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The 2 edge-disjoint 3-paths polyhedron

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Abstract Given an undirected graph, we study the problem of finding K edge-disjoint paths, each one containing at most L edges, between a given pair of nodes. We focus on the case of $K = 2$ and $L = 3$. For this particular case, previous known compact formulations are valid only for the case with non-negative edge costs. We provide the first compact linear description that is also valid for general edge costs. We describe new valid inequalities that are added to a well known extended formulation in a layered graph, to get a full description of the polyhedron for $K = 2$ and $L = 3$. We use a reduction of the problem to a size-2 stable set problem to prove this second property.

Keywords Network design, Survivability, Hop constraints, Combinatorial Optimization

1 Introduction

Consider an undirected graph $G = (V, E)$, with edge costs $c_{ij}, \{i, j\} \in E$, positive integers K and L and a pair of vertices o (the origin) and d (the destination). We study the problem of finding K edge-disjoint paths between o and d, of minimal total cost, such that each path is composed of a maximum of L edges. The hop limit L imposes a certain level of quality of service and the number of disjoint paths ensure survivability of the network in case of $K - 1$ simultaneous failures.

This problem arises in the context of the design of information systems organized in large-scale, complex and costly networks. The risk of failure in these networks must be

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controlled as well as possible by the network managers in order to guarantee the best service to the users. As a consequence, the development of survivable networks becomes a crucial field of research and investigation. A survivable network is often defined as a network in which the various demands can be routed without loss of service quality even in case of failure (link or node failure). In this paper, we impose that K different paths exist for each origin-destination pair. These K paths must be edge-disjoint to ensure it is always possible to reroute the demand if $K-1$ edges fail.

In general, the survivability constraints alone may not be sufficient to guarantee a cost effective routing with a good quality of service. The reason for this is that the routing paths may be too long, leading to unacceptable delays. Since in most of the routing technologies, delay is caused at the switching nodes, it is usual to measure the delay in a path in terms of its number of intermediate nodes, or equivalently, its number of arcs (or hops). Thus, to guarantee the required quality of service, we impose a limit on the number of arcs of the routing paths.

For the case $K = 1$, Dahl and Gouveia [7] gave a complete description of the polytope when $L \leq 3$. In [6], Dahl et al. provide extended exact formulations for the case when $L = 4$ and give some evidence that finding a complete description in the space of the design variables for this case might be hard to find.

Many other papers (e.g. [1,8,10]) consider the relaxation of the problem where a minimum cost subgraph containing K edge-disjoint path of maximum L edges is searched for. This relaxation has the same set of optimal solutions as the problem studied here when the costs are non-negative.

Itaï et al. [11] studied the complexity of finding a maximal number of disjoint paths with length constraints and showed that the edge-disjoint problem is polynomial for $L \leq 3$, open for $L \leq 4$ and NP-hard for $L \geq 5$. Later, Bley [2] showed that the single demand node-disjoint length-restricted paths is NP-hard for $L \geq 5$. Recently, Bley and Neto [3] showed that the edge-disjoint problem is APX-complete for $L = 4$, closing the until then unknown complexity cases.

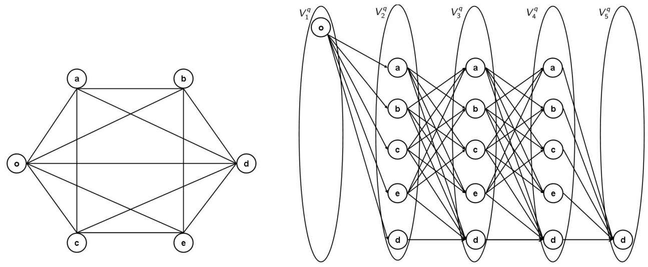
This paper focuses on the hop-indexed extended formulations that that has been used in the more general problem discussed in Botton et al. [5]. We show that the proposed formulation provides a complete description of the associated polyhedron for $L = 3$ and $K = 2$, after adding two new classes of valid inequalities. For the closely related arc-disjoint version of the problem, results similar to those in Section 3 are presented in [4].

This paper is organized as follows: In Section 2, we describe the formulation for the general case of the problem. In Section 3 we describe properties of this formulation when $L = 3$. New valid inequalities for the case $K = 2$ or $L = 3$ are introduced in Section 4. In Section 5, we relate the problem with the size-2 stable set problem to show that, with the new inequalities, we obtain a full description of the associated polyhedron.

2 Formulation

Our model is based on a hop-indexed extended formulation arising from a representation of the problem in a layered graph (see, e.g., Gouveia [9] and Botton et al. [5]).

