Functional spaces defined via Boyd functions

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joint work with T. Lamby, L. Loosveldt, T. Kleyntssens,...

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Road map

- The context
- Admissible sequences and Boyd functions
- Generalized Besov spaces
- Spaces of pointwise smoothness
- Multifractal analysis
- Applications

A locally bounded function f belongs to $\Lambda^\alpha(x_0)$ (with $\alpha \geq 0$ and $x_0 \in \mathbb{R}^n$) if there exist a constant C and a polynomial P_{x_0} of degree less than α such that

$$|f(x) - P_{x_0}(x)| < C|x - x_0|^\alpha,$$

in a neighborhood of x_0.

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The multifractal spectrum of f is defined as

$$D_\infty(h) = \dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_\infty(x_0) = h\}.$$

Under some general assumptions, there exist a function ϕ and $2^d - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^d}$, called wavelets, such that

$$\{\phi(x-k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}\}$$

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Any function $f \in L^2(\mathbb{R}^d)$ can be decomposed as follows,

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x-k) + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x) \psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbb{R}^d} f(x) \phi(x-k)\, dx.$$

Let $\lambda_{j,k}^{(i)}$ denote the dyadic cube $\frac{i}{2^{j+1}} + \frac{k}{2^j} + [0, \frac{1}{2^{j+1}})^d$ and set $c_\lambda = c_{\lambda_{j,k}^{(i)}} = c_{j,k}^{(i)}$.

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The notation Λ_j will stand for the set of dyadic cubes λ of $\mathbb{R}^d$ with side length 2^{-j}
and the unique dyadic cube from Λ_j containing the point $x_0 \in \mathbb{R}^d$ will be denoted $\lambda_j(x_0)$.

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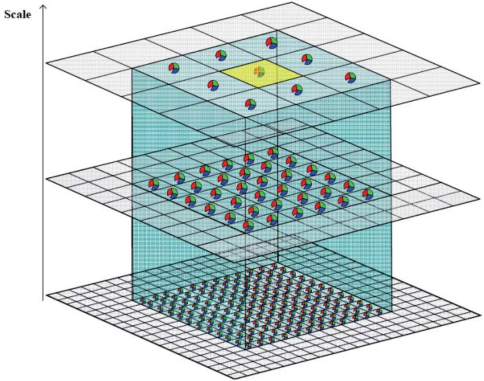


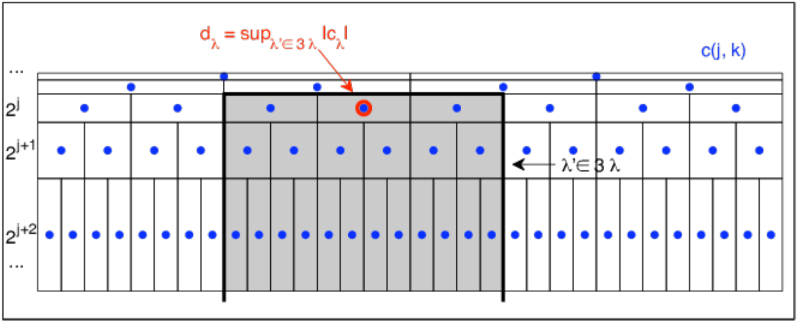
Given a dyadic cube $\lambda \in \Lambda_j$, the wavelet leader of λ is defined by

$$d_\lambda^\infty = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^\infty(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^\infty,$$

where 3λ denotes the set of the 3^d dyadic cubes adjacent to λ.


$d_\lambda = \sup_{\lambda' \in 3\lambda} |c_{\lambda'}|$
c(j, k)
...
2^j
2^{j+1}
2^{j+2}
...
$\lambda' \in 3\lambda$

Proposition (Jaffard, Meyer)

If f belongs to $\Lambda^h(x_0)$ then

$$(2^{jh} d_j^\infty(x_0))_j \in \ell^\infty \tag{1}$$

Conversely, if (1) is satisfied for a function $f \in B^\eta_{\infty,\infty}$ for some $\eta > 0$ then f belongs to $\Lambda^{h-\epsilon}(x_0)$ for any $\epsilon \in (0, h)$.

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Meyer gave an example of function $f \in B^\eta_{\infty,\infty}$ satisfying (1) such that $f \notin \Lambda^h(x_0)$.

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Nevertheless, the previous result gives a characterization of the Hölder exponent.

