Numerical simulation of Hermite processes 31st Annual Meeting of the Royal Statistical Society of Belgium – Ghent

Laurent Loosveldt

8th November 2024







Main definitions

- A stochastic process $\{X_t\}_{t \ge 0}$ has:
 - ▶ stationary increments if, for every h > 0, the stochastic processes

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▶ is *H*-self-similar if, for every c > 0, the stochastic processes

 $\{X_{ct}\}_{t\geq 0}$ and $\{c^H X_t\}_{t\geq 0}$

are equal in finite-dimensional distributions.





Long-range dependence

If $\{X_t\}_{t\geq 0}$ is centred with stationary increments, we consider the correlation kernel

$$\rho(k-\ell):=\mathbb{E}[(X_{k+1}-X_k)(X_{j+1}-X_j)] \quad j,k\in\mathbb{N}.$$



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The process $\{X_t\}_{t\geq 0}$ has long-range dependence if

$$\sum_{j\in\mathbb{Z}}|\rho(j)|=\infty.$$

Correlation decays slowly as the lag tends to infinity.





Applications

Such phenomena arise, for instance, in

- astronomy,
- biology,
- climatology,
- hydrology,
- image processing,
- internet traffic modelling,
- mathematical finance,
- physics.



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We need good models !



Definition

Given $H \in (0,1)$, a fractional Brownian motion of Hurst parameter H is a centred continuous Gaussian process $\{B_t^H\}_{t\geq 0}$ with covariance function

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$$\left\{\frac{1}{c_H}\left(\int_{-\infty}^0 ((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}) \, dB(u) + \int_0^t (t-u)^{H-\frac{1}{2}} \, dB(u)\right)\right\}_{t \ge 0}$$





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It has stationary increments and is *H*-self-similar. If $H > \frac{1}{2}$, it has long-range dependence.



Given $d \in \mathbb{N}$ and $H \in (1/2, 1)$, the Hermite process of order d and Hurst parameter H is defined as

$$\frac{1}{c_H} \left\{ \int_{\mathbb{R}^d} \left(\int_0^t \prod_{\ell=1}^d (s - x_\ell)_+^{\frac{H-1}{d} - \frac{1}{2}} ds \right) dB(x_1) \dots dB(x_d) \right\}_{t \ge 0}$$



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- ▶ $d \ge 2$, it is a non-Gaussian process;



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- $d \ge 2$, it is a non-Gaussian process;
- ▶ has stationary increments, is *H*-self-similar with long-range dependence.



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$d \ge 3$

Pipiras, in his paper from 2004, raised the question of providing a wavelet-type expansion for any Hermite processes.



Wavelet approximation for deterministic functions

Two smooth functions:

- \blacktriangleright the scaling function ϕ
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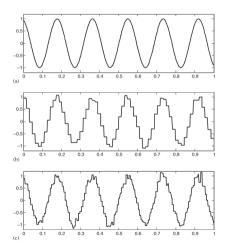
 $\begin{array}{c} f \in L^2(\mathbb{R}^d) \\ \swarrow \\ \{ \otimes_{\ell=1}^d 2^{J/2} \phi(2^J \cdot - k_\ell) \}_{\mathbf{k} \in \mathbb{Z}^d} \\ \Sigma_{\mathbf{k}} \langle f, \otimes_{\ell=1}^d 2^{J/2} \phi(2^J \cdot - k_\ell) \rangle \otimes_{\ell=1}^d 2^{J/2} \phi(2^J \cdot - k_\ell) \\ & \text{APPROXIMATION} \end{array}$

Projection onto the subspace spanned by $\begin{cases} \otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}) : \mathbf{k} \in \mathbb{Z}^{d}, \max_{1 \leq \ell \leq d} j_{\ell} \geq J \\ \\ \Sigma_{\mathbf{j},\mathbf{k}} \langle f, \otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}) \rangle \otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}) \\ \\ \\ \mathsf{DETAILS} \end{cases}$





Multiresolution analysis





Wavelet approximation for stochastic integral

$$I_d: L^2(\mathbb{R}^d) \to L^2(\Omega): f \mapsto \int_{\mathbb{R}^d} f \, dB(x_1) \dots dB(x_d)$$

is a quasi-isometry.





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$$\sum_{\mathbf{k}} (f, \bigotimes_{\ell=1}^{d} 2^{J/2} \phi(2^{J} \cdot -k_{\ell})) \bigotimes_{\ell=1}^{d} 2^{J/2} \phi(2^{J} \cdot -k_{\ell})$$

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$$\sum_{\mathbf{k}} \langle f, \otimes_{\ell=1}^{d} 2^{j/2} \phi(2^{j} \cdot -k_{\ell}) \rangle \frac{I_d(\otimes_{\ell=1}^{d} 2^{j/2} \phi(2^{j} \cdot -k_{\ell}))}{APPROXIMATION} \xrightarrow{I_d(f) \in L^2(\Omega)} \sum_{\mathbf{j}, \mathbf{k}} \langle f, \otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}) \rangle \frac{I_d(\otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}))}{DETAILS} \xrightarrow{L_d(f) \in L^2(\Omega)} \sum_{\mathbf{j}, \mathbf{k}} \langle f, \otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}) \rangle \frac{I_d(\otimes_{\ell=1}^{d} 2^{j\ell/2} \psi(2^{j\ell} \cdot -k_{\ell}))}{DETAILS}$$



