A Summary Indicator of Firm Concentration*

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Abstract

There is no summary indicator of firm concentration, but a multitude of *ad hoc* indicators that provide different and even contradictory results on the empirical distributions of firms. In this paper, I propose a definition of firm concentration and an Euclidean summary indicator of it. This indicator is then applied to a comprehensive database on the employment of firms in Belgium between 2006 and 2012.

Keywords: Competition, firm concentration, Hirschman-Herfindahl index, measurement.

JEL Classification: L11, L16, L25.

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1 Introduction

Firm concentration, or market structure, is the dispersion of economic aggregates such as aggregate sales, value added, assets or employment across firms or production units. All competitive market economies exhibit market structures where some firms account for much larger shares of economic activity than others. The smaller the number of firms, the larger their market shares, the more concentrated the supply. Firm concentration can be represented graphically by the distribution of firms ordered by size. This distribution, called the firm size distribution (FSD), is a function of the number and size of firms, two variables endogenous to the market dynamics. Since Gibrat (1931), it is well known that the aggregate FSD in market economies is skewed to the right, i.e., there are many small firms and few large firms.

The FSD is the outcome of three market forces: entry, exit, and growth of firms. These three forces depend, among other things, on the competition intensity of markets. However, even in perfect competition, markets can be concentrated because of barriers to entry. These include the economies of scale, the capital expenditures required by available technologies to make certain production processes profitable (Bain, 1956), and the sunk costs such as R&D and advertising expenditures sometimes necessary to survive competition (Sutton, 1991). Consequently, an aggregate skew distribution of firm sizes is theoretically compatible with perfect competition. Nevertheless, in the short or medium run, markets can deviate from perfect competition because some firms manage, even in a legal way, to increase in size and raise their market power by their ability to better meet demand, by buying out competitors, or by racing ahead technologically. These lawful market practices contribute to non-price competition and greater economic concentration. In the long run, the outperforming positions won by these firms can be eroded by old and new competitors but can also turn into monopolistic positions. The FSD is therefore the result, on the one hand, of the variability across markets of the minimum firm size to reach the break-even point and, on the other hand, of the growth of efficient firms.

If a concentrated market structure is not necessarily a signal of the presence of firms with strong market power, can the FSD still provide useful information for assessing market competition? Saving (1970) showed, in a dominant firm model, that there is a link between the concentration ratio, an indicator of market structure, and the Lerner index (Lerner, 1934), the textbook indicator of firms' market power, which is generally considered the relevant indicator for measuring market competition.[1] When the price elasticity of demand in the market is low and the price elasticity of supply of existing and potential rival firms to the large firms in that market is also low, then the Lerner index (market power) is highly dependent on the concentration ratio (market structure). In the same vein, Cowling and Waterson (1976) establish the link between the Hirschman-Herfindahl index, another indicator of market structure, and the Lerner index. Finally, Dickson (1979) shows that the two previous results are special cases of the general model proposed by Cowling and Waterson (1976).

Apart from the link that can be made between market power and market structure, there are other reasons why concentration indicators are widely used and often preferred to the Lerner index to assess market competition. First, a price above marginal cost as measured by the Lerner index does not necessarily mean that the firm is operating in an uncompetitive market. This may indicate that the firm is exploiting economies of scale efficiently or covering its fixed costs, but not necessarily that it is able to be a price maker.[2] In addition, the Lerner index is data intensive, since the marginal costs of firms must be known in order to calculate it. Finally, the Lerner index for a market requires the aggregation of the marginal price-cost of all firms in that market. There are different ways

[1] Giocoli (2012) discusses the precedence of the Lerner index and its extension to the dominant firm model.

[2] See Elzinga and Mills (2011) for a discussion of the pros and cons of the Lerner index.

of aggregating them and none is more relevant than another. All these shortcomings make it more convenient to use indicators of market structure to assess market competition.

The two market structure indicators mentioned above - the concentration ratio and the Hirschman-Herfindahl index - are widely used in the literature and by competition authorities. The concentration ratio is used to measure concentration in the upper tail of the distribution, while the Hirschman-Herfindahl index gives an approximate indicator of concentration in the entire distribution. However, there are many other indicators of concentration (Hart (1975), Curry and George (1983)). This proliferation indicates that, like the indicators of inequality, none is unanimously accepted. Authors have attempted to establish a certain number of criteria for identifying the best, but no indicator is free of criticism (Hall and Tideman (1967) and Hannah and Kay (1977)). Above all, these indicators are in fact *ad hoc*. Encaoua and Jacquemin (1978) show that they are all the product of market shares and an *ad hoc* weighting function. Hart (1975) goes further, showing that they are *ad hoc* functions of variance, and questions the real gain in terms of statistical information that they bring.

The multiplication of *ad hoc* indicators and their lack of theoretical underpinnings cast doubt on what these existing indicators actually measure. In addition, they miss the essential need of this literature: how to summarize the firm size distribution in a single number? The objective of the present paper is to address these shortcomings and to propose a summary indicator of firm concentration. The strengths of our contribution are threefold. First, a definition of minimum concentration is established to determine the distribution of firms with minimum concentration. Second, a summary indicator of firm concentration is created from the Euclidean difference between the observed empirical distribution and the theoretical distribution of firms with minimum concentration. Third, this new indicator is applied to a comprehensive database of firm sizes in Belgium at the national, regional and sectoral levels for each year from 2006 to 2012. The results are then compared with those obtained using the Hirschman-Herfindahl index.

The rest of the paper is organized as follows. Section II describes the parametric approach to measuring firm concentration. Section III discusses the *ad hoc* indicators of firm concentration and their shortcomings. Section IV presents a definition and an Euclidean summary indicator of firm concentration. Section V applies this new indicator to an exhaustive database of Belgian firms from 2006 to 2012. Section VI concludes.

2 The Parametric Approach to Measuring Firm Concentration

Let z denote a non-negative variable such as the size of a firm, which can be measured by employment, sales, value added or assets. Considering n firms, let z_i be the size of firm $i = 1, ..., n$, and s_i be the market share of firm i where $s_i = z_i / \sum_{i=1}^{n} z_i$. The firm size distribution, which orders frequencies of firms by size, is a realization of a (discrete or continuous) random variable Z, where Z is the size of firm. The probability of a firm to be of size z (or of an interval including z if the random variable is continuous) can be approximated by a parametric distribution. The scale parameter of the parametric distribution can be used to estimate the concentration of the distribution. This is precisely what Gibrat (1931) proposed and applied to employment data of French firms.