If we set

$$\eta(q) = \liminf_{j \to \infty} \frac{-1}{j} \log_2 (2^{-j} \sum_{\lambda \in \Lambda_j} (d_\lambda^\infty)^q),$$

a multifractal formalism is given by

$$D_\infty(h) = \inf_q \{d - \eta(q) + hq\}.$$

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The only result valid in the general case is the following inequality in $B^\eta_{\infty,\infty}$:

$$D_\infty(h) \leq \inf_q \{d - \eta(q) + hq\}.$$

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A sequence $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ of positive real numbers is called admissible if there exists a positive constant C such that

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$$\underline{\sigma}_j = \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j = \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k},$$

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$$\underline{s}(\sigma) = \lim_j \frac{\log_2 \underline{\sigma}_j}{j} \qquad \text{and} \qquad \overline{s}(\sigma) = \lim_j \frac{\log_2 \overline{\sigma}_j}{j}.$$

Given $\epsilon > 0$, there exists $C > 0$ s.t., for any j, k,

$$C^{-1}\, 2^{j(\underline{s}(\sigma)-\epsilon)} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C\, 2^{j(\overline{s}(\sigma)+\epsilon)}.$$

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If ψ satisfy

$$\lim_{s\to 0} \frac{\psi(st)}{\psi(s)} = 1$$

for any $t > 0$, then for $\sigma_j = 2^{js}\psi(2^j)$, we have $\underline{s}(\sigma) = \overline{s}(\sigma) = s$.

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For $j_0 = 0$, $j_1 = 1$, $j_{2n} = 2j_{2n-1} - j_{2x-2}$, $j_{2n+1} = 2^{j_{2n}}$, $\alpha > 0$,

$$\sigma_{j+1} = \begin{cases} \sigma_j & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ \sigma_j 2^\alpha & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases}$$

is such that $\underline{s}(\sigma) = 0$, $\overline{s}(\sigma) = 1$ and for all $\epsilon > 0$, $\sigma_j \leq C2^{j\epsilon}$ for some C.

A function $\phi : (0, \infty) \to (0, \infty)$ is a Boyd function if it is continuous, $\phi(1) = 1$ and

$$\overline{\phi}(t) = \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty.$$

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$$\underline{b}(\phi) = \lim_{t \to 0} \frac{\log \overline{\phi}(t)}{\log t} \qquad \text{and} \qquad \overline{b}(\phi) = \lim_{t \to \infty} \frac{\log \overline{\phi}(t)}{\log t}$$

It is well known that there is a connection between Boyd functions and admissible sequences. Many authors illustrate this link with the following example

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}) \\ \sigma_0 & \text{if } t \in (0, 1) \end{cases} .$$

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The Boyd indices are not preserved with this construction.Even with $\sigma_j = 2^{sj}$, we have $\underline{b}(\phi) < \underline{s}(\sigma) = s$.

Proposition (Lamby, N.)

For $\sigma_j = \phi(2^j)$ and $\gamma_j = 1/\phi(2^{-j})$, we have

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\gamma)\} \qquad \text{and} \qquad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\gamma)\}.$$

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An exemple of function preserving the Boyd indices:

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}) \\ 1/\phi(1/t) & \text{if } t \in (0,1) \end{cases} .$$

Proposition (Lamby, N.)

If σ is s.t. either $\underline{s}(\sigma) > 0$ or $\overline{s}(\sigma) < 0$, then there exists a C^∞ Boyd function ϕ such that

$$0 < \inf_{t>0} \frac{|\phi'(t)|}{\phi(t)} \leq \sup_{t>0} \frac{|\phi'(t)|}{\phi(t)} < \infty$$

and with the same indices as those of σ.

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Let γ be an admissible sequence s.t. $\underline{\gamma}_1 > 1$ and $\rho \in D(\mathbb{R})$ be a positive function s.t. $\rho(t) = 1$ for $|t| \leq 1$, ρ is decreasing for $t \geq 0$ and $\mathrm{supp}(\rho) \subset \{t \in \mathbb{R} : |t| \leq 2\}$. Given $J \in \mathbb{N}$, set

$$\varphi_j^{\gamma,J} = \rho(\gamma_j^{-1}|\cdot|) \quad \text{for } j \in \{0, \ldots, Jk_0 - 1\},$$

$$\varphi_j^{\gamma,J} = \rho(\gamma_j^{-1}|\cdot|) - \rho(\gamma_{j-Jk_0}^{-1}|\cdot|) \quad \text{for } j \geq Jk_0$$

and

$$\Delta_j^{\gamma,J} f = \mathcal{F}^{-1}(\varphi_j^{\gamma,J} \mathcal{F} f).$$

The generalized Besov space $B_{p,q}^{\gamma,\sigma}$ is defined by

$$B_{p,q}^{\sigma,\gamma} = \{f \in \mathcal{S}' : \|f\|_{B_{p,q}^{\gamma,\sigma}} = \left\|(\sigma_j \|\Delta_j^{\gamma,J} f\|_{L^p})_j\right\|_{\ell^q} < \infty\}.$$

Let $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x)$.