From the original undirected graph $G = (V, E)$, we create a directed layered graph $G' = (V', A')$ as illustrated in Fig. 1, where $V' = V'_1 \cup \ldots \cup V'_{L+1}$ with $V'_1 = \{o\}$, $V'_{L+1} = \{d\}$ and $V'_l = V \setminus \{o\}$, $l = 2, \ldots, L$. Let v_l be the copy of $v \in V$ in the *l-th* layer of graph G. Then, the arc sets are defined by $A' = \{(i_l, j_{l+1}) \mid ij \in E, i_l \in V'_l, j_{l+1} \in V'_{l+1}, l \in \{1, \ldots, L\}\} \cup \{d^l, d^{l+1} | l \in \{2, \ldots, L\}\}$, see Fig. 1.



(a) (b)

Fig. 1 A basic network (a) and its alternative layered representation (b) when $L = 4$.

Note that each path between o and d in the layered graph G is composed of exactly L arcs (hops), which correspond to a maximum of L edges (hops) in the original graph (the horizontal-loop arcs at the bottom of Figure 1 allow paths with less than L arcs). This transformation was first proposed by Gouveia [9] and is motivated by the fact that in the layered graph any path is feasible with respect to the hop-constraints. A set of K-paths satisfying the hop limit can be obtained by sending K units of flow in the layered graph from node o to node d.

Paths are represented by directed flow variables x^l_{ij} indicating if edge $\{i, j\}$ is used from i to j at position l in one of the K paths from o to d ($l \in \{1, \ldots, L\}$, $\{i, j\} \in E$), or equivalently, if arc $(i_l, j_{l+1}) \in A'$ is used in the layered graph. To represent paths with less than L edges, we also introduce variables x^l_{dd} that are set to one when l is larger than the actual length of the path, and we define $E' := E \cup \{d, d\}$.

With these variables, the K edge-disjoint L-paths problem can be formulated as (HOP):

$$\min \sum_{\{i,j\}\in E} \sum_{1\le l\le L} (c_{ij} x^l_{ij} + c_{ji} x^l_{ji}) \tag{1}$$

$$\text{s.t.} \sum_{j:\{o,j\}\in E} x^1_{oj} = K, \tag{2}$$

$$\sum_{j:\{i,j\}\in E'} \left(x^{l-1}_{ji} - x^l_{ij}\right) = 0 \qquad i \in V \setminus \{o\}, 2 \le l \le L, \tag{3}$$

$$\sum_{2\le l\le L-1} (x^l_{ij} + x^l_{ji}) \le 1, \qquad \{i, j\} \in E,\ i, j \neq d \tag{4}$$

$$\sum_{2\le l\le L} x^l_{id} \le 1, \qquad \{i, d\} \in E. \tag{5}$$

$$x^l_{ij}, x^l_{ji} \in \{0, 1\}, \qquad \{i, j\} \in E,\ 1 \le l \le L, \tag{6}$$

$$x^l_{dd} \ge 0 \text{ and integer}, \qquad 2 \le l \le L. \tag{7}$$

Constraints (2) and (3) are traditional flow conservation constraints, while (4) and (5) impose edge-disjointedness, and (6) and (7) are integrality constraints. Since in position 1, the only possible flows are coming from o, and in position L, flows are ending in d, in the remainder of the paper, we implicitly assume that variables x^1_{ij} for $i \neq o$ and x^L_{ij} for $j \neq d$ do not exist in the model. Similarly, variables x^l_{io} and x^l_{dj} can be removed for all $i \neq o$, $j \neq d$, $1 \le l \le L$.

3 Properties of the formulation for L=3

In this section, we consider the particular case $L = 3$. We first simplify the formulation, then we show that the linear relaxation is integral if costs are positive. Unfortunately, this result is not sufficient to have a complete description of the associated polyhedron. In the next section, we describe valid inequalities that, when added to the formulation, provide a complete description of this polyhedron when $K = 2$.

We first observe that in the particular case of $L = 3$, the problem has more structure as the arcs chosen in position 2 determine completely the solution. Indeed, if we go from i to j in position 2, we must use $\{o, i\}$ in position 1 and $\{j, d\}$ in position 3.

This suggests that variables x^1_{oi} and x^3_{jd} can be eliminated from formulation (HOP). We can do this by using flow conservation constraints:

$$x^1_{oi} = \textstyle\sum_{j:\{i,j\}\in E} x^2_{ij} \qquad i \in V \setminus \{o\},$$
$$x^3_{jd} = \textstyle\sum_{i:\{i,j\}\in E} x^2_{ij} \qquad j \in V \setminus \{d\},$$
$$x^3_{dd} = \textstyle\sum_{i:\{i,d\}\in E} x^2_{id} + x^2_{dd}.$$

Then, every occurrence of the variables x^1_{oi}, x^3_{jd} and x^3_{dd} can be eliminated from the model by using the corresponding righthand side.