However, this approach has two limitations. First, it requires a sufficient number of observations to be able to estimate concentration with statistical precision. This condition is often not met when one is interested in the concentration of the upper tail of the empirical distribution or of a small number of firms in a sector. Second, there has never been a strong consensus on the best parametric approximation of the empirical distribution of firm sizes, casting doubt on the information accuracy given by the parameters of parametric approximations. Since Gibrat (1931), the findings of the empirical literature have oscillated between two competing parametric approximations: the lognormal and the Pareto distributions. Both distributions are heavy-tailed but the Pareto law gives more weight to the largest firms and, hence, tends to overestimate the concentration of firms. Recently, two publications have unambiguously shown that the lognormal distribution provides a better fit to the firm size distribution in Belgium and the US, even at the upper tail (Artige and Bignandi (2023), Kondo et al. (2023)). Nevertheless, the empirical firm size distribution will never be exactly fitted by a lognormal distribution or any other parametric distribution with few parameters. Therefore, the measurement of the concentration depends closely on the choice of the parametric distribution.

3 *Ad hoc* Indicators of Firm Concentration

In the second half of the 20th century, the empirical literature on firm concentration flourished and multiple *ad hoc* indicators were proposed. Among these *ad hoc* indicators, two are frequently used in the economic literature to quantify firm concentration: the concentration ratio and the Hirschman-Herfindahl index.

3.1 Concentration Ratio

The concentration ratio CR_m measures the market share held by the m largest firms in the firm size distribution:

$$CR_m = \sum_{i=1}^{m} s_i = \sum_{i=1}^{m} \left(\frac{z_i}{\sum_{i=1}^{n} z_i} \right) = \sum_{i=1}^{m} \frac{z_i}{n\bar{z}}, \quad m \leq n, \tag{1}$$

where, I recall, s_i is the relative size of firm i or its market share, z_i is the size of firm i, $\bar{z}$ is the mean firm size and m is the number of the largest of the n firms ordered by size. The range of possible values is: $0 < CR_m \leq 1$. This indicator can be used to measure the market share of a single firm or the largest firms in a sector, a region or a nation.

By focusing on the largest firms, the concentration ratio is useful and appropriate for competition regulators who wish to assess the monopolistic positions and tendencies of large firms.

For a given number m, concentration is said to increase if the concentration ratio increases regardless of any distributional changes within the residual part of the firm size distribution. For small m values, the calculation of the concentration ratio requires little data and, usually, data on large firms that are generally readily available. Nevertheless, the statistical information provided by this indicator is limited. It provides information on the market weight of a group of large firms and its evolution over time, but cannot give any information on the concentration in the entire firm size distribution. Moreover, this indicator may increase when the number of firms increases, in the case where one or more firms entering the market are among the m largest firms (Hart, 1961).

3.2 The Hirschman-Herfindahl Index

The Hirschman-Herfindahl (HH) index was developed independently by Hirschman (1945) and Herfindahl (1950).[3] It provides an indicator of the concentration of the entire firm size distribution. Its most common formula[4] is

$$HH = \sum_{i=1}^{n} s_i^2 \tag{2}$$

which is the sum of the relative firm sizes, $s_i = \frac{z_i}{\sum_{i=1}^{n} z_i}$, $i = 1, ..., n$, weighted by themselves. The values for this indicator range from $1/n$ (lowest concentration) to 1 (highest concentration, i.e. monopoly).

Along with the concentration ratio, the Hirschman-Herfindahl index is the most widely

[3]The authorship of this simple index is rather obscure. See Rousseau (2018) for more information about the history of this index.

[4]The indicator proposed by Hirschman (1945) is the square root of (2).

used statistic by researchers and competition authorities to measure concentration.[5] The two indicators are in fact complementary. The concentration ratio is the appropriate indicator for measuring the concentration of a subset of a distribution, and the *HH* index provides what the CR_m cannot, namely an indicator of the concentration of the entire distribution. Despite the existence of many alternative indicators of concentration of the entire firm size distribution, the *HH* index remains the most commonly accepted indicator of firm concentration. The main reason for its success is the inclusion of the deconcentrating effect of the number n of firms in the calculation of the spread of the distribution.[6] Indeed, most of the competing indicators of the *HH* index are inequality indicators where an increase in the number n of firms has an uncertain effect on the value of the indicator. Mathematically, we should expect concentration to decrease as the number n of firms increases for a given number L of employees, regardless of the size of the additional firms. It is this desirable property that led Hirschman (1945) to discard inequality indicators in favor of the *HH* index. On page 158, Hirschman writes:

"Control of an industry by few producers can be brought about by inequality of the individual output shares when there are many producers or by the fact that only few producers exist. One of the well-known conditions of perfect competition is that no individual seller should command an important share of the total market supply; this condition implies the presence of both relative equality of distribution ***and*** *of large numbers. The notion of concentration is thus seen to be more complex than the concept of income concentration. Therefore, the methods which have been devised to measure the concentration of income are inadequate for the measurement of the concentration phenomenon with which we are here concerned. An extreme case is this: if we would try to read off from a Lorenz graph*

[5] The Antitrust Division of the US Department of Justice makes explicit use of the *HH* index: https://www.justice.gov/atr/herfindahl-hirschman-index

[6] As n increases, the lowest possible value of the *HH* index decreases. This property is lost if the normalized *HH* index, $\frac{HH-1/n}{1-1/n}$, is used instead.

the degree of concentration of an industry in which two firms divided between themselves the total output, we would have to conclude that, because the Lorenz curve would coincide with the equidistribution line, there is no concentration."

As noted by Hirschman (1945), we can rewrite (2) as follows:

$$\sum_{i=1}^{n} s_i^2 = \frac{1}{n}\left(\frac{\sigma_z^2}{\bar{z}^2} + 1\right) \tag{3}$$

where $\sigma_z^2 = Var(z)$ and $\sigma_z/\bar{z}$ is the coefficient of variation. When all sizes z_i (or relative sizes s_i) are equal, then the variance σ_z^2 is equal to zero and, as already mentioned, the *HH* index is equal to $1/n$. The condition for the HH index to decrease when n increases, as we should expect, is

$$\frac{\partial HH}{\partial n} < 0 \;\;\Rightarrow\;\; \frac{\partial(\sigma_z/\bar{z})}{\partial n} < \frac{1}{n}\left(\frac{(\sigma_z/\bar{z})^2 + 1}{2(\sigma_z/\bar{z})}\right) \Leftrightarrow n < \frac{(\sigma_z/\bar{z})^2 + 1}{2(\sigma_z/\bar{z})\left[\frac{\partial(\sigma_z/\bar{z})}{\partial n}\right]} \tag{4}$$

where $\frac{\partial HH}{\partial n}$ is the partial derivative of the HH index with respect to n. If the coefficient of variation does not increase too much after an increase in n, the *HH* index will vary inversely with n - the total number of employees being held constant - as a concentration indicator should. When σ^2 tends to zero, the condition (4) is always verified as n always takes a finite value. Despite this desirable property, the *HH* index is not an adequate indicator of the concentration of the firm size distribution because, like all indicators of inequality, it is not a strictly monotone function in its arguments. The condition (4) makes it clear that the partial derivative of the *HH* index with respect to n can take zero values. Therefore, the *HH* index does not assign a unique value to each variant of the firm size distribution for a given number of employees. In this respect, the *HH* index does not do any better than the inequality indicators such as, for instance, the Gini coefficient. By way of illustration, Table 1 shows two different firm size distributions for a given total number of employees equal to 6. The Hirschman-Herfindahl indices turn out to be identical, although our intuition tells us that employment is more concentrated in distribution 2.