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Proposition (Moura)

Let $p, q \in [1, \infty]$, σ and γ be such that $\underline{\gamma}_1 > 1$ and $0 < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} < n$ $(n \in \mathbb{N})$, we have

$$B_{p,q}^{\sigma,\gamma} = \{f \in L^p : (\sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta_h^n f\|_{L^p})_j \in \ell^q\}.$$

For $\sigma_j = 2^{sj}$ and $\gamma_j = 2^j$, we have $B_{p,q}^{\sigma,\gamma} = B_{p,q}^s$.

Proposition (Loosveldt, N.)

Let $p, q \in [1, \infty]$, σ and γ be such that $\underline{\gamma}_1 > 1$ and

$$k < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \leq \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n.$$

If $f \in B^{\sigma,\gamma}_{p,q}$ then $f \in W^k_p$ and for all $|\alpha| \leq k$,

$$(\gamma_j^{-|\alpha|}\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} f\|_{L^p})_j \in \ell^q, \qquad (2)$$

which means that $D^\alpha f \in B^{\gamma^{-|\alpha|}\sigma,\gamma}_{p,q}$.
Conversely, if $f \in W^k_p$ satisfies (2) for $|\alpha| = k$, then $f \in B^{\sigma,\gamma}_{p,q}$.

Proposition (Loosveldt, N.)

Let $p, q \in [1, \infty]$, σ and γ be such that $\underline{\gamma}_1 > 1$ and

$$n < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \leq \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n + 1.$$

TFAE:

- $f \in B^{\sigma,\gamma}_{p,q}$;
- $f \in W^n_p$ and for all $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$, we have

$$f(x+h) = \sum_{|\alpha| \leq n} D^\alpha f(x) \frac{h^\alpha}{|\alpha|!} + R_n(x,h) \frac{|h|^n}{n!},$$

  where $(\sigma_j \gamma_j^{-n} \sup_{|h| \leq \gamma_j^{-1}} \|R_n(\cdot, h)\|_{L^p})_j \in \ell^q$;
- Given a net of $\mathbb{R}^d$ made of cubes of diagonal γ^{-1}, there exists g_{π_j} s.t.
  - the trace of g_{π_j} in each cube is a polynomial of degree $\leq n$,
  - one has $(\sigma_j \|f - g_{\pi_j}\|_{L^p})_j \in \ell^q$.

Given a function ϕ defined on $\mathbb{R}^d$ and $\epsilon \neq 0$, we set $\phi_\epsilon = |\epsilon|^{-d}\phi(\cdot/\epsilon)$.

Proposition (Loosveldt, N.)

Let $p, q \in [1, \infty]$, σ and γ be such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\sigma) > 0$. We have

$$B^{\sigma,\gamma}_{p,q} = \{f \in L^p : \exists \phi \in D \text{ s.t. } (\sigma_j \|f * \phi_{\gamma_j^{-1}} - f\|_{L^p})_j \in \ell^q\}.$$

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Proposition (Merucci/Loosveldt, N.)

Let $p, q \in [1, \infty]$, σ and γ be such that $\underline{\gamma}_1 > 1$ and

$$k < \underline{s}(\sigma) \leq \overline{s}(\gamma)^{-1} \leq \overline{s}(\sigma) \leq \underline{s}(\gamma)^{-1} < n.$$

We have

$$B^{\sigma,\gamma}_{p,q} = [W^k_p, W^n_p]^{\sigma,\gamma}_q.$$

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Let $p, q \in [1,\infty]$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^\sigma_{p,q}(x_0)$ whenever

$$(\sigma_j 2^{jd/p} \sup_{|h|\leq 2^{-j}} \|\Delta_j^{\lfloor \overline{s}(\sigma)\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))})_j \in \ell^q,$$

where $B_h(x_0,r) = \{x : [x, x + (\lfloor \overline{s}(\sigma)\rfloor + 1)h] \subset B(x_0,r)\}$.

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Proposition (Loosveldt, N.)

