# Stability of return groups

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Joint work with F. Gheeraert, H. Goulet-Ouellet
and J. Leroy

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# Preliminary notions

# Basics

For a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and a word $u \in \mathcal{L}(X)$, a *return word* to $u$ is a word $r$ such that $ru \in \mathcal{L}(X)$ and $ru$ contains exactly two occurrences of $u$, one as a prefix and one as a suffix. By convention, all letters are return words to $\varepsilon$.

**Example**

Take the *Tribonacci substitution* $\sigma : a \mapsto ab, b \mapsto ac, c \mapsto a$. Let $X$ be the shift space generated by $\sigma$. The set of return words to $u = aba$ in $X$ is given by $\mathcal{R}_X(u) = \{ab, aba, abac\}$. All three of these return words can be seen in the factor

$$\ldots baabacababacabaabacabaca \ldots$$

where we underlined an occurrence for each one.

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# Applications

- F. Durand in *characterization of substitutive sequences using return words* (1998).
  $\longrightarrow$ Inductive coding by return words.
- Decision procedures for substitutive shift spaces by F. Durand and J. Leroy (2013 and 2022).
  $\longrightarrow$ Computable point of view in primitive substitutive shift spaces.
- L. Vuillon in *A characterization of sturmian words by return words* (2001).
- Appear naturally when studying critical exponents as studied by F. Dolce, Ľ. Dvořáková and E. Pelantová (2022 and 2023).

# Interpretation as subgroups

The *return group* to $u$ in $X$ is the subgroup of $F_{\mathcal{A}}$ generated by the return words to $u$. This subgroup is denoted: $\langle \mathcal{R}_X(u) \rangle$.

**Note**

The *free group* $F_{\mathcal{A}}$ is the set of reduced words over $\mathcal{A} \cup \mathcal{A}^{-1}$ endowed with concatenation.

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- $F_{\mathcal{A}}$

# Known results

- Return theorem and its converse by V. Berthé *et al.* (2015) and F. Gheearaert *et al.* (2024).
- Return groups of automatic sequences by V. Berthé and H. Goulet-Ouellet (2023) and E. Krawczyk and C. Müllner (2023).
- Welldoc property by Ľ. Dvořáková *et al.* (2016).
- Minimality of skew products by V. Berthé *et al.* (2024).
- …

# The stability property

# Main tool

$X$ has *eventually $\varphi$-stable return groups* if there exists $M \in \mathbb{N}$ such that, for all $w \in \mathcal{L}(X)$ of length $M$, and all $u \in \mathcal{L}(X)$ such that $w$ is a prefix of $u$, we have

$$\varphi \langle \mathcal{R}_X(u) \rangle = \varphi \langle \mathcal{R}_X(w) \rangle .$$

If $M = 0$, we moreover say that $X$ has *$\varphi$-stable return groups.*

**Remark**

This terminology allows for the following restatement of the *Return Theorem* and its converse. A shift space $X$ is dendric if, and only if, it has $\mathrm{id}$-stable return groups and exactly $d$ return words to $u$ for any $u \in \mathcal{L}(X)$.

# First observation

**Lemma (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

If $w$ is a factor of $v \in \mathcal{L}(X)$, then there exists $p \in \mathcal{L}(X)$ such that

$$\varphi \langle \mathcal{R}_X(w) \rangle \geq \varphi(p)^{-1} \varphi \langle \mathcal{R}_X(v) \rangle \varphi(p).$$

In particular, if $w$ is a prefix of $v$, then $\varphi \langle \mathcal{R}_X(w) \rangle \geq \varphi \langle \mathcal{R}_X(v) \rangle$.

**Proof.**

See the blackwall. □

# Satisfaction

We can restate the condition of having *eventually $\varphi$-stable return groups* as having only one return group to long enough words up to conjugacy.

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**Theorem (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

If $G$ is finite, abelian or free and $\varphi$ is onto, then TFAE:

1. $X$ has eventually $\varphi$-stable return groups;
2. $\{\varphi\langle\mathcal{R}_X(u)\rangle \mid u \in \mathcal{L}(X)\}$ is finite;
3. the subgroups $\varphi\langle\mathcal{R}_X(u)\rangle$ are conjugate for all but finitely many $u \in \mathcal{L}(X)$;
4. the subgroups $\varphi\langle\mathcal{R}_X(u)\rangle$ are conjugate for infinitely many $u \in \mathcal{L}(X)$.