	Distribution 1	**Distribution 2**
Firm size	Number of firms	Number of firms
1 employee	3	0
2 employees	0	3
3 employees	1	0
HH	1/3	1/3

Table 1: Hirschman-Herfindahl index (*HH*) for two different firm size distributions

The example in Table 1 precisely illustrates the *HH* index's main flaw. This indicator does not make it possible to rank the different distributions according to the level of firm concentration. As with indicators of inequality, the *HH* index measures the deviation from the equidistribution of firm sizes. However, the level of equality (minimum value of the *HH* index) depends on the number of employees and the number of firms. In Table 1, the number of employees is the same for both distributions but the number of firms is higher in distribution 1 than in distribution 2. Consequently, the egalitarian level of firm concentration is 1/4 in distribution 1 and 1/3 in distribution 2. In the end, the only information given by Table 1 is that the concentration is minimal in distribution 2 while it is higher than the minimum concentration in distribution 1 for a given number of employees. However, the *HH* index does not allow us to conclude on the comparison between the concentrations of distributions 1 and 2. If, in addition, the number of employees varies, then the *HH* index becomes useless: two distributions with different numbers of employees and firms have incomparable *HH* indices.

3.3 Theoretical Lineage of *ad hoc* Indicators of Concentration

Between the 1940s and the 1970s, a multitude of *ad hoc* indicators of concentration were proposed in the scientific literature (Marfels (1971), Hart (1975) and Curry and George (1983)). This multiplication of indicators puts the applied researcher and the practitioner in a quandary since these indices provide different or even contradictory values of the observed concentration of firms. Some authors have attempted to establish a list of criteria for sorting out the most relevant indicators from the others and identifying the optimal indicator (Hall and Tideman (1967), Marfels (1972), Hannah and Kay (1977)). Unfortunately, these indicators were crafted without theoretical foundations, hence the term '*ad hoc*' often associated with them (Hart, 1975).

This lack of a theoretical underpinning has made the scientific debate about finding the right indicator unnecessarily obscure. Thanks to the work of Hart (1961, 1971, 1975, 1979), the theoretical lineage of all these indicators has been made clear. Hart (1975) examined fifteen indicators of inequality and concentration of firms, including the concentration ratio, the Hirschman index, the Herfindahl index, the Gini index and the entropy indicator. He shows that these fifteen indicators are in fact all functions of the moments of the observed distribution. In particular, following Hart (1971, 1975), we can observe that the concentration ratio

$$CR_m = \sum_{i=1}^{m} s_i = \sum_{i=1}^{m} \left(\frac{z_i}{\sum_{i=1}^{n} z_i} \right) \tag{5}$$

is an upper quantile of the first moment distribution of z and the Herfindahl index

$$HH = \sum_{i=1}^{n} s_i^2 = \frac{\sum_{i=1}^{n} z_i^2}{\left(\sum_{i=1}^{n} z_i\right)^2} = \frac{\sum_{i=1}^{n} z_i^2}{\sum_{i=1}^{n} z_i} \times \frac{1}{n\bar{z}} \tag{6}$$

is the product of $\sum_{i=1}^{n} z_i^2 / \sum_{i=1}^{n} z_i$, the first moment about zero (i.e. the arithmetic mean) of the first moment distribution of z, and $1/n\bar{z}$, where $\bar{z}$ is the first moment about zero (i.e. the arithmetic mean) of the distribution of z.

The contribution of Hart (1975) is a decisive milestone in this literature. It makes it possible to establish a common theoretical basis for these two concentration indicators and the thirteen others that he studied, and to show that these measurements, rather than adding information to the knowledge of the distribution, all rely on the information provided by the moments of this distribution.

4 A Summary Indicator of Firm Concentration

In this section, I propose a definition and a summary indicator of firm concentration, where firm size is measured by the number of employees (or full-time equivalents). Therefore, the proposed indicator of firm concentration is an indicator of employment concentration, where the unit of measurement is a person employed or a full-time equivalent.

4.1 Firm Concentration: A Definition

In market economies, firms compete to supply goods and services to consumers within the constraints of technology and the competition rules set by regulators. The number of firms able to survive the competition is determined endogenously by the regulated markets. The smaller the number of firms, the more concentrated the supply. Consider a market economy with a labor force equal to L employees and assume that each individual works full-time for a single firm and cannot own more than one firm. The feasible number n of firms is therefore bounded:

$$n \leqslant L \tag{7}$$

The possible values for n range from a single firm with L employees to $n = L$ firms with a single employee, i.e. the maximum possible number of firms. This leads to our following definition of concentration:

Definition 1 (Firm concentration) *Firm concentration is the deviation of the empirical firm size distribution from the single-employee firm distribution (theoretical minimum concentration or atomistic market structure).*

The theoretical minimum firm concentration is obtained when the number of firms is at its maximum value $n = L$ where the firm size distribution is degenerate at the size equal to 1. As the number of firms decreases, the concentration of firms increases monotonically until a single firm acquires a monopoly on supply ($n = 1$), where the firm size distribution is again degenerate at the size equal to L.[7]

4.2 Firm Concentration: Measurement

As Proposition 1 suggests, measuring firm concentration consists of calculating the distance between the empirical firm size distribution and the theoretical single-employee firm distribution. How to measure this distance and sum it up in a single number? Using linear algebra, I propose to calculate the vector differences between the two distributions for each firm size and then to calculate, for each firm size, the Euclidean norm of this vector difference. Summing these Euclidean norms for all firm sizes observed in the empirical firm size distribution yields a summary indicator of firm concentration.

Consider the vector space $\Omega(z, n_z)$ where $z \in \mathbb{R}_+^*$ is firm size, measured by the number of employees or full-time equivalents, and $n_z \in \mathbb{N}_+$ is the frequency of firms of size z. The total number of firms is therefore equal to $n = \sum_{z=z_1}^{z_k} n_z$ where $k \in \mathbb{N}_+^*$ is the number of firm sizes. Let us now define the two distributions of interest in the vector space Ω:

i) The empirical firm size distribution e

For each firm size $z = 1, ..., z_k$, let $\mathbf{x}_z = (z, n_{e,z})$ the vector of firm size z and

[7] For the Hirschman-Herfindahl index, the theoretical minimum firm concentration is obtained when the firm size distribution is degenerate at the size equal to $1/n$, which varies with n. This distribution is the equidistribution as in the inequality indicators.

the frequency of firms $n_{e,z}$ of the empirical firm size distribution e, where k is the observed number of firm sizes in the empirical firm size distribution. The set of vectors $\mathbf{x}_z$ characterizes the empirical firm size distribution e in the vector space $\Omega(z, n_z)$.

ii) The theoretical single-employee firm size distribution t

The theoretical or minimum firm size distribution I defined in Proposition 1, i.e. the single-employee firm distribution, is characterized by the set of vectors $\mathbf{b}_z = (z, n_{t,z})$ in the vector space $\Omega(z, n_z)$ where $n_{t,z}$ is the frequency of firms for firm size z of the theoretical firm size distribution t. Since the distribution t is degenerate, there are only two possibilities for the vector $\mathbf{b}_z$: either $\mathbf{b}_{z=1} = (1, L)$ where all firms are single-employee, or $\mathbf{b}_{z\neq1} = (z, 0)$ where there are no firms for all the other sizes $z \neq 1$ observed in the empirical distribution e.