Let $p, q \in [1, \infty]$, $f \in L^p_{\text{loc}}$, $x_0 \in \mathbb{R}^d$ and σ be such that $0 \leq \lfloor \overline{s}(\sigma) \rfloor < \overline{s}(\sigma)$. f belongs to $T^\sigma_{p,q}(x_0)$ iff there exists a unique polynomial P_{x_0} of degree at most $\lfloor \overline{s}(\sigma) \rfloor$ such that

$$(\sigma_j 2^{jd/p} \|f - P_{x_0}\|_{L^p(B(x_0, 2^{-j}))})_j \in \ell^q.$$

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where $B_h(x_0, r) = \{x : [x, x + (\lfloor \overline{s}(\sigma) \rfloor + 1)h] \subset B(x_0, r)\}$.

For $p = q = \infty$ and $\sigma_j = 2^{jh}$, we have $T^{\sigma}_{p,q}(x_0) = \Lambda^h(x_0)$.

Given a dyadic cube $\lambda \in \Lambda_j$, the p-wavelet leader of λ ($p \in [1, \infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} 2^{(j-j')d/p} |c_{\lambda'}| \Big)^p \Big)^{1/p}.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p,$$

where 3λ denotes the set of the 3^d dyadic cubes adjacent to λ.

Let $p, q \in [1, \infty]$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$ and σ be s.t. $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞; f belongs to $T^{\sigma}_{p,q,\log}(x_0)$ whenever

$$\Big(\frac{2^{jd/p}\sigma_j}{\log_2(2^{-jd/p}\sigma_j^{-1})} \sup_{|h| \leq 2^{-j}} \|\Delta_j^{\lfloor \bar{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}\Big)_{j \geq J} \in \ell^q,$$

for some $J \in \mathbb{N}$.

Proposition (Loosveldt, N.)

If f belongs to $T^{\sigma}_{p,q}(x_0)$ then

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

Conversely, for $f \in L^p_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ and σ s.t. $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d/p}$, if f belongs to $\dot{X}^{\eta}_{p,q}(x_0)$ for some $\eta > 0$, then

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

implies $f \in T^{\sigma}_{p,q,\log}(x_0)$.

Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B + x) = 0$, for every $x \in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set is a prevalent set.

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- If E is finite-dimensional, B is Haar-null if and only if $\mathcal{L}(B)=0$;
- if E is infinite-dimensional, the compact sets of E are Haar-null;
- a translated of a Haar-null set is Haar-null;
- a prevalent set is dense in E;
- the intersection of a countable collection of prevalent sets is prevalent.

Proposition (Loosveldt, N.)

In many cases, the condition

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

implies $f \in T^\sigma_{p,q,\log}(x_0)$, but, from the prevalent point of view, almost every such function f does not belong to $T^\sigma_{p,q}(x_0)$.

Corollary (Loosveldt, N.)

From the prevalent point of view, a.e. $f \in B^{\eta}_{\infty,\infty}$ ($\eta > 0$) satisfying

$$(2^{jh} d_j^{\infty}(x_0))_j \in \ell^{\infty}$$

belongs to $\Lambda^h_{\log}(x_0) \setminus \Lambda^h(x_0)$, where

$$\Lambda^h_{\log}(x_0) = \{f \in L^{\infty}_{\mathrm{loc}} : |f(x) - P_{x_0}(x)| \leq C|x - x_0|^h \log|x - x_0|^{-1}$$
$$\text{in a nbh of } x_0 \text{ for some polynomial } P_{x_0} \text{ of degree} < h\}.$$

Corollary (Loosveldt, N.)

From the prevalent point of view, a.e. $f \in B^{\eta}_{\infty,\infty}$ ($\eta > 0$) satisfying

$$(2^{jh} d_j^{\infty}(x_0))_j \in \ell^{\infty}$$

belongs to $\Lambda^h_{\log}(x_0) \setminus \Lambda^h(x_0)$, where

$$\Lambda^h_{\log}(x_0) = \{f \in L^{\infty}_{\mathrm{loc}} : |f(x) - P_{x_0}(x)| \leq C|x - x_0|^h \log|x - x_0|^{-1}$$
$$\text{in a nbh of } x_0 \text{ for some polynomial } P_{x_0} \text{ of degree} < h\}.$$

Therefore, "the logarithmic correction is prevalent".

Let $T_p^\sigma = T_{p,\infty}^\sigma$. One can

- introduce the space t_p^σ and study the properties of these spaces (completeness, density, embedding,...);
- give a generalization of Whitney's extension theorem and study the Bessel operator;
- investigate the estimations that can be made if the derivatives belong to the spaces T_p^σ and t_p^σ;
- study the action of the convolution integral operator on T_p^σ and show how these spaces can be utilized to examine the regularity of the solutions of an elliptic partial differential equation.