Observe also that (4) reduces to:

$$x^2_{ij} + x^2_{ji} \leq 1, \qquad \{i, j\} \in E,$$

and (5) reduces to:

$$x^2_{id} + x^3_{id} \leq 1, \qquad \{i, d\} \in E.$$

or equivalently, removing x^3_{id} using flow conservation constraints,

$$x^2_{id} + \sum_{j:\{i,j\}\in E} x^2_{ji} \leq 1, \qquad \{i, d\} \in E.$$

Finally, $x^1_{oi} \leq 1$ leads to

$$\sum_{j:\{i,j\}\in E} x^2_{ij} \leq 1, \qquad i \in V \setminus \{o, d\}.$$

Since we are left only with variables at position 2, we ommit the position in the remainder of the paper by defining $x_{ij} := x^2_{ij}$, leading to model (HOP3) for $L = 3$:

$$\min \quad c'_{dd} x_{dd} + \sum_{ij\in E} (c'_{ij} x_{ij} + c'_{ji} x_{ji}) \tag{8}$$

$$\text{s.t.} \quad x_{dd} + \sum_{j:\{i,j\}\in E} \left(x_{ij} + x_{ji}\right) = K, \tag{9}$$

$$x_{ij} + x_{ji} \leq 1, \qquad \{i, j\} \in E,\ i, j \neq o, d \tag{10}$$

$$x_{id} + \sum_{j:\{i,j\}\in E} x_{ji} \leq 1, \qquad \{i, d\} \in E, \tag{11}$$

$$\sum_{j:\{i,j\}\in E} x_{ij} \leq 1 \qquad i \in V \setminus \{o, d\}, \tag{12}$$

$$x_{ij}, x_{ji} \in \{0, 1\}, \qquad \{i, j\} \in E, \tag{13}$$

$$x_{dd} \in \{0, 1\}. \tag{14}$$

where $c'_{ij} = c_{oi} + c_{ij} + c_{jd}$ if $i, j \neq d$, $c'_{id} = c_{oi} + c_{id}$, and $c'_{dd} = c_{od}$. Variable x_{dd} indicates if edge {o,d} is in the solution.

Note that this formulation can also be interpreted as an enumeration of all paths of length 3, since each path is uniquely determined by the second arc, and so the same formulation could be obtained by a specialization of a Dantzig-Wolfe path-based reformulation of the problem.

Another interpretation of the formulation is that we want to select K arcs in a directed version of the graph, $G' = (V, A)$, where $A = \{(i, j), (j, i) : \{i, j\} \in E\} \cup \{(d, d)\}$. Each variable in (HOP3) corresponds to an arc in A.

Proposition 1 *The linear relaxation of model (HOP3) has an optimal integral solution if $c_{ij} \geq 0$ for all $\{i, j\} \in E$, for any value of K.*

Proof The key observation to prove the result is that, when $c_{ij} \geq 0$, there exists an optimal solution to the linear relaxation of (HOP3) such that $x_{ij} = 0$ or $x_{ji} = 0$ for all $\{i, j\} \in E$, $i, j \notin \{o, d\}$. Indeed, consider an optimal solution x^*, of cost z, such that $x^*_{ij} > 0$ and $x^*_{ji} > 0$. Let $\epsilon = \min\{x^*_{ij}, x^*_{ji}\}$, and consider the solution x' identical to x^* except for $x'_{ij} = x^*_{ij} - \epsilon$, $x'_{ji} = x^*_{ij} - \epsilon$, $x'_{id} = x^*_{id} + \epsilon$ and $x'_{jd} = x^*_{jd} + \epsilon$. In other terms, we move ϵ units of flow from paths $o - i - j - d$ and $o - j - i - d$ to paths $o - i - d$ and $o - j - d$. Solution x' is feasible, and such that $x'_{ij} = 0$ or $x'_{ji} = 0$. Moreover, the cost of x' is $z - \epsilon(c_{ij} + c_{ji}) \leq z$, hence it must be optimal too.

From this observation, we can conclude that constraints (10) are redundant and can be removed from the formulation. The result then follows from the fact that the polyhedron defined by the linear relaxation of (9), (11) − (14) is integral.