From the vectors $\mathbf{x}_z$ and $\mathbf{b}_z$, we can obtain a new vector $\mathbf{h}_z = (z, n_{e,z} - n_{t,z})$ whose firm frequency for each size is equal to the difference in the firm size frequencies between the empirical and theoretical firm size distributions. The set of vectors $\mathbf{h}_z$ characterizes the distribution d of the differences in the frequency of firm sizes between the two distributions. We can now calculate the Euclidean norm of the vector $\mathbf{h}_z$, which is equal to the Euclidean distance, for a given value of z, between the vector $(z, n_{e,z})$ of the empirical distribution and the vector $(z, n_{t,z})$ of the theoretical distribution:

$$\|\mathbf{h}_z\| := \sqrt{\langle \mathbf{h}_z, \mathbf{h}_z \rangle} = \sqrt{z^2 + (n_{e,z} - n_{t,z})^2} \tag{8}$$

Since the firm size z is greater than or equal to 1, it is more convenient to measure the norm of the vector $\mathbf{h}_z$ from the point (1,0) than from the point (0,0). Therefore, we can rewrite (8) as

$$\|\mathbf{h}_z\| := \sqrt{\langle \mathbf{h}_z, \mathbf{h}_z \rangle} = \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2} \tag{9}$$

The sum of the Euclidean norms of the vectors $\mathbf{h}_z$ is the Euclidean norm of the distribution d:

$$\sum_{z=1}^{z_k} \|\mathbf{h}_z\| = \sum_{z=1}^{z_k} \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2}, \quad n_{e,z} \neq 0 \; \Lambda \; n_{t,z} \neq 0 \tag{10}$$

Note that the sum (10) excludes the vectors for which $n_{e,z}$ and $n_{t,z}$ are both zero for a given firm size z. Given that the theoretical firm size distribution has only two possible vectors $\mathbf{b}_z$, the sum (10) can be usefully written as follows:

$$\sum_{z=1}^{z_k} \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2} = \sqrt{0 + (n_{e,z=1} - L)^2} + \sum_{z \neq 1} \sqrt{(z-1)^2 + (n_{e,z\neq 1} - 0)^2} \tag{11}$$

In order to be able to compare the values of the Euclidean norm (10) for different empirical firm size distributions, whatever the number k of firm sizes and the total number of firms, it is necessary to normalize it by proceeding as follows: first, let us calculate the expression of (10) when the empirical size distribution is the one with maximum concentration (monopoly):

$$\sum_{z=1}^{z_k} \|\mathbf{h}_z\| = \sum_{z=1}^{z_k} \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2} = L + \sqrt{(L-1)^2 + 1} \tag{12}$$

Then, let us divide (10) by (12). This leads to the indicator of firm concentration I propose:

Definition 2 (Summary indicator of Firm Concentration) *The summary indicator of firm concentration when firm size is measured by the number of employees or full-time equivalents is*

$$I = \frac{\sum_{z=1}^{z_k} \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2}}{L + \sqrt{(L-1)^2 + 1}}, \quad n_{e,z} \neq 0 \; \Lambda \; n_{t,z} \neq 0 \tag{13}$$

where $\sum_{z=1}^{z_k} \sqrt{(z-1)^2 + (n_{e,z} - n_{t,z})^2} = \sqrt{(n_{e,z=1} - L)^2} + \sum_{z \neq 1} \sqrt{(z-1)^2 + n_{e,z \neq 1}^2}$.

$I \in (0, 1)$ is the summary indicator of firm concentration based on the Euclidean distance between the vectors of the empirical and single-employee firm size distributions. For all possible values of n and L, $I = 0$ when the empirical firm size distribution is equal to the distribution with the minimum concentration (i.e. the theoretical firm size distribution), and $I = 1$ when the empirical firm size distribution is equal to the distribution with the maximum concentration (monopoly). Table 2 repeats Table 1, adding the calculation of firm concentration with this new indicator. It can be seen that, unlike the Hirschman-Herfindahl index, it is able to distinguish between distributions 1 and 2, confirming the intuition that distribution 2 is the more concentrated.

	Distribution 1	**Distribution 2**
Firm size	Number of firms	Number of firms
1 employee	3	0
2 employees	0	3
3 employees	1	0
HH	1/3	1/3
I	0.471	0.825

Table 2: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration I for two different firm size distributions

In addition to its sound theoretical foundations, our summary indicator of firm concentration is able to rank distributions based on concentration level, regardless of the number of employees and firms.

5 Application to Belgian Employment 2006-2012

The aim of this section is to use this new concentration indicator I on data from Belgian firms to study the level and its evolution over time of firm concentration in Belgium. In addition, we propose to compare the results obtained with this indicator with those obtained with the Hirschman-Herfindahl index.

5.1 Data

Our database consists of all registered private firms and establishments of the NACE sectors A to N in Belgium from 2006 to 2012.[8] The database was obtained from the Belgian Ministry of Economy[9]. The variable of interest is firm size measured by the number of salaried employees. In Belgium, each enterprise must provide its list of employees to the social security administration every quarter. Our database contains the exact number of employees of the last quarter. In this study, the unit of observation is the firm, which may be a combination of several legal units if they share a common economic activity. This choice is justified by the fact that economic decisions, such as hiring and firing decisions, are made at the firm level. Table 3 presents summary statistics on Belgian firms between 2006 and 2012. The database also includes information on the regional location and the NACE sector of each firm.

5.2 Results

In this section, we calculate the HH and I concentration indicators from national, regional and sectoral data on firms in Belgium for the years 2006-2012. The results show that the two indicators provide different, even contradictory, information on levels and trends in firm concentration in Belgium. This was to be expected, since the quadratic

[8]For the list and description of the 2008 NACE sectors in the European Union, see Appendix A.
[9]Source: SPF Economie - Direction générale Statistique - Statistics Belgium

Year	Employment	Firms	Mean size	Median size
2006	2,080,570	201,677	10.31	3
2007	2,069,337	197,731	10.46	3
2008	2,264,683	204,563	11.07	3
2009	2,230,109	203,424	10.96	3
2010	2,262,225	203,963	11.09	3
2011	2,277,888	203,733	11.18	3
2012	2,280,598	202,480	11.26	3

Table 3: Firm demographics in Belgium (NACE sectors A to N) between 2006 and 2012

function of market shares of the HH index amplifies the weight of large firms, which can distort the reality of firm concentration and its evolution over time. But, above all, the erratic fluctuating values of the HH index illustrate the inability of this indicator to rank distributions according to the level of firm concentration (see Section 3.2).