Road map

- The context
- Admissible sequences and Boyd functions
- Generalized Besov spaces
- Spaces of pointwise smoothness
- Multifractal analysis
- Applications

Let $p \in [1, \infty]$; if, given $h > -d/p$, $\sigma^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \sigma^{(h)}$ is p-decreasing if it satisfies $\underline{s}(\sigma^{(h)}) > -d/p$, $\underline{\sigma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_p^{\sigma^{(h)}}(x_0) \subset T_p^{\sigma^{(h')}}(x_0)$$

We will always assume that $\sigma^{(\cdot)}$ is p-decreasing.

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$$T_p^{\sigma^{(h)}}(x_0) \subset T_p^{\sigma^{(h')}}(x_0)$$

We will always assume that $\sigma^{(\cdot)}$ is p-decreasing.

For example, we can take $\sigma_j^{(h)} = 2^{jh}\psi(2^j)$, with $\psi(t) = \sqrt{|\log|\log t^{-1}||}$.

For $\psi = 1$, we have $T_p^{\sigma^{(h)}}(x_0) = T_p^h(x_0)$.

Given $p \in [1, \infty]$ and a family of admissible sequences $\sigma^{(\cdot)}$, the generalized p-Hölder exponent associated to $f \in L^p_{\mathrm{loc}}$ and $\sigma^{(\cdot)}$ at x_0 is defined by

$$h_p(x_0) = \sup\{h > -d/p : f \in T_p^{\sigma^{(h)}}(x_0)\}.$$

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Given $p \in [1, \infty]$, a family of admissible sequences $\sigma^{(\cdot)}$ and a function $f \in L^p_{\text{loc}}$, we set

$$D_p(h) = \dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_p(x_0) = h\}.$$

Proposition (Loosveldt, N.)

Given $p \in [1, \infty]$, if $(\sigma^{(h)})_{h>-d/p}$ is a decreasing family of admissible sequences s.t. $h < h'$ implies $\sigma_j^{(h)} \in o(\sigma_j^{(h')})$ and $\overline{s}(\sigma^{(h)}) \leq \overline{s}(\sigma^{(h')})$, then, from the prevalence point of view, almost every function in $T_p^{\sigma^{(h)}}(x_0)$ is such that $h_p(x_0) = h$.

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Corollary (Loosveldt, N.)

Given $p \in [1, \infty]$ and $h > -d/p$, from the prevalence point of view, almost every function f in $T_p^h(x_0)$ satisfies $h_p(x_0) = h$.

Proposition (Loosveldt, N.)

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In particular, we have

Corollary (Hunt)

Given $h > 0$, from the prevalence point of view, almost every function f in $\Lambda^h(x_0) = T_\infty^h(x_0)$ satisfies $h_\infty(x_0) = h$.

An admissible sequence γ and a family of admissible sequences $\sigma^{(\cdot)}$ are compatible for $p, r, s \in [1, \infty]$ if

- $\underline{s}(\gamma) > 0$,
- $\underline{s}(\gamma) - d/r > -d/p$,
- the function ζ defined $(-d/p, \infty)$ by

$$\zeta(h) = \underline{s}(\frac{\sigma^{(h)}}{\gamma})$$

  is non-decreasing, continuous and such that $\{h > -d/p : \zeta(h) < -d/r\} \neq \emptyset$.

Proposition (Loosveldt, N.)

Let $p, r, s \in [1, \infty]$, γ be an admissible sequence and $\sigma^{(\cdot)}$ be a family of admissible sequences compatible with γ. From the prevalence point of view, for almost every $f \in B^{\gamma}_{r,s}$, the function D_p is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and

$$D_p(h) = d + r\zeta(h),$$

for any $h \in I$.

Road map

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The sample path $B = \{B_x\}_{x\in\mathbb{R}}$ of a Brownian does not belong to $\Lambda^{1/2}(\mathbb{R})$ and for a.e. $x_0 \in \mathbb{R}$,

$$|B_{x_0} - B_x| \leq C|x_0 - x|^{1/2} w(|x - x_0|),$$

with $w(h) = \sqrt{|\log|\log h^{-1}||}$.

The Weierstraß function

$$W(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(2^{2j} x \pi)$$

does belong to $\Lambda^{1/2}(\mathbb{R})$.

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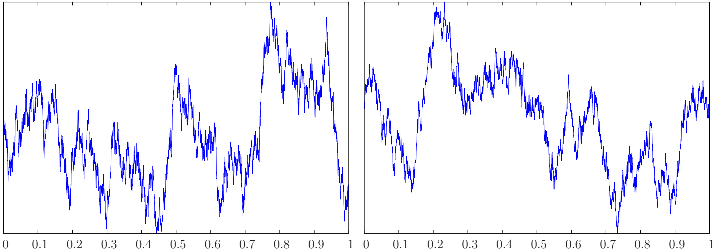
does belong to $\Lambda^{1/2}(\mathbb{R})$.

