SDPM - Source and Doublet Panel Method

Theory manual and quick reference guide

Adrien Crovato

Department of Aerospace & Mechanical Engineering

Abstract

This document provides the mathematical formulation of the main equations implemented in SDPM [1], version 1.2.0, December 2024. More details about the mathematical foundation can be found in the series of articles by Morino [1, 2, 3, 4] and by Dimitriadis et al. [5, 6, 7, 8].

This theory manual and quick reference guide is organized as follows. Section 1 presents the integral formulation of the linearized unsteady potential flow equation. Section 2 presents the panel method used to discretize and solve the boundary integral equation. Section 3 presents the gradients calculation procedure. Section 4 presents the transonic correction methodology. Finally, section 5 gives an overview of the available API as well as its configuration parameters.

[1] https://gitlab.uliege.be/am-dept/sdpm, Accessed November 2024.

Contents

1 Mathematical modeling

The unsteady compressible potential equation is selected in order to model unsteady flowfields with a computational cost that remains tractable and suitable for preliminary aircraft design. This equation is theoretically restricted to subcritical flows, *i.e.* flows that remain subsonic throughout the entire field. However, solutions of the potential equation can be corrected to account for nonlinear transonic and viscous effects. These corrections are typically obtained from higher-fidelity steady modeling.

Under the assumptions that the flow is inviscid and isentropic, it can be considered irrotational, and the continuity equation can be written as

$$\beta^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{2M_\infty}{a_\infty}\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x}+\frac{1}{a_\infty^2}\frac{\partial^2\phi}{\partial t^2}=0, \tag{1.1}$$

where $\beta=\sqrt{1-M_\infty^2}$ is the compressibility factor, M_∞ and a_∞ are the freestream Mach number and speed of sound, x, y and z are the space coordinates, t is the time, and ϕ is the perturbation potential, deriving from the perturbation velocity. The total velocity of the flow is the sum of the freestream velocity $\mathbf{U}_\infty$ and the perturbation velocity $\mathbf{u}$,

$$\mathbf{U}=\mathbf{U}_\infty+\mathbf{u}=\begin{bmatrix}U_{\infty,x}\\U_{\infty,y}\\U_{\infty,z}\end{bmatrix}+\begin{bmatrix}\partial\phi/\partial x\\\partial\phi/\partial y\\\partial\phi/\partial z\end{bmatrix}. \tag{1.2}$$

Two transformations are applied to equation (1.2) in order to obtain a solution. First, the Prandtl-Glauert transformation

$$\xi=\frac{x}{\beta},\eta=y,\zeta=z, \tag{1.3}$$

is used to rewrite equation (1.1) as

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{\partial^2\phi}{\partial\zeta^2}-\frac{2M_\infty}{\beta a_\infty}\frac{\partial}{\partial t}\frac{\partial\phi}{\partial\xi}+\frac{1}{a_\infty^2}\frac{\partial^2\phi}{\partial t^2}=0. \tag{1.4}$$

Then, the Fourier Transform is used to express equation (1.4) in the frequency domain,

$$\frac{\partial^2\phi(\omega)}{\partial\xi^2}+\frac{\partial^2\phi(\omega)}{\partial\eta^2}+\frac{\partial^2\phi(\omega)}{\partial\zeta^2}-\mathrm{i}\omega\frac{2M_\infty}{\beta a_\infty}\frac{\partial\phi(\omega)}{\partial\xi}+\omega^2\frac{1}{a_\infty^2}\phi(\omega)=0, \tag{1.5}$$

where ω is the circular frequency. Finally, using Green's third identity, the solution to equation (1.5) can be written as

$$\phi(\boldsymbol{\xi})=-\frac{1}{4\pi}\int_S\mathbf{n}(\boldsymbol{\xi}_\mathrm{S})\left(\nabla\phi(\boldsymbol{\xi}_\mathrm{S})\frac{E}{R}(\boldsymbol{\xi}_\mathrm{S})-\phi(\boldsymbol{\xi}_\mathrm{S})\nabla\left(\frac{E}{R}(\boldsymbol{\xi}_\mathrm{S})\right)\right)dS, \tag{1.6}$$

where $\boldsymbol{\xi}$ denotes a point located anywhere in the flow, $\boldsymbol{\xi}_\mathrm{S}$ denotes a point located anywhere on the body surface S, $\mathbf{n}$ is the unit normal to the surface, R is the distance between points ξ and $\boldsymbol{\xi}_\mathrm{S}$, and $E=e^{-\mathrm{i}\frac{\omega}{a_\infty\beta}(R-M_\infty(\xi-\xi_\mathrm{S}))}$. Note that the first term in the integral is known as the source function while the second term is called the doublet function. Also note that the gradient is expressed in the space defined by the transformed coordinates ξ, η, ζ.