# Behaviour of famous families

**Proposition (Berthé *et al.*, 2015)**

Every minimal *connected shift space* has $\mathrm{id}$-stable return groups.

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Every minimal *eventually connected shift space* has eventually $\mathrm{id}$-stable return groups.

**Corollary (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

Every minimal *eventually dendric shift space* has eventually $\mathrm{id}$-stable return groups.

# Behaviour of famous families

**Theorem (Krawczyk and Müllner, 2023)**

If $X$ is aperiodic and *k-automatic* then, for each $n \in \mathbb{N}$, there is a constant $K_n$ such that, for every $u \in \mathcal{L}_{\geq K_n}(X)$, any return word to $u$ has length multiple of $k^n$.

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**Corollary (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

If $X$ is aperiodic and $k$-automatic, then it does not have eventually $|\cdot|$-stable return groups. In particular, $X$ does not have eventually $\mathrm{id}$-stable return groups.

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**Corollary (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

A minimal eventually connected shift space that is $k$-automatic is periodic.

# Closure properties

# With respect to derivation

**Definition**

Let $\theta_u : \mathcal{B}^* \to \mathcal{A}^*$ defining a bijection between $\mathcal{B}$ and $\mathcal{R}_X(u)$. There is a shift space $Y$ over $\mathcal{B}$ such that $\mathcal{L}(Y) = \theta_u^{-1}(\mathcal{L}(X))$. This shift space is minimal and is unique up to a renaming of the letters. It is called the *derived shift space of X* with respect to $u$, and is denoted $D_u(X)$.

Let us look at an example:

$$\dots baabacabababacabaabacabaca \dots$$

becomes

$$\dots ogobo \dots$$

# With respect to derivation

**Proposition (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

If $X$ has eventually $\varphi$-stable return groups, then $D_u(X)$ has eventually $\psi$-stable return groups for all $\psi\colon F_{\mathcal{B}} \to H$ such that $\ker(\varphi\theta_u) \leq \ker(\psi)$.

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**Proposition (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

The family of shift spaces with (eventually) stable return groups is not closed under derivation. A counterexample is given by $X$ defined by the substitution

$$\sigma\colon a \mapsto baa, b \mapsto ca, c \mapsto bad, d \mapsto acd.$$

# With respect to substitutions

**Proposition (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

Let $\sigma\colon \mathcal{A}^* \to \mathcal{B}^*$ be a substitution. If $X$ has eventually $\varphi$-stable return groups, then $\sigma[X]$ has eventually $\psi$-stable return groups for any group morphism $\psi\colon F_{\mathcal{B}} \to H$ satisfying $\ker(\varphi) \leq \ker(\psi\sigma)$.

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**Corollary (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

1. The family of shift spaces with eventually stable return groups is closed under application of substitutions.
2. The family of shift spaces with stable return groups is closed under application of substitutions that are surjective when seen as group morphisms.

# Decidability and computability

# In finite groups

**Proposition (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

Let $X$ be substitutive and $G$ be finite. One can compute a subgroup $H$ such that, for any long enough $u$, $\varphi\langle\mathcal{R}_X(u)\rangle$ is $H$ (up to conjugacy).

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**Corollary (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

Let $X$ be substitutive and $G$ be finite. Then $\varphi$-stability of $X$ is decidable.

# In the free group

In free groups, the word and conjugacy problems are decidable and the rank is computable. These algorithmic techniques are detailed in *Stallings foldings and subgroups of free groups* by Kapovich and Myasnikov (2002). They are based on *Stallings foldings* introduced in Stallings' seminal paper *Foldings of g-tree* (1991).

# In the free group

**Proposition (Gheeraert, Goulet-Ouellet, Leroy and S., 2024+)**

Let $X$ be generated by a primitive substitution $\sigma$. If $\sigma$ is either derivating or bifix, one can:

1. decide whether $X$ has eventually $\mathrm{id}$-stable return groups;
2. compute the subgroup over which the return groups stabilize if they do;
3. decide whether $X$ has $\mathrm{id}$-stable return groups.

# Conclusion

# Open questions

- Are there shift spaces with (eventually) $\mathrm{id}$-stable return groups without any notion of connectedness?
- Is eventual $\varphi$-stability a dynamical property?
- Is there a meaningful parallel between return groups in shift spaces and fundamental groups?
- Are there families of shift spaces characterized by such properties?