The uniform Weierstraß function of parameters (a, b) is the classical Weierstraß function coupled with a random phase. More precisely, this process is defined by

$$W(x) = \sum_{n=0}^{+\infty} a^n \cos((b^n x + U_n)\pi),$$

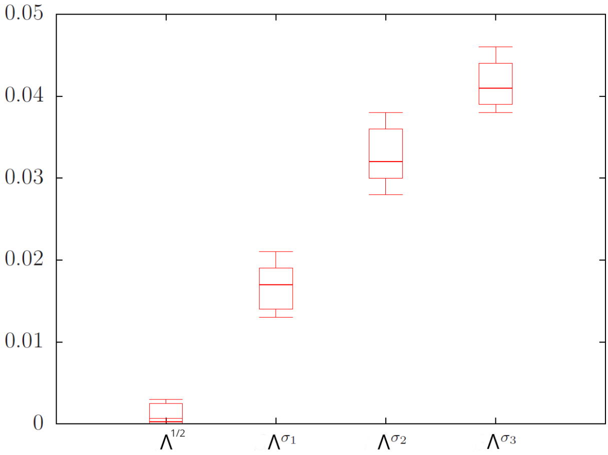
where $0 < a < 1 < b$ with $ab \geq 1$ and where each U_n is chosen independently with respect to the uniform probability measure on $[0, 1]$.

A realization of a BM vs a realization of a Weierstraß process



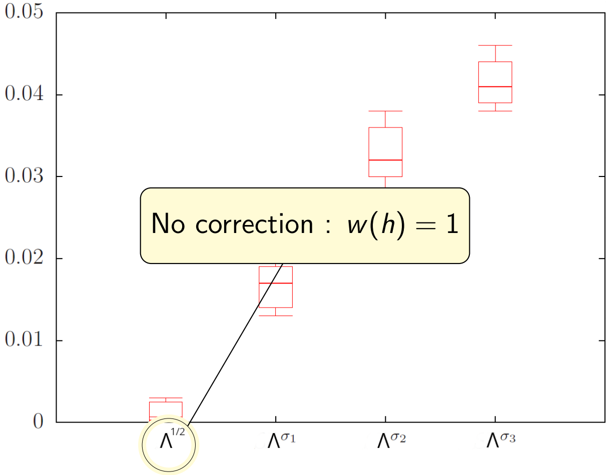
(for W (right), we set $a = 0.8$ and $b = 1.6$).

The picture...

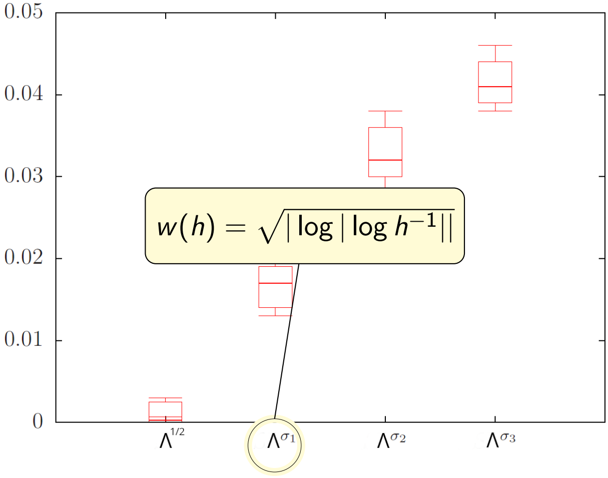


0.05
0.04
0.03
0.02
0.01
0
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}

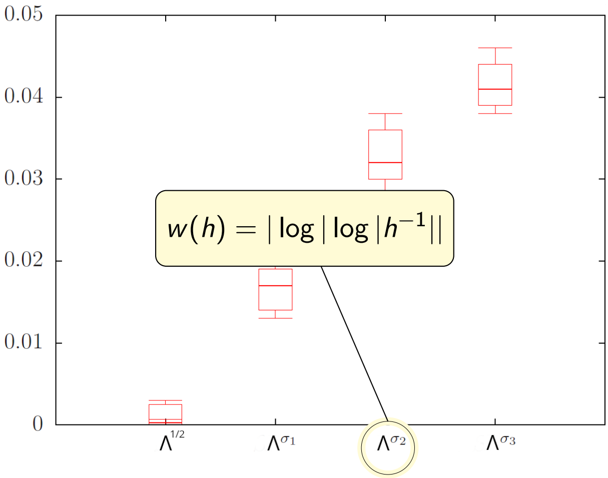
The spaces...