Two boundary conditions are required to solve equation (1.6): the farfield and the impermeability boundary conditions. Firstly, the perturbation potential should vanish far away from the body surface. This is enforced automatically by the source and doublet functions. Secondly, no flow should pass through the body surface. This is enforced by setting the normal mass flux to zero on the surface. For steady flows,

$$\nabla\phi(\boldsymbol{\xi}_{\mathrm{S}})\cdot\mathbf{n}(\boldsymbol{\xi}_{\mathrm{S}}) = -\mathbf{U}_{\infty}\cdot\mathcal{B}\odot\mathbf{n}(\boldsymbol{\xi}_{\mathrm{S}}), \tag{1.7}$$

where $\mathcal{B}$ is defined as

$$\mathcal{B} = \begin{bmatrix} 1/\beta \\ 0 \\ 0 \end{bmatrix}. \tag{1.8}$$

For unsteady flows,

$$\nabla\phi(\boldsymbol{\xi}_{\mathrm{S}})\cdot\mathbf{n}(\boldsymbol{\xi}_{\mathrm{S}}) = -\mathbf{U}_{\mathrm{c}}(\omega)\cdot\mathcal{B}\odot\mathbf{n}(\boldsymbol{\xi}_{\mathrm{S}}), \tag{1.9}$$

where $\mathbf{U}_{\mathrm{c}}(\omega)$ is the velocity induced by the motion of the body on the surface, and where $\odot$ denotes the Hadamard product. For a combination of rigid pitch and plunge, $\mathbf{U}_{\mathrm{c}}(\omega)$ writes

$$\mathbf{U}_{\mathrm{c}}(\omega) = \begin{bmatrix} 0 \\ 0 \\ U_{\infty,x}\tilde{\alpha} \end{bmatrix} + \mathrm{i}\omega \begin{bmatrix} -\tilde{\alpha}\,(z - z_{\mathrm{ref}}) \\ 0 \\ \tilde{\alpha}\,(x - x_{\mathrm{ref}}) + \tilde{h} \end{bmatrix}, \tag{1.10}$$

where $\tilde{\alpha}$ and $\tilde{h}$ are the pitch and plunge amplitudes, and where x_{ref} and z_{ref} are the x and z coordinates of the reference center. For a flexible motion imposed using a mode shape, $\mathbf{U}_{\mathrm{c}}(\omega)$ writes

$$\mathbf{U}_{\mathrm{c}}(\omega) = \tilde{q} \begin{bmatrix} U_{\infty,z}\theta_y \\ U_{\infty,z}\theta_x \\ -\left(U_{\infty,x}\theta_y + U_{\infty,y}\theta_x\right) \end{bmatrix} + \mathrm{i}\omega\tilde{q} \begin{bmatrix} 0 \\ 0 \\ \delta_z \end{bmatrix}, \tag{1.11}$$

where δ_z is the z-component of the modal displacements, θ_x and θ_y are the x and y-components of the modal rotations, and where $\tilde{q}$ is the modal amplitude. Note that equation (1.11) only takes into account the contribution of out-of-plane mode shapes, and is only valid for small rotations.

The nonlinear pressure coefficient, obtained from the momentum equation under the irrotational assumption, can be computed as

$$
\begin{aligned}
C_p(\omega) = 1 &- \frac{2}{|\mathbf{U}_\infty|^2}\mathrm{i}\omega\phi(\omega) \\
&- \frac{\mathcal{U}(\omega) \circledast \mathcal{U}(\omega)}{|\mathbf{U}_\infty|^2} \\
&+ \frac{M_\infty^2}{|\mathbf{U}_\infty|^2}\frac{\partial\phi(\omega)}{\partial x} \circledast \frac{\partial\phi(\omega)}{\partial x} \\
&+ \frac{M_\infty^2}{|\mathbf{U}_\infty|^3}(\mathrm{i}\omega\phi(\omega)) \circledast \frac{\partial\phi(\omega)}{\partial x} \\
&+ \frac{M_\infty^2}{|\mathbf{U}_\infty|^4}(\mathrm{i}\omega\phi(\omega)) \circledast (\mathrm{i}\omega\phi(\omega)).
\end{aligned}
\tag{1.12}
$$

where $\mathcal{U}(\omega)$ denotes the sum of the convolutions of the components of the unsteady perturbation velocity, and where $\circledast$ denotes the convolution product. For a steady flow, equation (1.12) reduces to

$$
C_p = 1 - \frac{|\mathbf{U}|^2 - M_\infty^2 u_x^2}{|\mathbf{U}_\infty|^2}. \tag{1.13}
$$

Integrating the pressure coefficient over the whole surface yields the resulting aerodynamic load coefficient

$$
\mathbf{C_R}(\omega) = -\frac{1}{S_{\text{ref}}}\int_S C_p(\omega)\mathbf{n}dS, \tag{1.14}
$$

which is in turn used to compute lift and drag coefficients as

$$
\begin{aligned}
C_L(\omega) &= C_{R_z}(\omega) - C_{R_x}(\omega) \circledast \alpha(\omega), \\
C_D(\omega) &= C_{R_x}(\omega) + C_{R_z}(\omega) \circledast \alpha(\omega),
\end{aligned}
\tag{1.15}
$$

where $\alpha(\omega)$ is the angle of attack expressed in the frequency domain, and where S_{ref} is the reference surface area of the body. Finally, the generalized aerodynamic force matrix at frequency ω is calculated by integrating the z-component of the surface displacement δ_{z_i} of mode i by the z-component of the pressure loads of mode j,

$$
Q_{ij}(\omega) = \int_S \delta_{z_i}(\omega)C_{p_j}(\omega)\mathbf{n}_z dS. \tag{1.16}
$$

More details about the calculation of the generalized aerodynamic force matrix can be found in the works of Dimitriadis [9, 10] and Güner [11].

2 Numerical method

The surface of the body is discretized into quadrilateral elements. The source and doublet functions required to solve equation (1.6) are discretized accordingly, and each panel holds a constant-valued source and doublet. Additionally, a wake must be shed from the trailing edge of a lifting body so that it can produce aerodynamic loads. The wake is also discretized into quadrilateral panels. Since the wake has zero thickness, it only holds a constant-valued doublet on each panel and no source. The geometry of the discretized body and wake, as well as the discretized sources and doublets, are illustrated in figure 2.1.