Indeed, let us consider an undirected bipartite graph $\overline{G} = (V_1 \cup V_2, \overline{E})$, where V_1 contains two copies of each vertex in $V \setminus \{o\}$, and V_2 contains a copy of each vertex in $V \setminus \{o\}$. For each vertex $i \in V \setminus \{o\}$, we denote i_1 and i_d its two copies in V_1, and i_2 its copy in V_2. The set of edges is defined as follows:

$$\begin{aligned} \overline{E} = &\{\{i_1, j_2\} : \{i, j\} \in E, i, j \notin \{o, d\}\} \\ &\cup \{\{i_d, i_2\} : \{i, d\} \in E, i \neq o\} \cup \{\{d_1, d_2\}\}. \end{aligned}$$

If we associate variable x_{ij} to edge $\{i_1, j_2\}$ for each $\{i, j\} \in E$, $i, j \notin \{o, d\}$, variable x_{id} to edge $\{i_d, i_2\}$ for each $\{i, d\} \in E$, $i \neq o$, and variable x_{dd} to edge $\{d_1, d_2\}$, then it is easy to see that the linear relaxation of (9), (11) − (14) corresponds to the polyhedron of incidence vectors of matchings of cardinality K in the bipartite graph $\overline{G}$, and is therefore integral. □

Proposition 1 is valid for any value of K and is closely related to the work of Bendali et al. [1]. They consider the problem of finding a subgraph that *contains* K edge-disjoint paths of length less than or equal to $L = 3$ between two nodes o and d, and give a complete description of the polyhedron in the space of natural variables. The problem studied by Bendali et al. is a relaxation of the problem studied in this paper, but the optimal solutions of the two problems coincide when costs are non-negative. It was shown in [5] that the projection of (HOP) on the space of natural variables implies all the inequalities presented in [1]. Combining this result with Proposition 1 gives an alternative proof of the main result presented in [1].

4 Valid inequalities for K=2 and L=3

Even when $K = 2$, the linear programming relaxation of (HOP3) has fractional extreme points. Figure 2 represents one such point with a solution composed of the paths $o - i - j - d$,

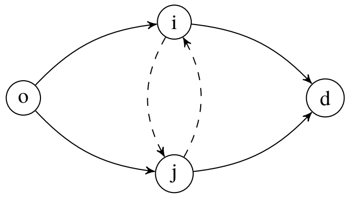


Fig. 2 A fractional extreme point for (HOP3)

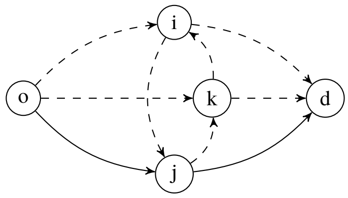


Fig. 3 A fractional extreme point for (HOP3)+(15)

$o-j-i-d$, $o-i-d$, $o-j-d$ on which a flow of 0.5 is sent. The corresponding fractional solution is $x_{ij} = x_{ji} = x_{id} = x_{jd} = 0.5$

This fractional point can be easily cut off by the following set of inequalities that result from strenghtening inequality (10).

Proposition 2 *Inequalities*

$$x_{ij} + x_{ji} + x_{id} \leq 1 \qquad \{i,j\} \in E,\ i,j \notin \{o,d\}, \tag{15}$$

are valid for $(HOP3)$, *for any value of* K.

Proof We already know that in any feasible solution, $x_{ij} + x_{ji} \leq 1$ holds. Now assume that $x_{ij} = x_{id} = 1$. This solution corresponds to paths $o-i-j-d$ and $o-i-d$, which are not edge-disjoint ($\{o,i\}$ is used twice). Similarly, $x_{ji} = x_{id} = 1$ corresponds to paths $o-j-i-d$ and $o-i-d$, and $\{i,d\}$ is used twice. The inequality is therefore valid. □

Observe that for $K = 2$ and $L \geq 4$, paths $o-j-i-k-d$ (for an arbitrary node $k \neq o,i,j,d$ and $o-i-d$) form a feasible solution that violates (15), hence these inequalities are only valid when $L = 3$.

These new inequalities are not sufficient to eliminate all fractional extreme points. Consider the point depicted in Figure 3. In this solution, a flow of 0.5 is sent on paths $o-i-j-d$, $o-j-d$, $o-j-k-d$ and $o-k-i-d$. It is valid for the linear programming relaxation of (HOP3) augmented with (15).

To cut off this fractional point, we introduce a new class of inequalities for the original (HOP) model, that are valid for any L but $K = 2$ only.