0.05
0.04
0.03
0.02
0.01
0
No correction : $w(h) = 1$
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}



0.05
0.04
0.03
0.02
0.01
0
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}
$w(h) = \sqrt{|\log|\log h^{-1}||}$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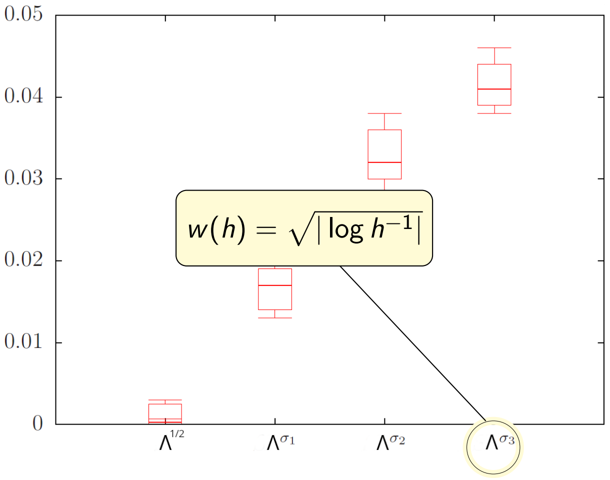
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}
$w(h) = |\log|\log|h^{-1}||$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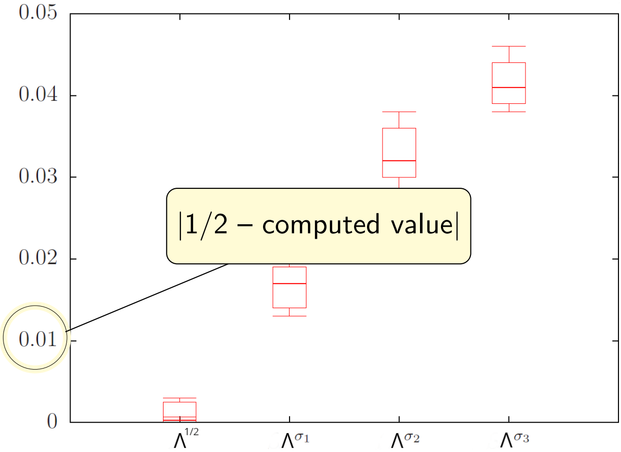
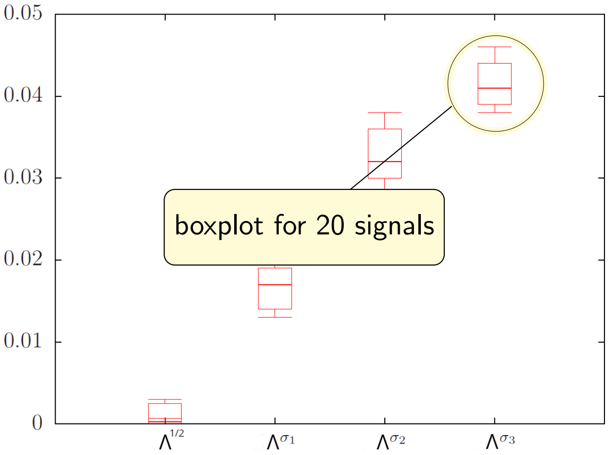
0
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}
$w(h) = \sqrt{|\log h^{-1}|}$

The error...

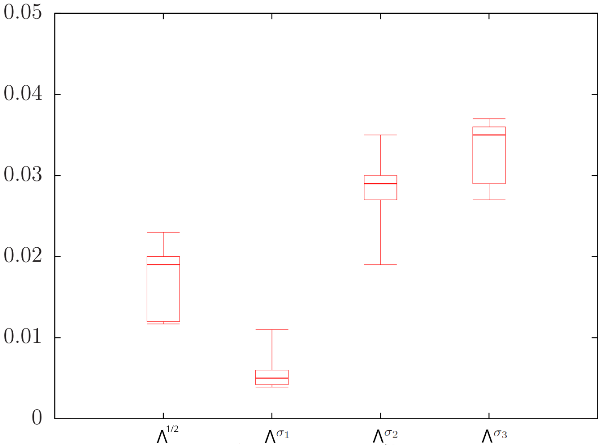

0.05
0.04
0.03
0.02
0.01
0
|1/2 – computed value|
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}

The boxplot...