5.2.1 Aggregate Firm Concentration in Belgium (2006-2012)

Belgium is a member country of the European Union with more than 11 million inhabitants. Between 2006 and 2012, Belgium had around 200,000 firms belonging to NACE sectors A to N (Table 3). The average size of these firms was about 10-11 employees while the median size was 3 employees, confirming the stylized fact that the distribution of firm sizes is skewed to the right (Table 4). The values of the mean and standard deviation between 2006 and 2012 suggest that the distribution of firm sizes is relatively stable over time. Although these two statistics increase slightly over time, it is not possible to conclude that the distribution of firms in Belgium becomes more concentrated since the number of employees also increases over the period. Hence the need to use concentration indicators such as the Hirschman-Herfindahl index (HH) or our summary concentration index (I). Both indicators show that the concentration of firms in sectors A to N decreased over the period. Nevertheless, there is a notable difference between the two indicators: the HH index fluctuates much more than the I index, which shows that the concentration of firms, although decreasing, is very stable over the period (Table 4 and Figure 1). As argued at the end of Section 3.2, the decreasing trend observed from the values of the *HH* index should be considered with caution because these values are not comparable with each other due to the variation in the number of employees and firms over the period.

	Mean size	Sd	Median size	*HH*		*I*	
				Value	Index number 2006=100	Value	Index number 2006=100
2006	10.31	55.27	3	0.0001473	100.00	0.6088	100.00
2007	10.46	55.75	3	0.0001486	100.88	0.6079	99.84
2008	11.07	59.33	3	0.0001453	98.64	0.6065	99.61
2009	10.96	57.25	3	0.000139	94.37	0.6050	99.37
2010	11.09	57.03	3	0.0001346	91.38	0.6045	99.28
2011	11.18	57.87	3	0.0001364	92.60	0.6054	99.43
2012	11.26	57.62	3	0.0001342	91.11	0.6040	99.21

Table 4: Aggregate firm concentration in Belgium (2006-2012)

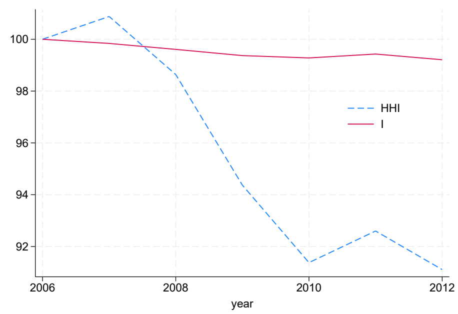
	HH			*I*		
	Flanders	Wallonia	Brussels	Flanders	Wallonia	Brussels
	Value			Value		
2006	0.0002264	0.0004277	0.0016787	0.6125	0.6268	0.7282
2007	0.0002276	0.0004992	0.0015336	0.6130	0.6262	0.7265
2008	0.0002387	0.0004329	0.0013825	0.6132	0.6263	0.7153
2009	0.0002246	0.0003788	0.0014535	0.6096	0.6219	0.7234
2010	0.0002136	0.00039	0.0014428	0.6102	0.6204	0.7252
2011	0.000218	0.0003885	0.0014626	0.6106	0.6242	0.7299
2012	0.0002129	0.0003766	0.0015072	0.6110	0.6192	0.7227
	Index number			Index number		
2006	100.00	100.00	100.00	100.00	100.00	100.00
2007	100.53	116.72	91.36	100.08	99.91	99.77
2008	105.43	101.22	82.36	100.11	99.92	98.23
2009	99.20	88.57	86.58	99.52	99.22	99.34
2010	94.35	91.19	85.95	99.62	98.98	99.60
2011	96.29	90.83	87.13	99.68	99.58	100.24
2012	94.04	88.05	89.78	99.74	98.79	99.25

Table 5: Firm concentration in Belgian regions (2006-2012): Values and index numbers (2006=100).

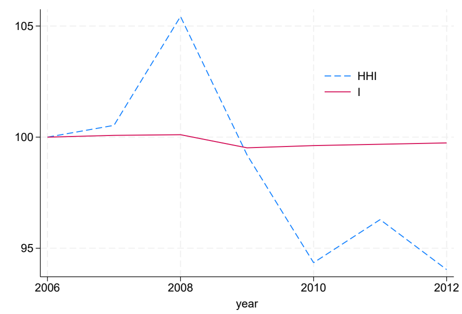
5.2.2 Firm Concentration in the Three Belgian Regions (2006-2012)

Belgium is made up of three regions of unequal demographic and economic size: Flanders, the most populous region, represents 58% of the Belgian population, while Wallonia and the Brussels region account for only 32% and 10% respectively. The Brussels region is also the capital city of Belgium, and is essentially an urban area.

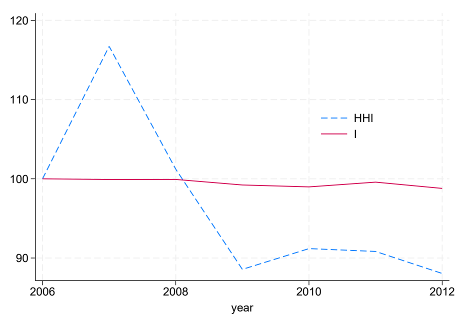
Table 5 shows the regional results for both concentration indicators. The I index shows that firm concentration is highest in the Brussels region. This result should come as no surprise, since the Belgian capital is home to the headquarters of major national and international firms. Furthermore, the I index gives very similar concentration values between Flanders and Wallonia. Finally, as with the national results, the I index shows great stability of concentration over time in all regions (Figure 1). As for the HH index, it indicates that the concentration of firms is highest in Brussels, followed by Wallonia. Again, these results should be taken with caution because the minimum values of the HH index are not the same in the three regions. The highest minimum value is in the Brussels region because there are fewer firms and the highest minimum value is in Flanders since it is the region with the most firms. Regarding the evolution of concentration over time, the HH index shows that the concentration of firms decreases in all three regions, with the strongest trend being observed in Wallonia after volatile fluctuations.



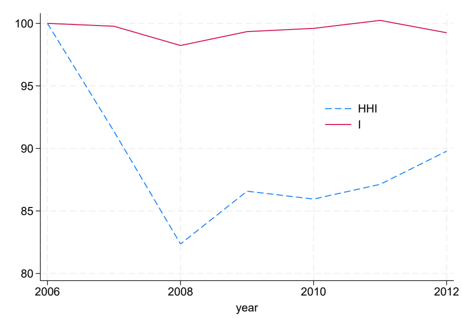
(a) Belgian firms in NACE sectors A to N



(b) Flanders-based firms in NACE sectors A to N



(c) Wallonia-based firms in NACE sectors A to N



(d) Brussels-based firms in NACE sectors A to N

Figure 1: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration (I), 2006-2012, 2006=100.