Proposition 3 *Inequalities*

$$x^2_{ji} + x^2_{jd} + x^1_{oi} + x^3_{jd} \leq 2, \quad \{i,j\} \in E,\ i,j \notin \{o,d\}, \tag{16}$$

are valid for HOP *for* $K = 2$ *and any value of* L.

Proof Since a solution is composed of edge-disjoint paths, x^2_{jd} and x^3_{jd} cannot be both equal to 1. Similarly, if $x^2_{ji} = x^2_{jd} = 1$, then $x^1_{oj} = 2$ by flow conservation, which is infeasible. It follows immediately if $x^2_{jd} = 1$, $x^2_{ji} = x^3_{jd} = 0$ and the inequality is satisfied. If $x^2_{jd} = 0$, a solution violating the inequality has $x^1_{oi} = x^3_{jd} = x^2_{ji} = 1$. Obviously, $o - i$ and $j - i$ must belong to different paths, and so do $j - i$ and $j - d$. As a solution is made of exactly two paths ($K = 2$), it follows that $o - i$ and $j - d$ must be on the same path. Since $o - i$ is in position 1 and $j - d$ in position 3, the only possibility is to have $i - j$ in position 2, which would lead to use edge $\{i, j\}$ twice, a contradiction. □

For $K \geq 3$ and $L = 3$, a feasible solution, which clearly violates (16), can be composed by the three paths $o - i - k - d$, $o - j - i - d$ and $o - l - j - d$, with i, j, k and l four arbitrary nodes distinct from o and d. Therefore, inequalities (16) are only valid for $K = 2$.

5 The 2 edge-disjoint 3-paths polyhedron

In this section, we show that the linear relaxation of (HOP3), with the addition of the valid inequalities from the previous section, completely describes the polyhedron of incidence vectors of 2 edge-disjoint 3-paths. One consequence of this result is an implicit description of the polyhedron in the space of natural variables of [1] (that can be obtained by projection). To the best of our knowledge, only the description of the polyhedron associated to the relaxation of the problem for a subgraph containing 2 edge-disjoint 3-paths was known from the literature.

We first summarize the formulation obtained by adding (15) and (16) to ($HOP3$) in the next subsection, then show in Subsection 5.2 that the problem of finding K edge-disjoint 3-paths can be reduced to finding a minimum cost stable set of cardinality K in a particular conflict graph. Subsection 5.3 gives a complete description of the convex hull of incidence vectors of stable sets, for the particular case of sets of cardinality 2. Finally, in Subsection 5.4, we apply the results for the stable set to the 2 edge-disjoint 3-paths problem.

5.1 Formulation

As we did in Section 3, when $L = 3$, we can rewrite (16) by eliminating x^1_{oi} and x^3_{jd}, leading to

$$x_{ji} + x_{jd} + \sum_{k:\{i,k\}\in E} x_{ik} + \sum_{k:\{k,j\}\in E} x_{kj} \leq 2.$$

Variable x_{ij} appears in both sums, so by isolating it we obtain

$$2x_{ij} + x_{ji} + x_{jd} + \sum_{k:\{i,k\}\in E, k\neq j} x_{ik} + \sum_{k:\{k,j\}\in E, k\neq i} x_{kj} \leq 2. \tag{17}$$

Adding (17) to (HOP3) and replacing (10) by the stronger (15) leads to formulation (2HOP3) below for the 2 edge-disjoint 3-paths problem:

$$\min \quad c'_{dd}x_{dd} + \sum_{ij\in E}(c'_{ij}x_{ij} + c'_{ji}x_{ji}) \tag{18}$$

$$\text{s.t.} \quad x_{dd} + \sum_{j:\{i,j\}\in E}\left(x_{ij} + x_{ji}\right) = 2, \tag{19}$$

$$x_{ij} + x_{ji} + x_{id} \leq 1, \qquad \{i,j\}\in E, i,j \notin \{o,d\} \tag{20}$$

$$x_{id} + \sum_{j:\{i,j\}\in E} x_{ji} \leq 1, \qquad \{i,d\}\in E, \tag{21}$$

$$\sum_{j:\{i,j\}\in E} x_{ij} \leq 1 \qquad i\in V\setminus\{o,d\}, \tag{22}$$

$$2x_{ij} + x_{ji} + x_{jd} + \sum_{k:\{i,k\}\in E, k\neq j} x_{ik} + \sum_{k:\{k,j\}\in E, k\neq i} x_{kj} \leq 2, \qquad \{i,j\}\in E, i,j\notin\{o,d\}, \tag{23}$$

$$x_{ij}, x_{ji} \in \{0,1\}, \qquad \{i,j\}\in E, \tag{24}$$

$$x_{dd} \in \{0,1\}. \tag{25}$$

5.2 Reduction of the 2 edge-disjoint 3-paths problem to a size-2 stable set problem

From the results of Section 3, we know that the problem of finding K edge-disjoint 3-paths can be reduced to the selection of K arcs in the directed graph $G' = (V, A)$ where $A = \{(i,j),(j,i) : \{i,j\}\in E\} \cup \{(d,d)\}$. These arcs must be such that the paths obtained when using them in position 2 are edge-disjoint. This leads us to the definition of compatible arcs.