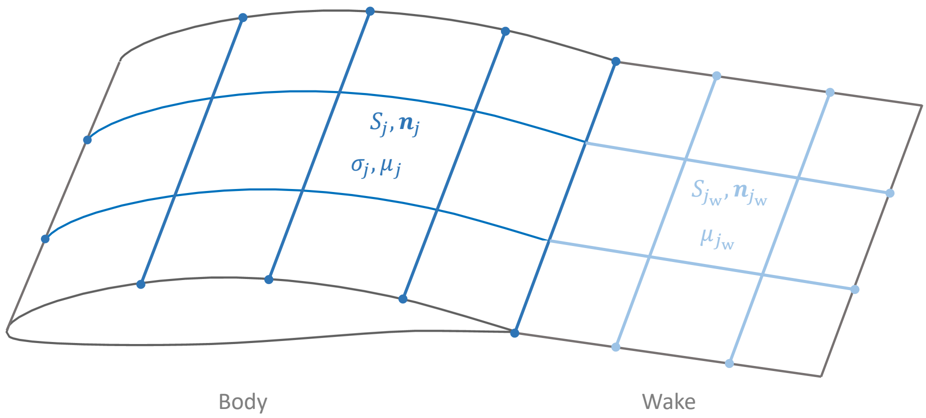

Figure 2.1: Discretized geometry with panel of surface area S, unit normal $\mathbf{n}$, sources σ and doublets μ.

Since the value of the source and doublets are constant on each panel, equation (1.6) can be written as

$$\phi_i = \sum_j \left[-\frac{\sigma_j}{4\pi} \int_{S_j} \frac{E}{R} dS_j + \frac{\mu_j}{4\pi} \int_{S_j} \mathbf{n}_j \cdot \nabla \left(\frac{E}{R} \right) dS_j \right] + \sum_{j_\mathrm{w}} \frac{\mu_{j_\mathrm{w}}}{4\pi} \int_{S_{j_\mathrm{w}}} \mathbf{n}_{j_\mathrm{w}} \cdot \nabla \left(\frac{E}{R} \right) dS_{j_\mathrm{w}}, \quad (2.1)$$

where ϕ_i is perturbation potential on panel i, dS_j represents the area of the body panel j placed on the body surface S, and dS_{j_w} represents the area of a wake panel j_w placed on the wake surface S_w, and where σ_j and μ_j denote the value of the source and doublet at the control point of panel j. The integral terms appearing in equation (2.1) are known as the unsteady aerodynamic influence coefficients. In the present work, they are approximated by

$$\begin{aligned}
A_{ij}(\omega) &= \frac{1}{4\pi}\int_{S_j} \mathbf{n}_j \cdot \nabla \left(\frac{e^{-\mathrm{i}\frac{\omega}{a_\infty \beta}(R_{ij}-M_\infty(\xi_i-\xi_j))}}{R_{ij}} \right) dS_j \\
&\simeq \frac{1}{4\pi}(1+\mathrm{i}\frac{\omega}{a_\infty\beta}R_{ij})e^{-\mathrm{i}\frac{\omega}{a_\infty \beta}(R_{ij}-M_\infty(\xi_i-\xi_j))}\int_{S_j} \mathbf{n}_j \cdot \nabla \left(\frac{1}{R_{ij}}\right) dS_j \\
&+ \frac{\mathrm{i}}{4\pi}\frac{\omega}{a_\infty\beta}M_\infty n_{\xi_j} e^{-\mathrm{i}\frac{\omega}{a_\infty \beta}(R_{ij}-M_\infty(\xi_i-\xi_j))}\int_{S_j}\frac{1}{R_{ij}}dS_j, \\
B_{ij}(\omega) &= -\frac{1}{4\pi}\int_{S_j}\frac{e^{-\mathrm{i}\frac{\omega}{a_\infty \beta}(R_{ij}-M_\infty(\xi_i-\xi_j))}}{R_{ij}}dS_j \\
&\simeq -\frac{1}{4\pi}e^{-\mathrm{i}\frac{\omega}{a_\infty \beta}(R_{ij}-M_\infty(\xi_i-\xi_j))}\int_{S_j}\frac{1}{R_{ij}}dS_j,
\end{aligned} \tag{2.2}$$

where n_{ξ_j} is the ξ-component of the unit normal to panel j. The remaining integrals only depend on the geometry and are identical to steady aerodynamic coefficients. Their analytical expressions are given by Katz and Plotkin [12].

The solution procedure can be summarized as follows. Firstly, the body sources are computed to enforce impermeability following equations (1.7) and (1.9),

$$\begin{aligned}
\sigma_j(0) &= -\mathbf{U}_\infty \cdot \mathcal{B} \odot \mathbf{n}_j, \\
\sigma_j(\omega) &= -\mathbf{U}_{\mathrm{c}_j}(\omega) \cdot \mathcal{B} \odot \mathbf{n}_j,
\end{aligned} \tag{2.3}$$

where $\mathbf{U}_{\mathrm{c}_j}$ is the motion-induced velocity on panel j. Secondly, equation (2.1) is written for the potential just outside and just inside the body surface and the expressions are subtracted, yielding

$$A_{ij}\mu_j - B_{ij}\sigma_j = 0, \tag{2.4}$$

which is solved for the body doublets for each value of the frequency. Note that the wake contribution is included in the matrix $\mathbf{A}$. Thirdly, the tangential velocity is computed by differentiating the doublets on the surface, while the normal velocity is readily obtained from the value of the sources. Fourthly, the steady and unsteady pressure coefficients are computed on each panel using equations (1.13) and (1.12). Assuming sinusoidal motion at frequency ω, the velocity will have frequency components at $-\omega$, 0 and $+\omega$, and the terms in equation (1.12) can be written as vectors with three components,