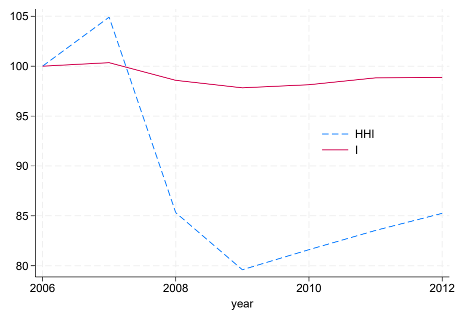
5.2.3 Firm Concentration in the Manufacturing Sector in Belgium (2006-2012)

Like all developed countries, Belgium is experiencing a deindustrialisation of its economy. It is therefore interesting to observe whether this deindustrialization is accompanied by a greater or lower concentration of firms in the manufacturing sector. Table 6 shows the values and index numbers for both indicators.

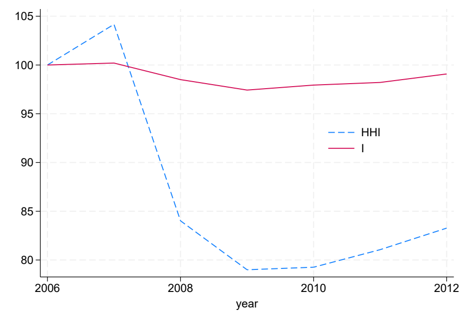
	HH				*I*			
	Aggregate	Flanders	Wallonia	Brussels	Aggregate	Flanders	Wallonia	Brussels
	Value				Value			
2006	0.0013046	0.0019169	0.0039591	0.0449795	0.7212	0.7404	0.7766	0.8427
2007	0.0013686	0.0019968	0.0053267	0.020975	0.7238	0.7419	0.7861	0.8287
2008	0.0011131	0.0016106	0.0043619	0.0199155	0.7110	0.7293	0.7760	0.8505
2009	0.0010385	0.0015143	0.0038872	0.0212053	0.7055	0.7213	0.7690	0.8516
2010	0.0010647	0.0015194	0.0042354	0.0217688	0.7078	0.7251	0.7686	0.8515
2011	0.0010898	0.0015539	0.0042917	0.0242173	0.7128	0.7271	0.7718	0.8435
2012	0.0011122	0.0015962	0.0042535	0.0264229	0.7130	0.7335	0.7729	0.8552
	Index number				Index number			
2006	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2007	104.91	104.17	134.54	46.63	100.35	100.20	101.22	98.34
2008	85.32	84.02	110.17	44.28	98.58	98.50	99.91	100.92
2009	79.60	79.00	98.18	47.14	97.83	97.43	99.01	101.06
2010	81.61	79.26	106.98	48.40	98.14	97.94	98.96	101.04
2011	83.54	81.06	108.40	53.84	98.83	98.21	99.37	100.09
2012	85.25	83.27	107.44	58.74	98.86	99.08	99.52	101.48

Table 6: Firm concentration in the Belgian manufacturing sector (NACE *C* sector) - 2006-2012. Values and index numbers (2006=100).

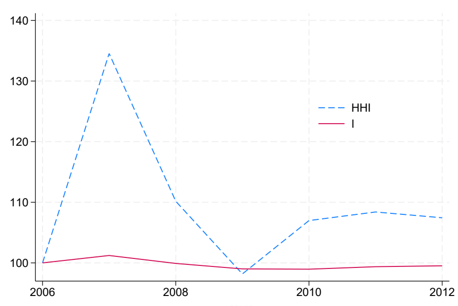
Again, the *I* index shows a very high degree of stability during the period of concentration of manufacturing firms at both national and regional levels (Figure 2). In terms of level, we can see that the concentration is higher in Brussels while that of Wallonia is a little higher than that of Flanders. The *HH* index is again much more volatile, especially in Wallonia. The ranking of the regions is the same as that observed with the *I* index. But, as before, caution must be exercised in the conclusions since the minimum values of the *HH* index vary with the number of firms.



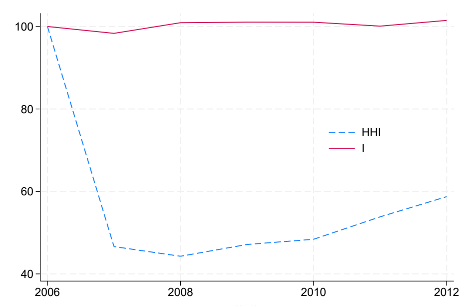
(a) Manufacturing firms in Belgium



(b) Flanders-based manufacturing firms



(c) Wallonia-based manufacturing firms



(d) Brussels-based manufacturing firms

Figure 2: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration (I), 2006-2012, 2006=100.

5.2.4 Firm Concentration in Services in Belgium (2006-2012)

The service sector is a heterogeneous sector. Therefore, we propose to measure the concentration of firms in the widest scope of services (NACE sectors G, H, I, K, M and N) but also in two subsets of this sector: the services sector employing mainly high-skilled workers (NACE sectors J, K, M and N) and the services sector employing more low-skilled workers (NACE sectors G, H and I). Tables 7, 8 and 9 show the values and index numbers for the two indicators of the three groups of services.

	HH				*I*			
	Aggregate	Flanders	Wallonia	Brussels	Aggregate	Flanders	Wallonia	Brussels
	Value				Value			
2006	0.0001477	0.0001693	0.0003616	0.0017971	0.6030	0.5983	0.6094	0.7329
2007	0.000155	0.0001778	0.0003826	0.0018608	0.6024	0.5996	0.6076	0.7345
2008	0.0001844	0.0002953	0.0003113	0.0017217	0.6071	0.6074	0.6096	0.7274
2009	0.0001859	0.0002902	0.0003108	0.0017958	0.6075	0.6054	0.6080	0.7340
2010	0.0001764	0.00027	0.0003103	0.0017705	0.6076	0.6052	0.6077	0.7373
2011	0.0001788	0.0002776	0.0002979	0.0017905	0.6059	0.6056	0.6069	0.7404
2012	0.0001728	0.0002632	0.0002972	0.001836	0.6054	0.6058	0.6047	0.7339
	Index number				Index number			
2006	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2007	104.94	105.02	105.81	103.54	99.90	100.21	99.70	100.21
2008	124.85	174.42	86.09	95.80	100.69	101.53	100.02	99.24
2009	125.86	171.41	85.95	99.93	100.76	101.19	99.76	100.15
2010	119.43	159.48	85.81	98.52	100.77	101.16	99.72	100.60
2011	121.06	163.97	82.38	99.63	100.50	101.21	99.59	101.03
2012	116.99	155.46	82.19	102.16	100.40	101.26	99.22	100.13

Table 7: Firm concentration in the Belgian service sector (NACE sectors G, H, I, K, M and N) - 2006-2012. Values and index numbers (2006=100).