Definition 1 (Compatible arcs) Let $G' = (V, A)$ where $A = \{(i,j),(j,i) : \{i,j\}\in E\}\cup\{(d,d)\}$. Two arcs (i,j) and $(k,l)\in A$ are *compatible* if and only one of the following conditions holds:

1. $(i,j) = (d,d)$;
2. $i \neq d$, $j = d$, $k \neq i$ and $l \neq i$;
3. $i,j \neq d$, $i \neq k$, $j \neq l$ and $(i,j) \neq (l,k)$.

Lemma 1 *The set of feasible solutions to (HOP3) coincide with incidence vectors of subsets of K pairwise compatible arcs.*

Proof Consider a feasible solution x to (HOP3). By (9), x is the incidence vector of a subset of K arcs. Consider two arcs (i,j) and (k,l) in this subset. We show that (i,j) and (k,l) are compatible. First note that (12) implies that $k \neq i$.

If $(i,j) = (d,d)$, the result holds by the first condition of the definition of compatible arcs. If $i \neq d$ and $j = d$, $x_{id} = 1$ and (11) imply that $l \neq i$, and the second condition is satified. Finally, if $i,j \neq d$, (10) implies that $(i,j) \neq (l,k)$ and (11) enforces $l \neq j$. Therefore the third condition is satisfied.

Conversely, let x be the incidence vector of a subset of K pairwise compatible arcs. It is easy to see that the definition of compatible arcs imply that x is a feasible solution of (HOP3). □

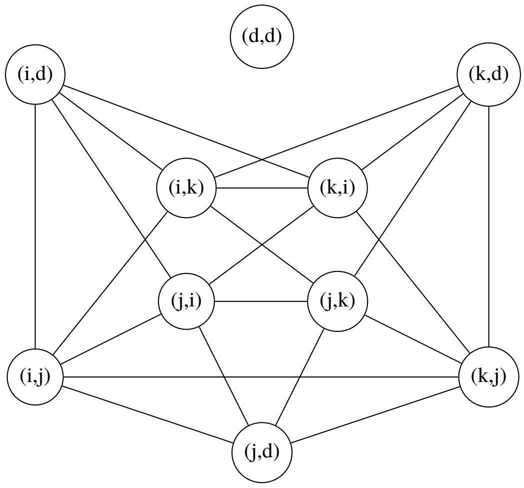


Fig. 4 Conflict graph G^c corresponding to a complete graph $V = \{o, d, i, j, k\}$

Finding K compatible arcs in G' is the same as looking for a size-K stable set in $G^c = (A, E^c)$, an auxiliary conflict graph. The set of nodes in G^c is the set A of arcs in G', and there is an edge in E^c for each incompatible pair of arcs in A. The structure of G^c is illustrated in Figure 4 for a complete graph G on 5 nodes. First, (d, d) is isolated as the corresponding path (the direct $o - d$ link) is edge-disjoint with any other path. Then, for any $i \neq o, d$, (i, d) (corresponding to the path $o - i - d$) is incompatible with (i, j) and (j, i) for all $j : \{i, j\} \in E$, $j \neq d$. Finally, for $j \neq d$, (i, j) (corresponding to $o - i - j - d$) is incompatible with (j, i), (j, d), (i, k) for all $k \neq j$, $(i, k) \in A$ and (k, j) for all $k \neq i$, $(k, j) \in A$.

In the particular case of $K = 2$, we can use the result of the next subsection to show that $(2HOP3)$ completely describes the polyhedron of the 2 edge-disjoint 3-paths problem. This is shown in Subsection 5.4.

5.3 The polyhedron of size-2 stable sets

The above reduction is useful because a full description of the convex hull of incidence vectors of stable sets of cardinality 2 is known, and can then be translated to the 2 edge-disjoint 3-paths problem. The results of this subsection are direct consequences of results by Janssen and Kilakos in [12]. We provide an alternative and shorter proof below.

Consider an undirected graph $G = (V, E)$, and let y be the incidence vector of a stable set of cardinality 2 in G. Given a clique P in G, we define $Q(P)$ as the set of nodes outside P, adjacent to all nodes in P, i.e. $Q(P) = \{i \in V \setminus P : \forall j \in P, ij \in E\}$.