Wavelet approximation for chaotic processes

We consider stochastic processes of the form

 $\{I_d(K(t, \bullet))\}_{t \geq 0}$

with, for all $t \ge 0$, $K(t, \bullet) \in L^2(\mathbb{R}^d)$. For the Hermite process, we have $K(t, \mathbf{x}) = \frac{1}{c_H} \int_0^t \prod_{\ell=1}^d (s - x_\ell)_+^{\frac{H-1}{d} - \frac{1}{2}} ds.$





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$$\sum_{\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j/2}\phi(2^{j}\cdot-k_{\ell})\rangle I_{d}} \bigotimes_{\ell=1}^{\langle g_{\ell}^{2}2^{j/2}\phi(2^{j}\cdot-k_{\ell})\rangle} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})\rangle I_{d}} \bigotimes_{\ell=1}^{\langle g_{\ell}^{2}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})\rangle} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})\rangle I_{d}} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})\rangle I_{d}} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})}} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})}} \sum_{\mathbf{j},\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi(2^{j}\ell\cdot-k_{\ell})}} \sum_{\mathbf{k}^{\langle K(t,\bullet),\otimes_{\ell=1}^{d}2^{j}\ell^{\prime 2}\psi$$



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But, generally, we want a stronger convergence: almost surely uniformly on compact sets.





(A. Ayache, J. Hamonier, L. L. - 2024)

The random series

$$X_{H,J}^{(d)}(t) = 2^{-J(H-1)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sigma_{J,\mathbf{k}}^{(H)} \int_0^t \prod_{\ell=1}^d \Phi_{\Delta}^{(1/2 + \frac{H-1}{d})} (2^J s - k_\ell) \, ds, \tag{1}$$

is almost surely, for each $J \in \mathbb{N}$, uniformly convergent in t on each compact interval $I \subset \mathbb{R}_+$. Moreover, for all such I, there exists an almost surely finite random variable (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X_{H}^{(d)} - X_{H,J}^{(d)}\|_{I,\infty} = \left\| \sum_{\substack{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^{d})^{2} \\ \max_{\ell \in [[1,d]]} j_{\ell} \geq J}} 2^{(\frac{H-1}{d})(j_{1}+\dots+j_{d})} \varepsilon_{\mathbf{j},\mathbf{k}} \int_{0}^{t} \prod_{\ell=1}^{d} \psi_{h_{\ell}}(2^{j_{\ell}}s - k_{\ell}) \, ds \right\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(H-1/2)}.$$



 Stochastic processes
 Few words about wavelets

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 $\sigma^{\scriptscriptstyle (H)}_{J,{f k}}$

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathscr{P}_m^{(d)}} \prod_{r=1}^m \mathbb{E}[Z_{J,k_{\ell_r}}^{(1/2+\frac{H-1}{d})} Z_{J,k_{\ell_r'}}^{(1/2+\frac{H-1}{d})}] \prod_{s=m+1}^{d-m} Z_{J,k_{\ell_s'}}^{(1/2+\frac{H-1}{d})}.$$
 (2)

where, for $J \in \mathbb{Z}$, we consider the sequence $(g_{J,k}^{\phi})_{k \in \mathbb{Z}}$ of i.i.d. $\mathcal{N}(0,1)$ Gaussian random variable defined, for all $k \in \mathbb{Z}$, by $g_{J,k}^{\phi} := 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k) \, dB(x)$. and for $\delta \in (-1/2, 1/2)$, the Gaussian FARIMA $(0, \delta, 0)$ sequence $(Z_{J,\ell}^{(\delta)})_{\ell \in \mathbb{Z}}$ associated to $(g_{J,k}^{\phi})_{k \in \mathbb{Z}}$ is given, for all $\ell \in \mathbb{Z}$, by

$$Z_{J,\ell}^{(\delta)} := g_{J,\ell}^{\phi} + \sum_{p=1}^{+\infty} \gamma_p^{(\delta)} g_{J,\ell-p}^{\phi}, \text{ with } \delta_p^{(\delta)} := \frac{\delta \Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}$$



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"Approximation process"

$$X_{H,J}^{(d)}(t) = 2^{-J(H-1)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sigma_{J,\mathbf{k}}^{(H)} \int_0^t \prod_{\ell=1}^d \Phi_{\Delta}^{(1/2 + \frac{H-1}{d})} (2^J s - k_\ell) \, ds,$$

with $\widehat{\Phi}_{\Delta}^{(\delta)}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi)$ Not computable, only because of the series.