$$
\begin{aligned}
\mathrm{i}\omega\phi(\omega) &= [(\mathrm{i}\omega\mu(\omega))^{*}, 0, \mathrm{i}\omega\mu(\omega)] \\
\frac{\partial\phi(\omega)}{\partial x} &= \left[\left(\frac{\partial\phi(\omega)}{\partial x}\right)^{*}, \frac{\partial\phi(0)}{\partial x}, \frac{\partial\phi(\omega)}{\partial x}\right] \\
\tilde{u}(\omega) &= \left[\left(U_{c,x} + \frac{\partial\phi(\omega)}{\partial x}\right)^{*}, U_{\infty,x} + \frac{\partial\phi(0)}{\partial x}, U_{c,x} + \frac{\partial\phi(\omega)}{\partial x}\right] \\
\tilde{v}(\omega) &= \left[\left(U_{c,y} + \frac{\partial\phi(\omega)}{\partial y}\right)^{*}, U_{\infty,y} + \frac{\partial\phi(0)}{\partial y}, U_{c,y} + \frac{\partial\phi(\omega)}{\partial y}\right] \\
\tilde{w}(\omega) &= \left[\left(U_{c,z} + \frac{\partial\phi(\omega)}{\partial z}\right)^{*}, U_{\infty,z} + \frac{\partial\phi(0)}{\partial z}, U_{c,z} + \frac{\partial\phi(\omega)}{\partial z}\right] \\
\mathcal{U}(\omega) \circledast \mathcal{U}(\omega) &= \tilde{u} \circledast \tilde{u} + \tilde{v} \circledast \tilde{v} + \tilde{w} \circledast \tilde{w},
\end{aligned}
\tag{2.5}
$$

where the superscript $*$ denotes the complex conjugate. Working out the convolutions, the first harmonic of the pressure coefficient can be written as

$$
\begin{aligned}
C_p(\omega) = &-2\frac{u_x(0)u_x(\omega) + u_y(0)u_y(\omega) + u_z(0)u_z(\omega)}{|\mathbf{U}_\infty|^2} - \frac{2\mathrm{i}\omega}{|\mathbf{U}_\infty|^2}\mu(\omega) \\
&+ \frac{2M_\infty^2}{|\mathbf{U}_\infty|^2}\frac{\partial\phi(0)}{\partial x}\frac{\partial\phi(\omega)}{\partial x} + \frac{2\mathrm{i}\omega M_\infty^2}{|\mathbf{U}_\infty|^3}\frac{\partial\phi(0)}{\partial x}\mu(\omega)
\end{aligned}
\tag{2.6}
$$

Finally, the pressure coefficient is integrated over the wing to obtain the resulting aerodynamic load following equation (1.14), which is then used to compute the lift and drag coefficients using equation (1.15). The generalized aerodynamic force matrix at frequency ω is calculated by discretizing equation (1.16)

$$
Q_{ij}(\omega) = \boldsymbol{\delta}_{z_i}(\omega) \cdot \mathbf{C}_{\mathbf{p}_j}(\omega) \odot \mathbf{n}_z \odot \mathbf{S}_{\mathrm{p}},
\tag{2.7}
$$

where $\boldsymbol{\delta}_{z_i}$ is the vector containing the z-component of the surface displacement of mode i on each panel, $\mathbf{C}_{\mathbf{p}_j}$ is the vector containing the pressure coefficient due to mode j on each panel, $\mathbf{n}_z$ is the vector containing the z-component of the unit normal to the body surface and $\mathbf{S}_{\mathrm{p}}$ is the vector containing the surface area of the panels.

3 Calculation of the gradients

Two gradient calculation techniques are implemented in `SDPM`: automatic and analytical differentiation.

3.1 Automatic differentiation

Automatic differentiation (AD) can be used in order to compute the gradients of the steady and unsteady aerodynamic loads with respect to the aerodynamic and shape variables. AD uses the fact that any numerical output of a computer code can be expressed as a sequence of statements, each of which can be differentiated programmatically. The derivative of the output is then recovered by using the chain rule to combine the statements and their derivatives. The implementation of AD in `SDPM` relies on `CoDiPack` [13][2], which uses operator overloading in order to compute the gradients. `CoDiPack` features forward and reverse accumulation modes. The forward mode computes the gradient of any number of outputs with respect to one input per evaluation, while the reverse mode computes the gradients of one output with respect to any number of inputs. Since the code will be used to solve optimization problems formulated using the adjoint method, the reverse mode has been selected. The main advantage of AD is that the computation of any derivative is straightforward and requires little additional programming. On the other hand, using AD results in an increase of memory usage. This is particularly the case with panel methods, for which the memory scales quadratically with respect to the number of panels.

3.2 Analytical derivatives

The computational time and memory cost required by AD can be prohibitive in some cases. In order to alleviate this issue, the gradients of the generalized aerodynamic load matrices with respect to the generalized coordinates defining the wing motion have been derived analytically and implemented in `SDPM`. These gradients are typically used to calculate and constrain flutter in an optimization problem.