Then y satisfies the following system of inequalities (STAB-2):

$$\sum_{i \in V} y_i = 2, \qquad (26)$$

$$2\sum_{i \in P} y_i + \sum_{i \in Q(P)} y_i \leq 2, \qquad P \in C, \qquad (27)$$

$$y_i \geq 0, \qquad i \in V, \qquad (28)$$

where C denotes the set of all cliques in G.

Constraint (26) ensures the stable set has cardinality 2, while constraints (27) are strengthened clique inequalities. Note that if P is a maximal clique, then $Q(P)$ is empty and the inequality is the classical clique inequality for the stable set problem.

Theorem 1 (Janssen and Kilakos [12]) *STAB-2 gives a complete description of the convex hull of incidence vectors of size-2 stable sets.*

Proof It is easy to see that the integer solutions of *STAB-2* correspond to the incidence vectors of stable sets of cardinality 2. It remains to show that all extreme points of the polyhedron are integer. To do this, we use the method introduced by Lovász (see section 9.2.2. in [13]). If, for any objective vector $w \in \mathbb{R}^{|V|}$, the set of size-2 stable sets maximizing $\sum_{i \in V} w_i y_i$ is included in the face defined by an inequality of *STAB-2*, then *STAB-2* is the convex hull of incidence vectors of size-2 stable sets.

Let w be an objective vector and z^* the objective value of an optimal size-2 stable set w.r.t. w.

1. First, assume that there exists $j \in V$ such that $y_j = 0$ for all optimal solutions, then the constraint $y_j \geq 0$ is active for all optimal solutions.
2. Otherwise for all $i \in V$, there exists an optimal solution such that $y_i = 1$. Then there exists $j(i) \in V$ such that $(w_i + w_{j(i)}) = z^*$ and $ij(i) \notin E$. Let $w_{\max} = \max_{i \in V} w_i$, and $P = \{i \in V : w_i = w_{\max}\}$ the set of nodes of maximal weight.
   Let $\overline{w} = (z^* - w_{max})$. If $\overline{w} = w_{max}$, i.e. $z^* = 2w_{max}$, it follows from the fact that each node belongs to an optimal solution that $w_i = w_{max}$ for every node $i \in V$. But then all feasible solutions are optimal, i.e. the optimal face coincides with the polyhedron.
   Thus we can assume from now on that $\overline{w} < w_{max}$. It follows that P is a clique. Indeed, if it is not the case an optimal solution can be built by taking two vertices from P leading to a total weight $2w_{max} > z^*$, a contradiction since z^* is the value of the optimal solution.
   (a) Consider an arbitrary optimal solution composed of vertices i and j such that $i \in P$. Then $j \notin P \cup Q(P)$ as i and j are not adjacent. Since $i \in P$ and $j \notin P \cup Q(P)$, inequality (27) is satisfied at equality.
   (b) The last case to consider is an optimal solution composed of two vertices k and l such that $k, l \notin P$. If $k \notin Q(P)$ then there exists $i \in P$ such that $ik \notin E$ and $\{i, k\}$ is a feasible solution. The total weight of this solution is $w_k + w_{\max} > w_k + w_l = z^*$ which leads to a contradiction. Thus $k \in Q(P)$ and by symmetry, $l \in Q(P)$ and inequality (27) is satisfied at equality.

□

Note that some inequalities in (STAB-2) might be dominated by other inequalities in the system. The lemma below characterizes some dominated inequalities. This characterization is useful in the proof of Theorem 2.

Lemma 2 *Let P be a non-maximal clique in G. If $P' := P \cup Q(P)$ is a clique, then this clique is maximal and the inequality (27) induced by P is dominated by the inequality (27) induced by P'*

Proof If P' is not maximal, there exists a vertex $v \in V \setminus P'$ adjacent to all nodes in P', and thus to all nodes in P. But then by definition of $Q(P)$, $v \in Q(P) \subset P'$, wich leads to a contradiction.

Hence, P' is a maximal clique and $Q(P') = \emptyset$. Inequality (27) induced by P' can thus be written as

$$2 \sum_{i \in P'} y_i \leq 2$$

or equivalently,

$$2 \sum_{i \in P} y_i + 2 \sum_{i \in Q(P)} y_i \leq 2$$

which clearly dominates inequality (27) induced by P. □

5.4 The polyhedron of the 2 edge-disjoint 3-paths problem

By applying Theorem 1 to the conflict graph G^c defined in Subsection 5.2, we can now prove the major result of this paper.