Strategy

To deduce the inequality

$$\|X_{H}^{(d)} - X_{H,J}^{(d)}\|_{I,\infty} = \left\| \sum_{\substack{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^{d})^{2} \\ \max_{\ell \in [[1,d_{l}]]} j_{\ell} \geq J}} 2^{(\frac{H-1}{d})(j_{1}+\dots+j_{d})} \varepsilon_{\mathbf{j},\mathbf{k}} \int_{0}^{t} \prod_{\ell=1}^{d} \psi_{h_{\ell}}(2^{j_{\ell}}s - k_{\ell}) \, ds \right\|_{I,\infty} \leq CJ^{\frac{d}{2}} 2^{-J(H-1/2)} .$$

we somehow remarked that this rate of convergence is mainly determined by the terms in the series for which the corresponding indices belongs to

$$D_j^1(t) := \{k \in \mathbb{Z} : [k2^{-j} - 2^{-ja}, k2^{-j} + 2^{-ja}] \subseteq [0, t]\}$$
 with $a \in (1/2, 1)$





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Set

$$\mathcal{J}_{I}^{1}(t) := \{ \mathbf{k} \in (D_{I}^{1}(t))^{d} : \max_{\ell, \ell' \in [[1,d]]} |k_{\ell} - k_{\ell'}| \le 2^{\varepsilon I} \} \text{ with } \varepsilon > 0.$$

L. Loosveldt

8th November 2024



(A. Ayache, J. Hamonier, L. L. - 2024)

For all $J \in \mathbb{N}$, the simulation process at scale J of the generalized Hermite process $\{X_{H}^{(d)}(t)\}_{t \in \mathbb{R}_{+}}$ is the process defined, for all $t \in \mathbb{R}_{+}$, by

$$S_{H,J}^{(d)}(t) = 2^{-J(H-1)} \sum_{\mathbf{k} \in \mathcal{J}_{J}^{1}(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_{\Delta}^{(1/2 + \frac{H-1}{d})}(s - k_{\ell}) \, ds.$$
(3)

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X_{H}^{(d)} - S_{H,J}^{(d)}\|_{I,\infty} \le C J^{\frac{d}{2}} 2^{-J(H-1/2)}.$$
(4)





LIÈGE université Mathématique	Stochastic processes	Few words about wavelets	Our results	17
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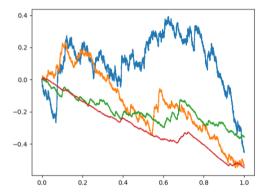


Figure: Paths of the Rosenblatt process of Hurst parameter 0,6 (blue), 0,7 (orange), 0,8 (green) and 0,9 (red).

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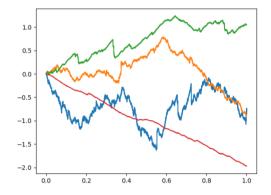


Figure: Paths of the Hermite process of order 3 of Hurst parameter 0,6 (blue), 0,7 (orange), 0,8 (green) and 0,9 (red).

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Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

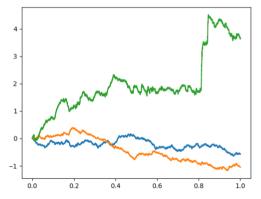


Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter 0,6.

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Python routine (using some R packages) for the Hermite processes of order 1,2 and 3 available upon request.

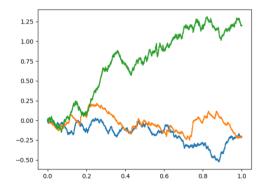


Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter 0,7.

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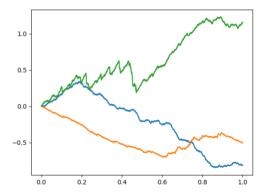


Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter 0,8.

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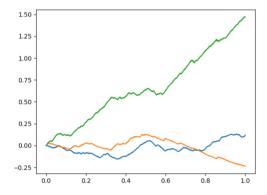


Figure: Paths of the Fractional Brownian Motion (blue), the Rosenblatt process (orange) and the Hermite process of order 3 (green) with Hurst parameter 0,9.



 test the numerical efficiency of statistical estimator for the Hurst parameters of the Hermite processes



- test the numerical efficiency of statistical estimator for the Hurst parameters of the Hermite processes
- statistical inference for the order of the process.



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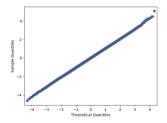


Figure: QQplot for the distribution of a fractional Brownian motion, compared with a $\mathcal{N}(0,1)$



- test the numerical efficiency of statistical estimator for the Hurst parameters of the Hermite processes
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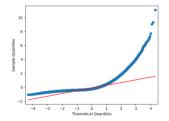


Figure: QQplot for the distribution of a Rosenblatt process, compared with a $\mathcal{N}(0,1)$



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- statistical inference for the order of the process.

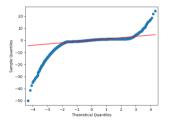


Figure: QQplot for the distribution of a Hermite process of order 3, compared with a $\mathcal{N}(0,1)$