Differentiating equation (2.7) with respect to the generalized coordinates composed of flexible displacements along the z-direction, δ_z, and of flexible rotations about the x and y directions, θ_x and θ_y, the derivative of generalized aerodynamic loads matrix at frequency ω with respect to the z-component of the surface displacement of mode k can then be expressed

$$\begin{aligned}\frac{\partial Q_{ij}(\omega)}{\partial_{\delta_{z_k}}} &= \delta_{ik} \cdot \mathbf{C}_{\mathbf{p}_j}(\omega) \odot \mathbf{n}_z \odot \mathbf{S}_\mathrm{p} \\ &+ \boldsymbol{\delta}_{z_i} \cdot \left(\mathrm{i}\omega \partial_{\dot{\delta}_z} \mathbf{C}_{\mathbf{p}_j}(\omega) + (\mathrm{i}\omega)^2 \partial_{\ddot{\delta}_z} \mathbf{C}_{\mathbf{p}_j}(\omega)\right) \delta_{jk} \odot \mathbf{n}_z \odot \mathbf{S}_\mathrm{p},\end{aligned} \tag{3.1}$$

where $\boldsymbol{\delta}_{z_i}$ is the vector containing the z-component of the surface displacement of mode i on each panel, $\mathbf{C}_{\mathbf{p}_j}$ is the vector containing the pressure coefficient due to mode j on each

[2] `https://github.com/SciCompKL/CoDiPack`, accessed November 2024.

panel, $\mathbf{n}_z$ is the vector containing the z-component of the unit normal to the body surface, $\mathbf{S}_\mathrm{p}$ is the vector containing the surface area of the panels, ω is the frequency, and where δ_{ij} is the Kronecker delta which value is 1 if i equals j and zero otherwise. Since the displacements are independent of the rotations, the derivative of the generalized aerodynamic force matrix with respect to the x and y-component of the surface rotation of mode k can be expressed

$$\frac{\partial Q_{ij}(\omega)}{\partial \theta_{x_k}} = \boldsymbol{\delta}_{z_i} \cdot \left(\partial_{\theta_x} \mathbf{C_p}_j(\omega) + \mathrm{i}\omega \partial_{\dot{\theta}_x} \mathbf{C_p}_j(\omega) \right) \delta_{jk} \odot \mathbf{n}_z \odot \mathbf{S}_\mathrm{p}, \tag{3.2}$$

and

$$\frac{\partial Q_{ij}(\omega)}{\partial \theta_{y_k}} = \boldsymbol{\delta}_{z_i} \cdot \left(\partial_{\theta_y} \mathbf{C_p}_j(\omega) + \mathrm{i}\omega \partial_{\dot{\theta}_y} \mathbf{C_p}_j(\omega) \right) \delta_{jk} \odot \mathbf{n}_z \odot \mathbf{S}_\mathrm{p}. \tag{3.3}$$

Equations (3.1) to (3.3) require to calculate the pressure derivatives: $\partial_{\boldsymbol{\delta}_z} \mathbf{C_p}$, $\partial_{\dot{\boldsymbol{\delta}}_z} \mathbf{C_p}$, $\partial_{\boldsymbol{\theta}_x} \mathbf{C_p}$, $\partial_{\dot{\boldsymbol{\theta}}_x} \mathbf{C_p}$, $\partial_{\boldsymbol{\theta}_y} \mathbf{C_p}$ and $\partial_{\dot{\boldsymbol{\theta}}_y} \mathbf{C_p}$. Substituting equations (2.4), (2.3) and (1.11) successively into equation (2.6), the first harmonic of the unsteady pressure coefficient can be expressed as a function of the generalized coordinates, which can then be differentiated to obtain the pressure derivatives. The partial gradients of the pressure coefficient with respect to the flexible displacements along z are given by

$$\begin{aligned}
\partial_{\boldsymbol{\delta}_z} \mathbf{C_p} &= \frac{2}{\beta} \left(\mathbf{u}_x^D(0) - M_\infty^2 \partial_x \boldsymbol{\phi}^D(0) \right) \odot \left(\mathbf{n}_z^D \odot \mathbf{n}_x^D + \nabla \mathbf{N}_x \mathbf{A}^{-1} \mathbf{B} \mathbf{n}_z^D \right) \\
&\quad + 2\mathbf{u}_y^D(0) \odot \left(\mathbf{n}_z^D \odot \mathbf{n}_y^D + \nabla \mathbf{N}_y \mathbf{A}^{-1} \mathbf{B} \mathbf{n}_z^D \right) \\
&\quad + 2\mathbf{u}_z^D(0) \odot \left(-\mathbf{I} + \mathbf{n}_z^D \odot \mathbf{n}_z^D + \nabla \mathbf{N}_z \mathbf{A}^{-1} \mathbf{B} \mathbf{n}_z^D \right), \\
\partial_{\dot{\boldsymbol{\delta}}_z} \mathbf{C_p} &= 2 \left(\mathbf{I} - M_\infty^2 \partial_x \boldsymbol{\phi}^D(0) \right) \odot \mathbf{A}^{-1} \mathbf{B} \mathbf{n}_z^D,
\end{aligned} \tag{3.4}$$

where β is the compressibility factor, M_∞ is the freestream Mach number, $\mathbf{I}$ is the identity matrix and $\mathbf{n}_x$ is the vector containing the x-component of the unit normal to the body surface panels. $\mathbf{u}_x(0)$, $\mathbf{u}_y(0)$ and $\mathbf{u}_z(0)$ are the vectors containing the x, y and z components of the steady velocity on each panel, respectively, and $\partial_x \boldsymbol{\phi}(0) = \mathbf{u}_x(0) - U_{\infty,x}$. $\nabla \mathbf{N}_x$., $\nabla \mathbf{N}_y$ and $\nabla \mathbf{N}_z$ are the matrices containing the x, y and z components of the surface gradient operators. Note that $\mathbf{v}^D$ denotes a diagonal matrix where the diagonal entries are those of vector $\mathbf{v}$. Note that the pressure derivatives depend on the frequency ω through the aerodynamic influence coefficient matrices $\mathbf{A}$ and $\mathbf{B}$.