0.05
0.04
0.03
0.02
0.01
0
boxplot for 20 signals
$\Lambda^{1/2}$
Λ^{σ_1}
Λ^{σ_2}
Λ^{σ_3}

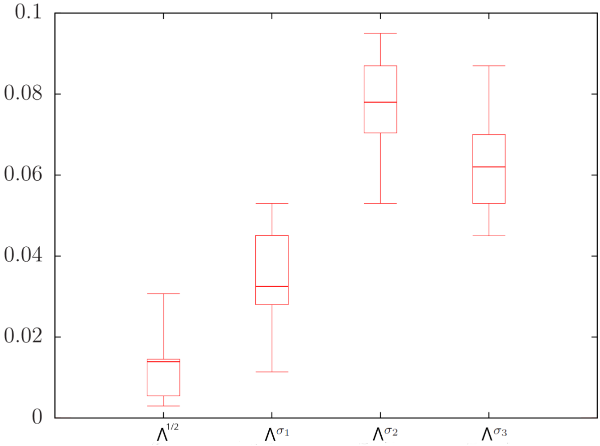
The Brownian motion vs the uniform Weierstraß function.

For 20 simulations of a BM, we get



The Brownian motion vs the uniform Weierstraß function.

For 20 simulations of W, we get



The Brjuno (Брюно) function is an arithmetic $\mathbb{Z}$-periodic function defined on irrational numbers as follows:

$$B : \mathbb{R} \setminus \mathbb{Q} \to \bar{\mathbb{R}} \quad x \mapsto \sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \log \frac{1}{\alpha_n},$$

where α_0 is the fractional part of x, and α_{n+1} is the fractional part of $1/\alpha_n$.

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Yoccoz took the nearest integer instead the integer part for the definition of B.

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These functions are of Bounded Mean Oscillation.

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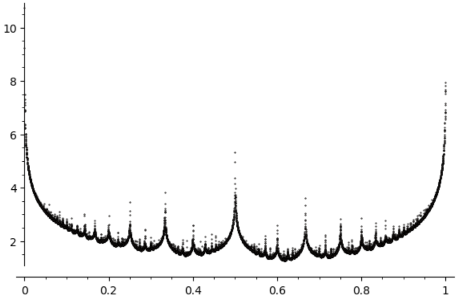

Proposition (Jaffard, Martin/Lamby, Martin, N.)

Concerning the Brjno functions, for $p \in [1, \infty)$, we have

$$h_p(x_0) = 1/\tau(x_0),$$

where $\tau(x_0)$ is the irrationality exponent of x_0 and

$$D_p(h) = 2h,$$

for $h \in [0, 1/2]$.

- Almeida, A.: Wavelet bases in generalized Besov spaces. J. Math. Anal. Appl. 304, 198-211 (2005)
- Calderón, A.P., Zygmund, A.: Local properties of solutions of elliptic partial differential equations. Studia Math. 20, 181-225 (1961)
- Daubechies, I.: Ten Lectures on Wavelets. CBMS-NSF Regional Conference Series in Applied Mathematics (1992)
- Hunt, B.: The prevalence of continuous nowhere differentiable functions. Am. Math. Soc. 122, 711-717 (1994)
- Kleyntssens, T., Nicolay, S.: A multifractal formalism for non-concave and non-increasing spectra: the leaders profile method, Appl. Comput. Harmon. Anal. 43 269-291 (2017)
- Lamby, T., Nicolay, S.: Some remarks on the Boyd functions related to the admissible sequences, Z. Anal. Anwend. 41 211-227 (2022)
- Lamby, T., Nicolay, S.: Interpolation with a function parameter from the category point of view, . Funct. Anal. 286 110249 (2024)
- Loosveldt, L., Nicolay, S.: Some equivalent definitions of Besov spaces of generalized smoothness. Math. Nachr. 292, 2262-2282 (2019)
- Loosveldt, L., Nicolay, S.: Generalized T_u^p spaces: On the trail of Calderón and Zygmund. Diss. Math., 554 (2020).
- Loosveldt, L., Nicolay, S.: Generalized spaces of pointwise regularity: to a general framework for the WLM. Nonlinearity 34, 6561-6586 (2021)
- Loosveldt, L., Nicolay, S.: Some Prevalent Sets in Multifractal Analysis: How Smooth is Almost Every Function in $T_p^\alpha(x)$. J. Fourier Anal. Appl. 28:58 (2022)
- Merucci, C.: Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. In M. Cwikel and J. Peetre, editors, Interpolation Spaces and Allied Topics in Analysis, pages 183-201. Springer (1984).
- Moura, S.D.: On some characterizations of Besov spaces of generalized smoothness. Math. Nachr. 280, 1190-1199 (2007)
- Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators, North-Holland Publishing Company (1978)