The I index confirms the conclusions drawn previously. The concentration of firms is very stable over time regardless of the group of services considered (Figures 3, 4 and 5). However, there was a slight decrease in the concentration of firms over time in the high-skilled subset and a slight increase in the low-skilled subset. The highest concentration level is found in the Brussels region while the levels of Flanders and Wallonia are very close. The values of the HH index for services are the most erratic of all and therefore difficult to interpret. The concentration of firms is increasing considerably in Flanders,

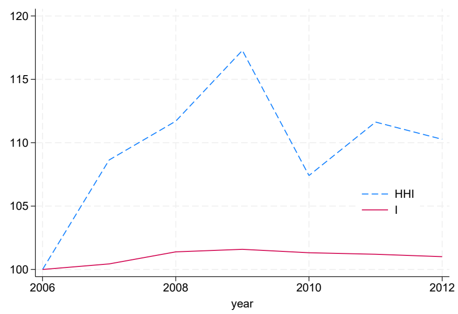
	HH				*I*			
	Aggregate	Flanders	Wallonia	Brussels	Aggregate	Flanders	Wallonia	Brussels
	Value				Value			
2006	0.0005184	0.0006006	0.0012666	0.003903	0.6539	0.6494	0.6689	0.7950
2007	0.0005113	0.0005786	0.0013054	0.0038823	0.6523	0.6492	0.6624	0.7894
2008	0.0006683	0.0012212	0.0010889	0.0037054	0.6552	0.6525	0.6626	0.7908
2009	0.000655	0.0011824	0.001065	0.0036913	0.6542	0.6524	0.6579	0.7901
2010	0.000613	0.001067	0.0010676	0.0036945	0.6557	0.6552	0.6631	0.7967
2011	0.0006137	0.0010733	0.000969	0.0037539	0.6542	0.6523	0.6580	0.7999
2012	0.0005881	0.0009864	0.0009746	0.0038414	0.6521	0.6505	0.6523	0.7909
	Index number				Index number			
2006	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2007	98.63	96.34	103.06	99.47	99.76	99.97	99.03	99.30
2008	128.92	203.33	85.97	94.94	100.19	100.48	99.07	99.47
2009	126.35	196.87	84.08	94.58	100.04	100.46	98.36	99.39
2010	118.25	177.66	84.29	94.66	100.27	100.90	99.14	100.22
2011	118.38	178.70	76.50	96.18	100.04	100.45	98.37	100.62
2012	113.45	164.24	76.95	98.42	99.72	100.17	97.52	99.49

Table 8: Firm concentration in the Belgian high-skilled service sector (NACE sectors *J*, *K*, *M* and *N*) - 2006-2012. Values and index numbers (2006=100).

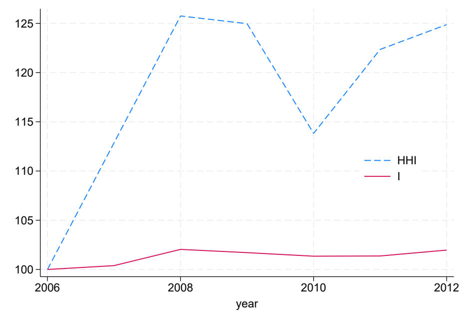
	HH				*I*			
	Aggregate	Flanders	Wallonia	Brussels	Aggregate	Flanders	Wallonia	Brussels
	Value				Value			
2006	0.000147	0.0001946	0.0004697	0.0022636	0.5886	0.5914	0.6062	0.6854
2007	0.0001597	0.0002196	0.0004935	0.0023865	0.5911	0.5937	0.6072	0.6917
2008	0.0001642	0.0002447	0.0003898	0.0023527	0.5967	0.6034	0.6069	0.6966
2009	0.0001724	0.0002432	0.000394	0.0027742	0.5979	0.6015	0.6042	0.7059
2010	0.0001579	0.0002215	0.0003824	0.0025414	0.5963	0.5993	0.6027	0.7044
2011	0.0001641	0.0002381	0.0003918	0.0025172	0.5957	0.5995	0.6006	0.7031
2012	0.0001621	0.000243	0.0003876	0.0024812	0.5945	0.6030	0.5986	0.6968
	Index number				Index number			
2006	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2007	108.64	112.85	105.07	105.43	100.44	100.39	100.16	100.91
2008	111.70	125.75	82.99	103.94	101.39	102.04	100.11	101.64
2009	117.28	124.97	83.88	122.56	101.59	101.71	99.66	102.99
2010	107.41	113.82	81.41	112.27	101.32	101.35	99.41	102.77
2011	111.63	122.35	83.41	111.20	101.20	101.37	99.07	102.58
2012	110.27	124.87	82.52	109.61	101.01	101.97	98.74	101.65

Table 9: Firm concentration in the Belgian low-skilled service sector (NACE sectors *G*, *H* and *I*) - 2006-2012. Values and index numbers (2006=100).

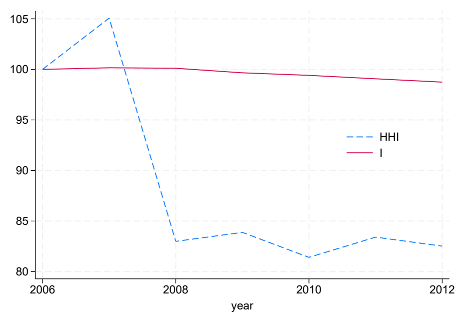
while it is decreasing sharply in Wallonia. In Brussels, concentration is more stable over the same period. The differences in trends in the three regions make little economic sense in a small country that is very open to trade in goods, services, and capital. These results confirm, once again, that the values of the HH index are not comparable between different economies.