The partial gradients of the pressure coefficient with respect to the flexible rotations around x and y are given by

$$\begin{aligned}
\partial_{\boldsymbol{\theta}_x} \mathbf{C_p} &= \frac{2}{\beta} \left(\mathbf{u}_x^D(0) - M_\infty^2 \partial_x \boldsymbol{\phi}^D(0) \right) \odot \left(\left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \odot \mathbf{n}_x^D + \nabla \mathbf{N}_x \mathbf{A}^{-1} \mathbf{B} \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \right) \\
&\quad + 2\mathbf{u}_y^D(0) \odot \left(-U_{\infty,z} \mathbf{I} + \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \odot \mathbf{n}_y^D + \nabla \mathbf{N}_y \mathbf{A}^{-1} \mathbf{B} \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \right) \\
&\quad + 2\mathbf{u}_z^D(0) \odot \left(U_{\infty,y} \mathbf{I} + \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \odot \mathbf{n}_z^D + \nabla \mathbf{N}_z \mathbf{A}^{-1} \mathbf{B} \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right) \right), \\
\partial_{\dot{\boldsymbol{\theta}}_x} \mathbf{C_p} &= 2 \left(\mathbf{I} - M_\infty^2 \partial_x \boldsymbol{\phi}^D(0) \right) \odot \mathbf{A}^{-1} \mathbf{B} \left(U_{\infty,z} \mathbf{n}_y^D - U_{\infty,y} \mathbf{n}_z^D \right),
\end{aligned} \tag{3.5}$$

and by

$$\begin{aligned}
\partial_{\boldsymbol{\theta}_y}\mathbf{C_p} &= -2\mathbf{u}_x^D(0)U_{\infty,z}\mathbf{I} \\
&+ \frac{2}{\beta}\left(\mathbf{u}_x^D(0) - M_\infty^2\partial_x\boldsymbol{\phi}^D(0)\right)\odot\left(\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\odot\mathbf{n}_x^D + \nabla\mathbf{N}_x\mathbf{A}^{-1}\mathbf{B}\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\right) \\
&+ 2\mathbf{u}_y^D(0)\odot\left(\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\odot\mathbf{n}_y^D + \nabla\mathbf{N}_y\mathbf{A}^{-1}\mathbf{B}\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\right) \\
&+ 2\mathbf{u}_z^D(0)\odot\left(U_{\infty,x}\mathbf{I} + \left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\odot\mathbf{n}_z^D + \nabla\mathbf{N}_z\mathbf{A}^{-1}\mathbf{B}\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right)\right), \\
\partial_{\dot{\boldsymbol{\theta}}_y}\mathbf{C_p} &= 2\left(\mathbf{I} - M_\infty^2\partial_x\boldsymbol{\phi}^D(0)\right)\odot\mathbf{A}^{-1}\mathbf{B}\left(\frac{U_{\infty,z}}{\beta}\mathbf{n}_x^D - U_{\infty,x}\mathbf{n}_z^D\right),
\end{aligned} \tag{3.6}$$

where $\mathbf{n}_y$ is the vector containing the y-component of the unit normal to the body surface panels. If the wing is subject to a rigid pitch or plunge, the pressure derivatives $\partial_{\tilde{\alpha}}\mathbf{C_p}$, $\partial_{\dot{\tilde{\alpha}}}\mathbf{C_p}$, $\partial_{\ddot{\tilde{\alpha}}}\mathbf{C_p}$, $\partial_{\dot{\tilde{h}}}\mathbf{C_p}$ and $\partial_{\ddot{\tilde{h}}}\mathbf{C_p}$ are also needed. They are given by

$$\begin{aligned}
\partial_{\tilde{\alpha}}\mathbf{C_p} &= \frac{2}{\beta}\left(\mathbf{u}_x(0) - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\left(U_{\infty,x}\mathbf{n}_z\odot\mathbf{n}_x + \nabla\mathbf{N}_x\mathbf{A}^{-1}\mathbf{B}\left(U_{\infty,x}\mathbf{n}_z\right)\right) \\
&+ 2\mathbf{u}_y(0)\odot\left(U_{\infty,x}\mathbf{n}_z\odot\mathbf{n}_y + \nabla\mathbf{N}_y\mathbf{A}^{-1}\mathbf{B}\left(U_{\infty,x}\mathbf{n}_z\right)\right) \\
&+ 2\mathbf{u}_z(0)\odot\left(-U_{\infty,x}\mathbf{1} + U_{\infty,x}\mathbf{n}_z\odot\mathbf{n}_z + \nabla\mathbf{N}_z\mathbf{A}^{-1}\mathbf{B}\left(U_{\infty,x}\mathbf{n}_z\right)\right), \\
\partial_{\dot{\tilde{\alpha}}}\mathbf{C_p} &= 2\mathbf{u}_x(0)\odot(\mathbf{z} - z_{\text{ref}}\mathbf{1}) \\
&+ \frac{2}{\beta}\left(\mathbf{u}_x(0) - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\left(\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\odot\mathbf{n}_x\right. \\
&\left.+\nabla\mathbf{N}_x\mathbf{A}^{-1}\mathbf{B}\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\right) \\
&+ 2\mathbf{u}_y(0)\odot\left(\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\odot\mathbf{n}_y\right. \\
&\left.+\nabla\mathbf{N}_y\mathbf{A}^{-1}\mathbf{B}\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\right) \\
&+ 2\mathbf{u}_z(0)\odot\left(-(\mathbf{x} - x_{\text{ref}}\mathbf{1}) + \left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\odot\mathbf{n}_z\right. \\
&\left.+\nabla\mathbf{N}_z\mathbf{A}^{-1}\mathbf{B}\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\right) \\
&+ 2\left(\mathbf{1} - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\mathbf{A}^{-1}\mathbf{B}U_{\infty,x}\mathbf{n}_z \\
\partial_{\ddot{\tilde{\alpha}}}\mathbf{C_p} &= 2\left(\mathbf{1} - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\left(\mathbf{A}^{-1}\mathbf{B}\left(-\frac{1}{\beta}(\mathbf{z} - z_{\text{ref}}\mathbf{1})\odot\mathbf{n}_x + (\mathbf{x} - x_{\text{ref}}\mathbf{1})\odot\mathbf{n}_z\right)\right),
\end{aligned} \tag{3.7}$$

