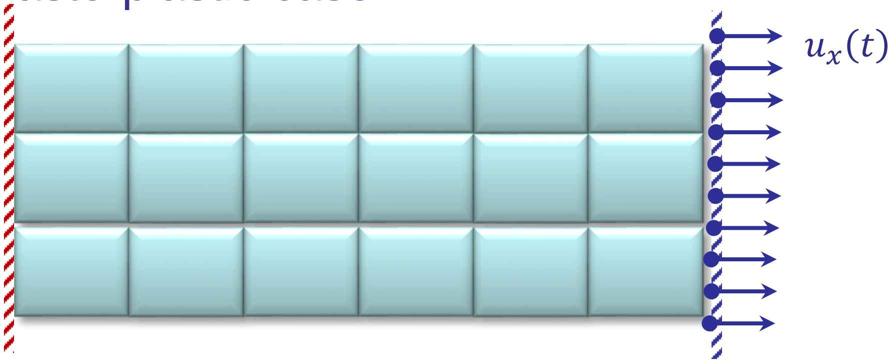
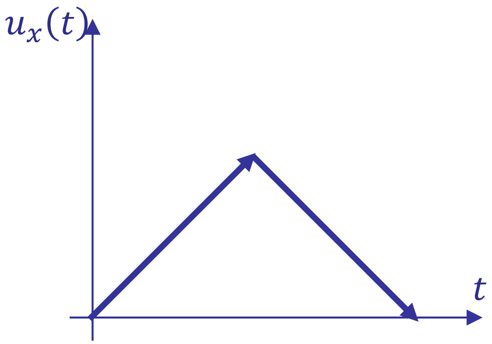
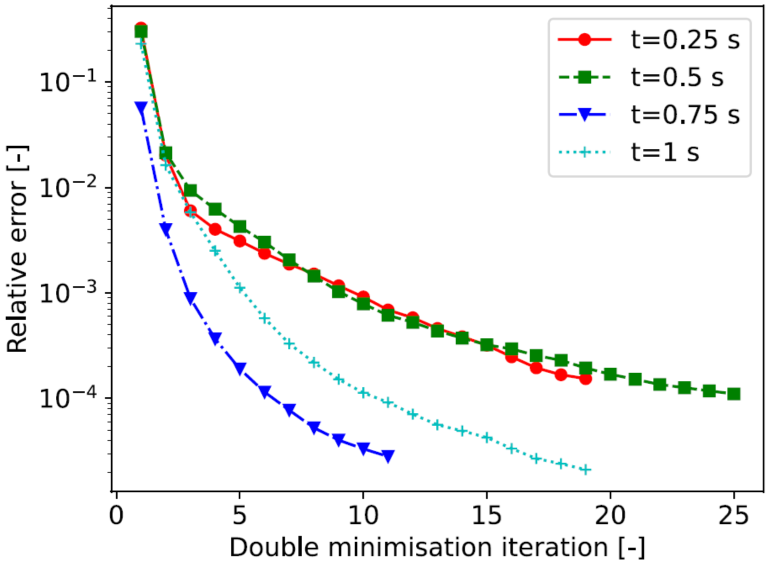
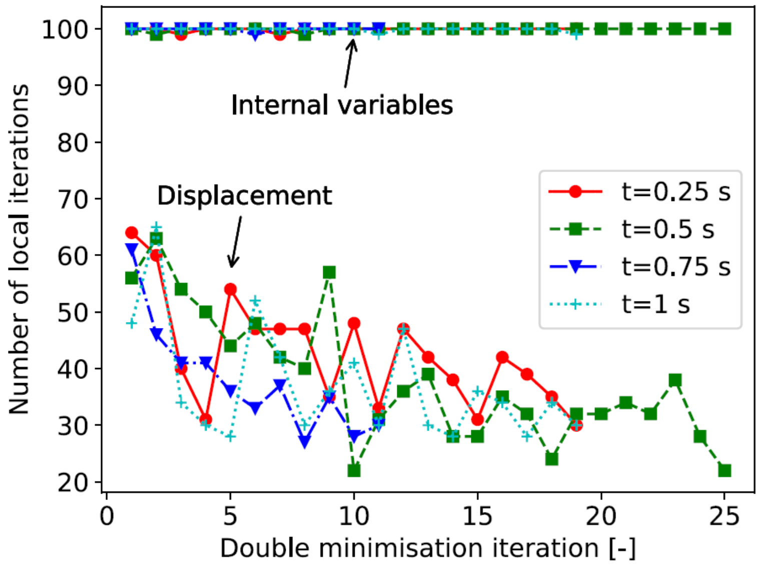




  - Effect of double-minimization & local iterations





Conclusions

- Application of QC to FEM
  - FE resolution needs to be rethought
  - It will probably stay advantageous to solve part of the problem on classical computers

- Quantum annealing
  - Real annealers can now be used
  - Efficient to solve optimization problem…. FEM is actually a minimization problem
  - Main current limitation is the number of connected qubits

- Publication
  - V. D. Nguyen, F. Remacle, L. Noels. A quantum annealing-sequential quadratic programming assisted finite element simulation for non-linear and history-dependent mechanical problems. *European Journal of Mechanics – A/solids* 105, 105254 10.1016/j.euromechsol.2024.105254
- Data and code on
  - Doi: 10.5281/zenodo.10451584