Reaching non-primary resonances via multiharmonic forcing

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Summary. Nonlinear systems exhibit a wide variety of resonances of primary or non-primary type. Among the latter, some can appear as isolated branches of solutions, making them difficult to obtain with traditional numerical and experimental approaches. In this work, we propose to leverage multiharmonic forcing to bridge different resonances associated with the same mode together. The method starts with a given harmonic forcing, and progressively increases the amplitude of a second harmonic component, all the while decreasing the amplitude of the first. It eventually reaches another harmonic forcing situation where only the second harmonic component subsists. By doing so, this multiharmonic transition method can continuously go from an easily attainable primary resonance to a (possibly isolated) non-primary resonance by transiting through a simultaneous resonance state. The method is illustrated numerically with a Duffing oscillator and a doubly-clamped Von Kármán beam.

Introduction

Resonances play a central role in vibration theory because they are associated with high-amplitude structural responses for a harmonically forced nonlinear system. Primary resonances are akin to their linear counterparts and can in general easily be located and characterized. However, nonlinear systems feature a much wider variety of resonances. Among them, non-primary resonances (of super-, sub- or ultrasubharmonic type) arise when the forcing frequency and the frequency of the resonant motion are close to be rationally related [1]. Depending on the nonlinearities in the system, these non-primary resonances may appear as isolated branches of solutions [2], making them difficult to reach with traditional numerical and experimental methods. Yet they are important to characterize as they can constitute dangerous high-amplitude attractors in the operational range of the system, or conversely, they can be exploited to optimize performance [3].
A similarity between primary and non-primary resonances is oftentimes assumed in analytical methods [1] and was recently made explicit for subharmonic resonances of weakly dissipative, weakly forced systems [4]. In this work, we leverage this similarity between primary and non-primary resonances to attain the latter starting from the former using multiharmonic forcing as a proxy.

Muti-harmonic transition between resonances

Duffing oscillator

We first illustrate the rationale behind the method with a Duffing oscillator and highlight the connection between its 1:1 primary, 3:1 superharmonic and 1:3 subharmonic resonances. We set the fundamental harmonic at ω to an amplitude of 0.25α and the one on the third harmonic to $0.25(1-\alpha)$, where $\alpha \in [0,1]$. The ordinary differential equation governing the behavior of the Duffing oscillator with this multiharmonic forcing is then

$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) = 0.25(\alpha \sin(\omega t) + (1-\alpha)\sin(3\omega t)). \tag{1}$$

Figure 1a depicts the nonlinear frequency response (NFR) of the oscillator computed with the harmonic balance method for $\alpha = 0$. For $\alpha = 1$, the exact same NFR curve is obtained, but is stretched by a factor 3 along the frequency axis. A prominent feature of this NFR is the primary resonance of the oscillator featuring large response amplitudes. Superharmonic resonances are also observable as small peaks at lower frequencies.
A question then arises: what happens to these resonances if one progressively increases (or decreases) α from 0 to 1 (or 1 to 0)? Figure 1b depicts such scenarii. Starting from $\alpha = 0$ to 1, the primary resonance at $\alpha = 0$ gradually shrinks in amplitude and transits as a simultaneous primary and superharmonic resonance, to finally become the 3:1 superharmonic resonance at $\alpha = 1$. Conversely, the primary resonance at $\alpha = 1$ eventually becomes the 1:3 subharmonic resonance at $\alpha = 0$. Interestingly, the multiharmonic forcing offers a continuous path toward this 1:3 subharmonic resonance, which is isolated from the main branch of the NFR when one considers a single-harmonic forcing at $\alpha = 0$. This suggests that a homotopy between two different single-harmonic forcing cases might allow us to connect isolated arbitrary resonances.

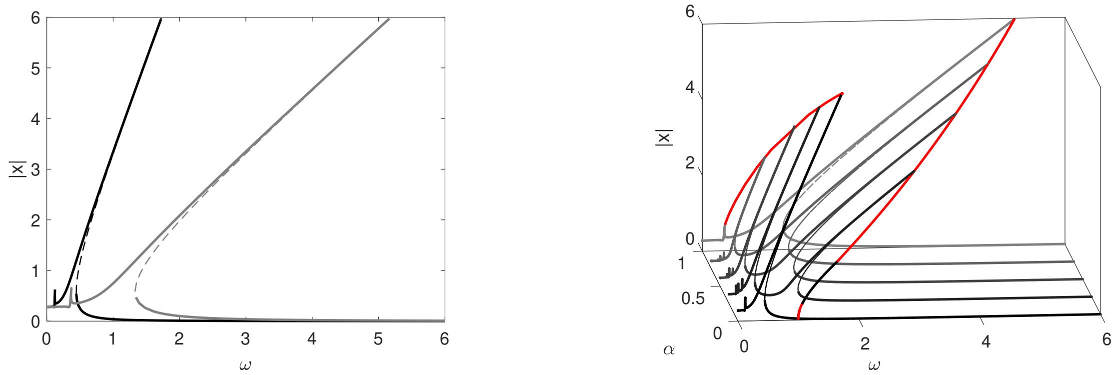
Doubly-clamped thin beam

We now move to the more complex case of a doubly-clamped Von Kármán beam and aim to attain a more exotic 3:5 ultrasubharmonic resonance. The nonlinear finite element model with 27 degrees of freedom is identical to that considered in [5]. The evolution of the vector of generalized degrees of freedom $\mathbf{x}$ is governed by the following equations