and by

$$\begin{aligned}
\partial_{\dot{\tilde{h}}}\mathbf{C_p} &= \frac{2}{\beta}\left(\mathbf{u}_x(0) - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\left(\mathbf{n}_z\odot\mathbf{n}_x + \nabla\mathbf{N}_x\mathbf{A}^{-1}\mathbf{B}\mathbf{n}_z\right) \\
&+ 2\mathbf{u}_y(0)\odot\left(\mathbf{n}_z\odot\mathbf{n}_y + \nabla\mathbf{N}_y\mathbf{A}^{-1}\mathbf{B}\mathbf{n}_z\right) \\
&+ 2\mathbf{u}_z(0)\odot\left(-\mathbf{1} + \mathbf{n}_z\odot\mathbf{n}_z + \nabla\mathbf{N}_z\mathbf{A}^{-1}\mathbf{B}\mathbf{n}_z\right), \\
\partial_{\ddot{\tilde{h}}}\mathbf{C_p} &= 2\left(\mathbf{1} - M_\infty^2\partial_x\boldsymbol{\phi}(0)\right)\odot\mathbf{A}^{-1}\mathbf{B}\mathbf{n}_z,
\end{aligned} \tag{3.8}$$

where $\mathbf{1}$ is the identity vector, $\mathbf{x}$ and $\mathbf{z}$ are the vectors containing the x and z coordinates of each panel center of gravity, and where x_{ref} and z_{ref} are the x and z coordinates of the reference center.

4 Transonic correction

The panel method is based on a linear equation. Therefore it cannot model the boundary layer and shocks, which are highly nonlinear. A correction methodology has been implemented in `SDPM` to account for these effects following the approach proposed by Dimitriadis [7]. The main idea is to correct the doublet aerodynamic coefficient matrix $\mathbf{A}$ by matching the derivative of the steady pressure coefficient with respect to the angle of attack expressed using the panel method to that obtained using higher-fidelity data. The linearized steady pressure coefficient on the panels can be expressed as

$$\mathbf{C_p}(0) \simeq \frac{2}{\beta}\left(\mathbf{n}_\xi \odot \boldsymbol{\sigma}(0) + \nabla\mathbf{N}_x \boldsymbol{\mu}(0)\right), \tag{4.1}$$

where β is the compressibility factor, $\mathbf{n}_\xi$ is the vector containing the ξ-component of the unit normal to the surface panels transformed using the Prandtl-Glauert correction, $\nabla\mathbf{N}_x$ is the matrice containing the x component of the surface gradient operator, and $\boldsymbol{\sigma}(0)$ and $\boldsymbol{\mu}(0)$ are the steady sources and doublets. Assuming that the freestream velocity is given by

$$\mathbf{U}_\infty = \begin{bmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{bmatrix}, \tag{4.2}$$

where the angle of attack α is small, injecting equations (2.3) and (2.4) into equation (4.1) and taking the derivative with respect to the angle of attack, the pressure derivative is expressed as

$$\partial_\alpha \mathbf{C_p} \simeq \frac{2}{\beta}\left(\mathbf{n}_\xi \odot \mathbf{n}_\zeta + \nabla\mathbf{N}_x \mathbf{A}^{-1}(0)\mathbf{B}(0)\mathbf{n}_\zeta\right), \tag{4.3}$$

where $\mathbf{n}_\zeta$ is the vector containing the ζ-component of the unit normal to the surface panels transformed using the Prandtl-Glauert correction. This expression can be matched to higher-fidelity data, typically obtained by solving the Reynolds-Averaged Navier-Stokes equations, by introducing the diagonal correction matrix $\mathbf{D}^{\mathrm{corr}}$,

$$\partial_\alpha \mathbf{C_p}^{\mathrm{HF}} \simeq \frac{2}{\beta}\left(\mathbf{n}_\xi \odot \mathbf{n}_\zeta + \nabla\mathbf{N}_x \mathbf{D}^{\mathrm{corr}} \mathbf{A}^{-1}(0)\mathbf{B}(0)\mathbf{n}_\zeta\right), \tag{4.4}$$

where $\partial_\alpha \mathbf{C_p}^{\mathrm{HF}}$ is the vector containing the higher-fidelity steady pressure coefficient on each panel.