Theorem 2 *The linear relaxation of (2HOP3) gives a complete description of the 2 edge-disjoint 3-paths polyhedron.*

Proof Using Theorem 1 and showing that constraints (26)-(27) correspond to constraints (19) to (23) for the conflict graph G^c, implies that the linear relaxation of (2HOP3) provides a complete description of the polyhedron of stable sets of size 2 in G^c.

Obviously, constraint (26) translates to (19). It then remains to show that inequalities (20) to (23) are of the form (27) for some cliques in G^c, and that all the other cliques in G^c lead to dominated inequalities.

Since (d,d) is isolated in G^c, the only clique containing (d,d) is the singleton $P = \{(d,d)\}$ and $Q(P) = \emptyset$ leading to the trivial inequality $x_{dd} \leq 1$.

Let us now consider cliques that contain (i,d) for some $i \neq d$. Since (i,d) is adjacent to (i,j) and (j,i) for all $j : \{i,j\} \in E$, $j \neq d$, the singleton $P = \{(i,d)\}$ is such that $Q(P) = \{(i,j),(j,i) : \{i,j\} \in E\}$, leading to the inequality

$$2x_{id} + \sum_{j:\{i,j\} \in E, j \neq d} x_{ij} + \sum_{j:\{i,j\} \in E, j \neq d} x_{ji} \leq 2$$

which is dominated as it is the sum of (21) and (22).

Any other clique contains at least one node (i,j) of the conflict graph with $i,j \notin \{o,d\}$. We consider all possible cases.

- If the clique is the singleton $P = \{(i,j)\}$, then $Q(P)$ contains the arcs incompatible with (i,j), i.e. (j,i), (j,d), (i,k) for all $k \neq j$, $(i,k) \in A$ and (k,j) for all $k \neq i$, $(k,j) \in A$, leading to (23).
- If the clique P contains both (i,j) and (j,i), then $Q(P)$ is a subset of $\{(i,d),(j,d)\}$ as these two arcs are the only ones that are incompatible with both (i,j) and (j,i). Since (i,d) and (j,d) are compatible, we are only left with three possible cliques:

  - $P = \{(i,j),(j,i),(i,d)\}$ leads to (20).
  - By symmetry, $P = \{(i,j),(j,i),(j,d)\}$ also leads to (20) with the role of i and j exchanged.
  - $P = \{(i,j),(j,i)\}$ leads to the sum of the two previous inequalities and is dominated.
- If P contains (i,j) and (j,d), then $P \cup Q(P) = P' := \{(i,j),(j,i),(j,d)\}$ is a maximal clique and, by Lemma 2, the only non-dominated inequality arises when $P = P'$ leading to (20).
- If P contains (i,j) and (i,k) for some $k \notin \{j\}$, and does not contain (j,i), then $P \cup Q(P) = P' := \{(i,l) : (i,l) \in A\}$. But it is easy to see that P' is a (maximal) clique and, by Lemma 2, the only non-dominated inequality arises when $P = P'$ leading to (22).
- If P contains (i,j) and (k,j) for some $k \neq i$, then $P \cup Q(P) = P' := \{(j,d)\} \cup \{(l,j) : (l,j) \in A\}$. But again P' is a (maximal) clique and, by Lemma 2, the only non-dominated inequality arises when $P = P'$ leading to (21).

In the argument above we have considered all possible cliques in G^c. We showed that clique inequalities (27) either correspond to inequalities in the system (20)-(23) or are redundant. It follows that the linear relaxation of (2HOP3) provides a complete description of the polyhedron of stable sets of size 2 in G^c. Consequently, the linear relaxation of (2HOP3) gives a complete description of the 2 edge-disjoint 3-paths polyhedron. □

6 Conclusion

In this paper, we present two new classes of valid inequalities for the classical extended formulation of the K edge-disjoint L-paths. One class is valid for any L and $K = 2$, and the other one is valid for any K and $L = 3$. We further show that when $K = 2$ and $L = 3$ simultaneously, these inequalities are sufficient to completely describe the convex hull of incidence vectors of feasible solutions to the problem.

The result is based on the fact that, when $L = 3$, the problem is equivalent to finding a stable set of size K in a conflict graph constructed from the original network. A known compact description of the convex hull of size-2 stable sets is then used to derive our main result.

An interesting line of research would be to use a similar approach to derive new classes of valid inequalities for $L = 3$ and larger values of K. Unfortunately, some experimental results with the porta software package lead us to conjecture that an exponential number of inequalities is necessary to describe the associated polyhedron, even with $K = 3$.

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