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{\text{nl}}(\mathbf{x}(t)) = \mathbf{f}(f_3 \sin(3\omega t) + f_5 \sin(5\omega t - 5\pi/3)), \tag{2}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are linear structural mass, damping and stiffness matrices, respectively, $\mathbf{f}_{\text{nl}}$ is the vector of nonlinear forces and $\mathbf{f}$ is the spatial distribution of external forces[1]. The primary resonance of the first mode is first obtained by

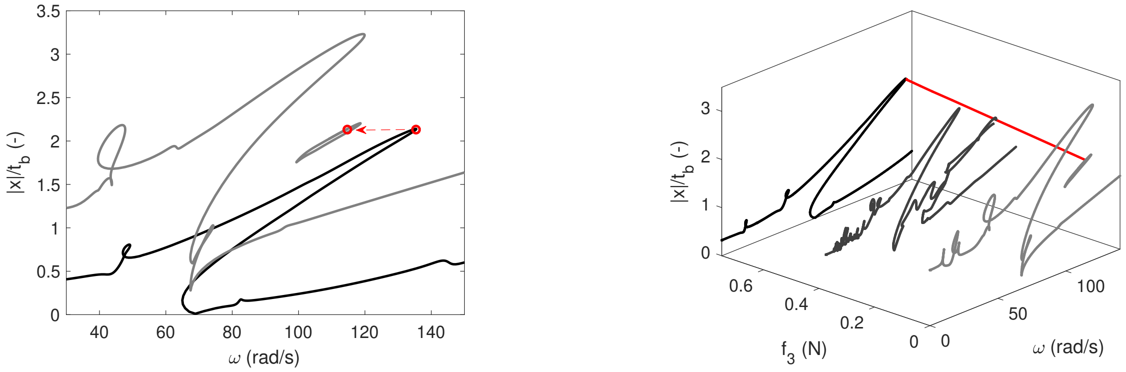
[1]The phase shift of the fifth harmonic is chosen to excite the same phase resonance mode as that excited by the third harmonic [2].



(a) (b)

Figure 1: NFR of the Duffing oscillator under harmonic forcing for $\alpha = 0$ (▬) and $\alpha = 1$ (▬) (a), and set of NFRs for different values of α (in different levels of gray) with phase resonance locus (▬) (b).

setting $f_3 = 0.75$ N and $f_5 = 0$. Instead of directly imposing f_3 and f_5 as in the previous example, we rather constrain the total amplitude of the forced degree of freedom to remain constant [5], which implicitly constrains f_3 and f_5. We also constrain the third harmonic at the forced degree of freedom to be in phase resonance [2]. As f_3 decreases, f_5 progressively increases to maintain the resonant motion. The 3:5 ultrasubharmonic resonance is attained when f_3 crosses zero. Figure 2a shows the two limits with pure tonal forcing (the primary resonance the method starts from, and the ultrasubharmonic resonance it reaches), while Figure 2b depicts the transition followed by the method.



(a) (b)

Figure 2: Driving-point NFRs (normalized with the thickness of the beam t_b) of the clamped-clamped beam at quarter span for $f_3 = 0.75$ N and $f_5 = 0$ (▬) and for $f_3 = 0$ and $f_5 = 5.1$ N (▬) with the transition from the 1:1 primary resonance to the 3:5 ultrasubharmonic resonnace displayed in red (a), and set of NFRs for different values of f_3 and f_5 (in different levels of gray) with phase resonance locus (▬) (b).

Conclusions

A method leveraging multiharmonic forcing was proposed to connect different (primary and non-primary) resonances associated with the same mode. An appealing feature of this simple procedure is its ability to reach isolated response curves from an easily obtainable primary resonance. This ad-hoc method is based on the hypothesis that the starting and target resonances are somewhat similar, but cannot predict in advance if this hypothesis is verified. It can also be realized experimentally rather simply, leveraging phase-locked loop controllers to maintain the excitation frequency in a state of phase resonance.

References

[1] Nayfeh A. H., Mook D. (1995) Nonlinear Oscillations. Wiley, NY.

[2] Volvert M., Kerschen G. (2021) Phase resonance nonlinear modes of mechanical systems. *J. Sound Vib.* **511**:116355.

[3] Huguet T., Badel A., Druet O., Lallart M. (2018) Drastic bandwidth enhancement of bistable energy harvesters: Study of subharmonic behaviors and their stability robustness *Appl. Energy* **226**:607–617.

[4] Cenedese M., Haller G. (2020) How do conservative backbone curves perturb into forced responses? A Melnikov function analysis. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2234**:20190494.

[5] Raze G., Volvert M., Kerschen G. (2023) Tracking amplitude extrema of nonlinear frequency responses using the harmonic balance method. *Int. J. Numer. Methods Eng.* e7376.