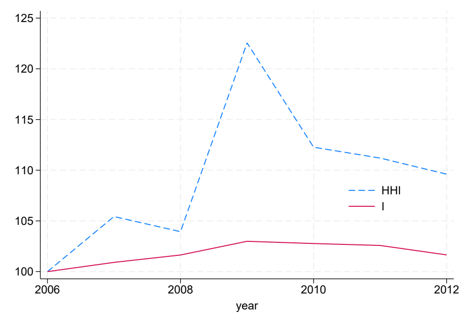
(a) Belgian firms in services



(b) Flanders-based firms in services

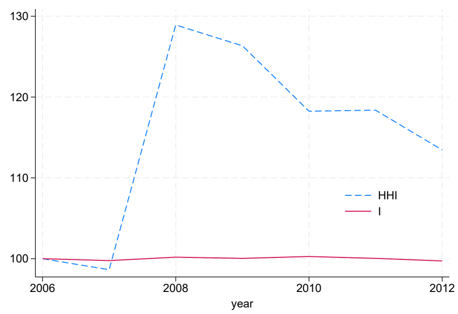


(c) Wallonia-based firms in services

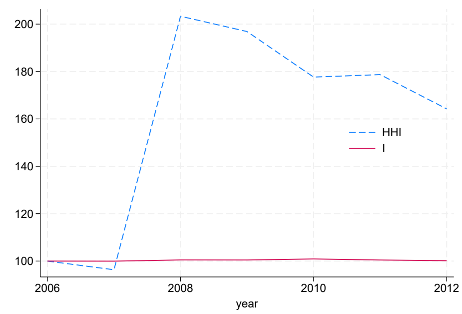


(d) Brussels-based firms in services

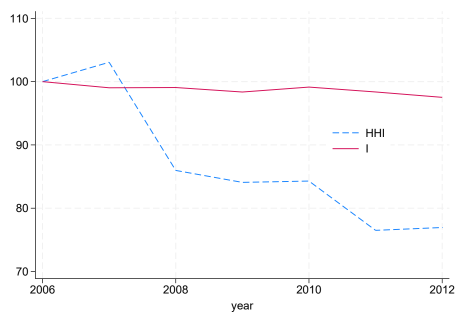
Figure 3: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration (I), 2006-2012, 2006=100.



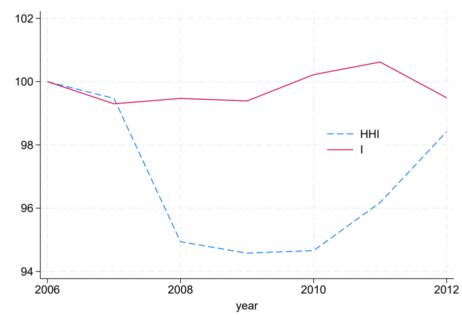
(a) Belgian firms in high-skilled services



(b) Flanders-based firms in high-skilled services

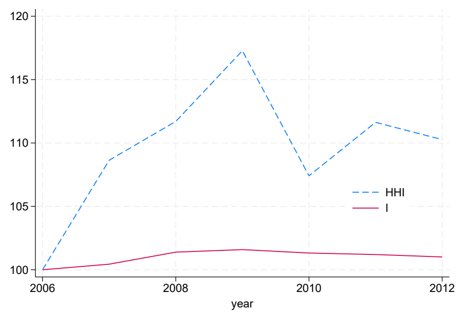


(c) Wallonia-based firms in high-skilled services

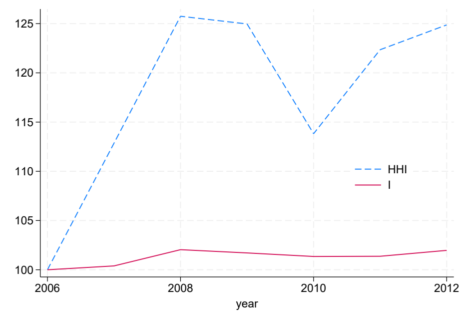


(d) Brussels-based firms in high-skilled services

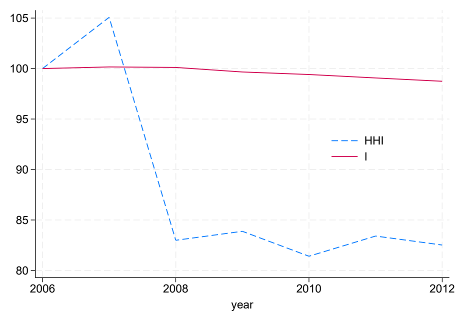
Figure 4: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration (I), 2006-2012, 2006=100.



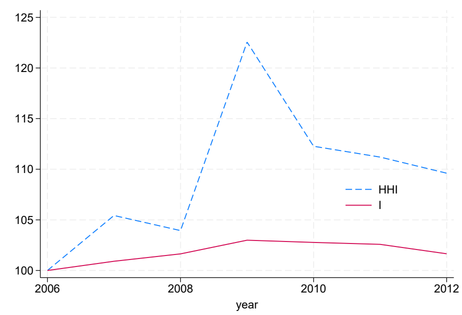
(a) Belgian firms in low-skilled services



(b) Flanders-based firms in low-skilled services



(c) Wallonia-based firms in low-skilled services



(d) Brussels-based firms in low-skilled services

Figure 5: Hirschman-Herfindahl index (HH) and summary indicator of firm concentration (I), 2006-2012, 2006=100.

6 Conclusion

This paper answers an essential research question in the literature of firm concentration : how to summarize the firm size distribution in a single number? So far, the indicators that this literature has proposed are all *ad hoc* indicators of the firm size distribution causing statistical distortions that are difficult to control and interpret. For example, the values of the Hirschman-Herfindahl index measuring two distributions with different numbers of firms and employees are incomparable because the minimum values of this index differ between the two distributions. What can be concluded from these values with regard to firm concentration?

To get out of the impasse of these *ad hoc* indicators, I have proposed in this paper a summary indicator of firm concentration based on the Euclidean distance between the firm size distribution measured by the number of employees and the single-employee firm distribution. It is free of statistical distortions and allows for comparing firm size distributions with different numbers of firms and employees.

The values taken by this summary indicator on data from Belgian firms confirm the results obtained with the parametric approach to measuring firm concentration: the level of concentration is higher in an urbanised capital region such as Brussels than in regions with more diversified economies such as Wallonia and Flanders, and higher in manufacturing than in services. Moreover, the time series of this indicator show a high degree of stability in the concentration of firms over a short period of 7 years at the national, regional and sectoral levels. Nevertheless, it seems likely that this summary indicator would result in less stable values of firm concentration over time on more disaggregated sectoral data or on the upper tail of distributions, as for example in the digital products or digital services sector. Confirmation is left for future research.

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A Appendix 1

Code	Economic Area
A	Agriculture, Forestry and Fishing
B	Mining and Quarrying
C	Manufacturing
D	Electricity, Gas, Steam and Air Conditioning Supply
E	Water Supply; Sewerage, Waste Management and Remediation Activities
F	Construction
G	Wholesale and Retail Trade; Repair of Motor Vehicles and Motorcycles
H	Transportation and Storage
I	Accommodation and Food Service Activities
J	Information and Communication
K	Financial and Insurance Activities
L	Real Estate Activities
M	Professional, Scientific and Technical Activities
N	Administrative and Support Service Activities
O	Public Administration and Defence; Compulsory Social Security
P	Education
Q	Human Health and Social Work Activities
R	Arts, Entertainment and Recreation
S	Other Service Activities
T	Activities of Households as Employers; Undifferentiate Goods and Services Producing Activities of Households for Own Use
U	Activities of Extraterritorial Organisations and Bodies

Table 10: Statistical classification of economic activities in the European Community Rev. 2 (2008): Level 1 codes.