The transonic correction methodology starts by expressing the corrected derivative of the steady doublets as

$$\nabla\mathbf{N}_x \partial_\alpha \boldsymbol{\mu}^{\mathrm{corr}}(0) = \frac{\beta}{2}\partial_\alpha \mathbf{C_p}^{\mathrm{HF}} - \mathbf{n}_\xi \odot \mathbf{n}_\zeta. \tag{4.5}$$

This equation is solved for the unknown corrected derivative, subject to the boundary condition that no correction is applied on the upper trailing edge panels, *i.e.* $\partial_\alpha \boldsymbol{\mu}^{\mathrm{corr}} = \partial_\alpha \boldsymbol{\mu} = \mathbf{A}^{-1}(0)\mathbf{B}(0)\mathbf{n}_\zeta$. The diagonal of the correction matrix $\mathbf{D}^{\mathrm{corr}}$ can then be obtained by dividing the corrected doublet derivatives by the original derivatives,

$$\mathrm{diag}(\mathbf{D}^{\mathrm{corr}}) = \partial_\alpha \boldsymbol{\mu}^{\mathrm{corr}}(0) \oslash \partial_\alpha \boldsymbol{\mu}(0), \tag{4.6}$$

where $\oslash$ denotes the Hadamard division operator. Once the correction factors have been calculated, they can be used to correct the unsteady aerodynamic coefficient matrices at each frequency of interest, and the unsteady velocities, pressure distribution and generalized aerodynamic force matrices can be obtained as explained in section 2.

5 Quick reference guide

SDPM is configured and accessed using an Application Programming Interface (API). The complete documentation is available at https://gitlab.uliege.be/am-dept/sdpm/-/wikis/home, accessed November 2024.

The list of parameters required to configure SDPM is provided below:

```
cfg = {
    # Model (geometry or mesh)
   'File': str, # input Gmsh file
   'Pars': dict, # parameters for Gmsh file (optional)
   # Markers
   'Wing': str, # name of the marker defining the lifting body (wing)
   'Te': str, # name of the marker defining the trailing edge of the lifting body
   # Freestream
   'Mach': float, # Mach number (optional)
   'AoA': float, # angle of attack (optional)
   'AoS': float, # angle of side slip (optional)
   # Unsteady motion
   'Num_wake_div': int, # number of cells per chord length in the wake
   'Frequencies': array[float], # list of reduced frequencies
   'Modes': str, # input file containing the modes (optional)
   'Num_modes': int, # number of modes
   # Transonic correction
   'Transonic_pressure_grad': str, # input file containing the pressure derivative (optional)
   # Geometry
   'Symmetry': bool, # whether only half (symmetric) model is provided
   's_ref': float, # reference surface area
   'c_ref': float, # reference chord length
   'x_ref': float, # reference point for moment computation (x)
   'y_ref': float, # reference point for moment computation (y)
   'z_ref': float # reference point for moment computation (z)
}
```

SDPM can then be initialized using:

```
from sdpm.api.core import init_sdpm
sdpm = init_sdpm(cfg)
```

where sdpm is a Python dictionary containing:

- n_f: the number of reference reduced frequencies,
- n_m: the number of modes,
- msh: the mesh,
- wrt: the utility to write the mesh and results on disk,
- pbl: the formulation of the problem,
- bdy: the body of interest (wing),

- `sol`: the solver,
- `grd`: the gradient calculator.

References

[1] L Morino. A general theory of unsteady compressible potential aerodynamics. Technical report, NASA, 1974.

[2] L Morino and P.C. Chen. Steady and oscillatory subsonic and supersonic aerodynamics around complex configurations. *AIAA Journal*, 13:368–374, 1975.

[3] L Morino and P.C. Chen. Indicial compressible potential aerodynamics around complex aircraft configurations. In *Aerodynamic Analysis Requiring Advanced Computers Conference*, pages 1067–1110, Hampton, VA, USA, 1975. NASA.

[4] L Morino. *Steady, Oscillatory and Unsteady Subsonic and Supersonic Aerodynamic Forces - Production version (SOUSSA-P 1.1) - Volume I - Theory Manual*. NASA, 1980.

[5] Mariano Sánchez Martínez and Grigorios Dimitriadis. Subsonic source and doublet panel methods. *Journal of Fluids and Structures*, 113:103624, 2022.

[6] Grigorios Dimitriadis, Pericles Panagiotou, Thomas Dimopoulos, and Kyros Yakinthos. Aerodynamic Stability Derivative Calculations Using the Compressible Source and Doublet Panel Method. *Journal of Aircraft*, 61(4), 2024.

[7] Grigorios Dimitriadis, Spiridon Kilimtzidis, Vassilis Kostopoulos, Vito Laraspata, and Leonardo Soria. Flutter calculations using the unsteady source and doublet panel method. *Journal of Aircraft*, In print, 2024.

[8] Grigorios Dimitriadis, Adrien Crovato, Mariano Sánchez Martínez, Spiridon Kilimtzidis, Vassilis Kostopoulos, Vito Laraspata, and Leonardo Soria. Transonic Corrections for the Unsteady Compressible Source and Doublet Panel Method. *Journal of Aircraft*, In print, 2024.

[9] G. Dimitriadis. *Introduction to Nonlinear Aeroelasticity*. John Wiley & Sons, first edition edition, 2017.

[10] G. Dimitriadis, N. Giannelis, and G. Vio. A modal frequency-domain generalised force matrix for the unsteady Vortex Lattice method. *Journal of Fluids and Structures*, 76:216–228, 2018.

[11] Hüseyin Güner. *Unsteady Aerodynamic Modeling Methodology for Transonic Flutter Calculations*. PhD thesis, University of Liège, May 2020.

[12] J. Katz and A. Plotkin. *Low-Speed Aerodynamics*. Cambridge Aerospace Series. Cambridge University Press, 2001.

[13] M. Sagebaum, T. Albring, and N.R. Gauger. High-Performance Derivative Computations using CoDiPack. *ACM Transactions on Mathematical Software (TOMS)*, 45(4), 12 2019.