Locally Stable Roommate Problem

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HEC LIÈGE
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Joint work

Joint work with two colleagues from HEC Liege (Belgium):

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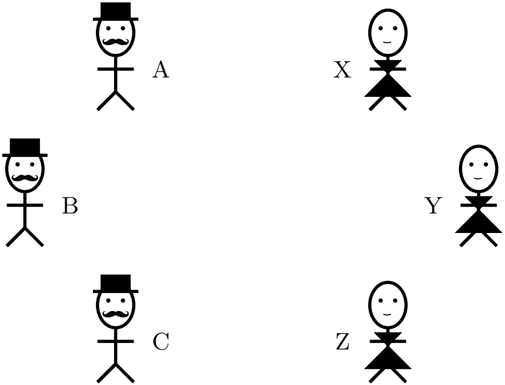
Professor

Marie Baratto
PhD looking for a postdoc

Disclaimer : This is a work in progress.

Stable Marriage Problem (SMP)

- Introduced by Gale and Shapley in 1962
- Aim : to match men and women based on their preferences for all members of the opposite gender



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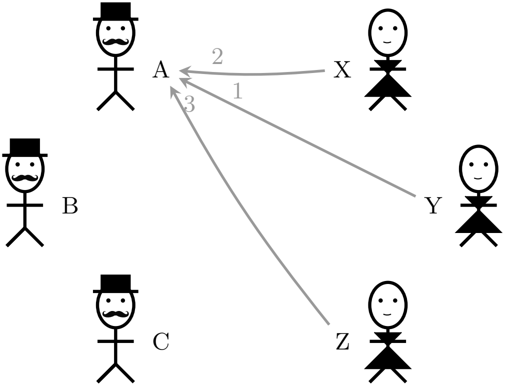

A: Y X Z

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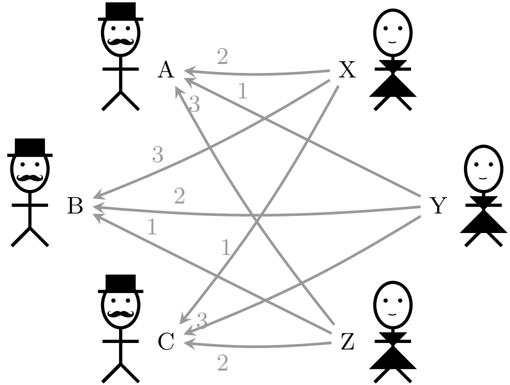

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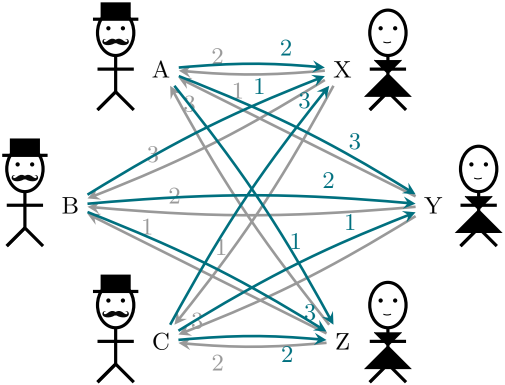

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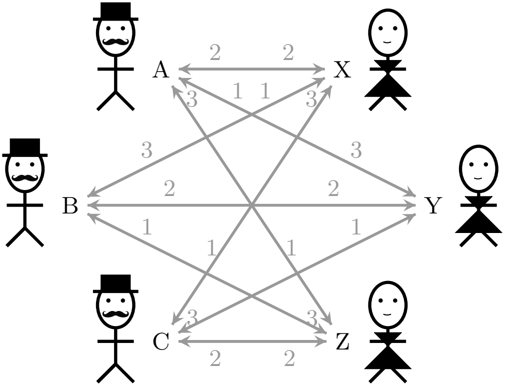

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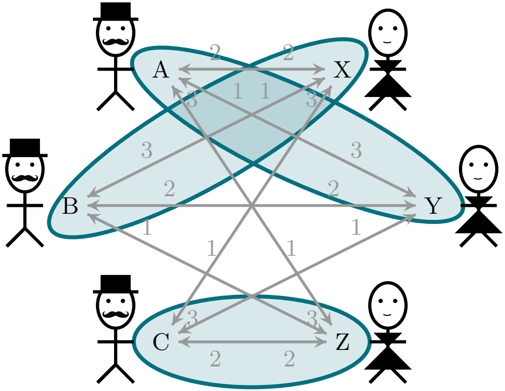

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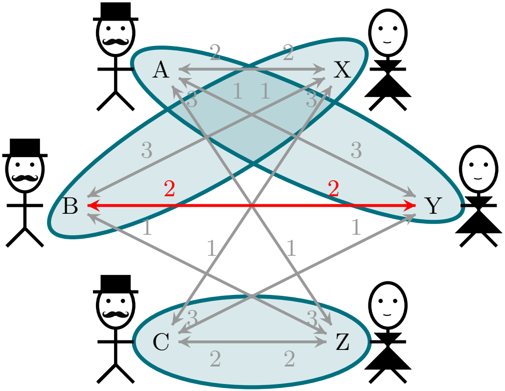

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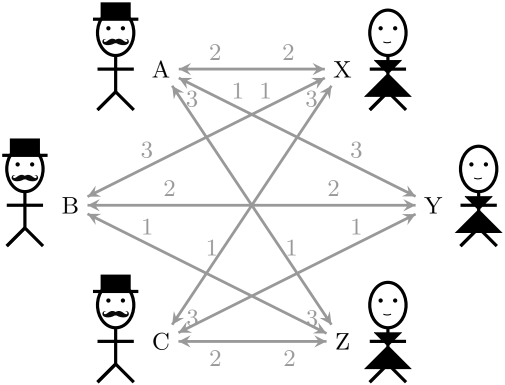

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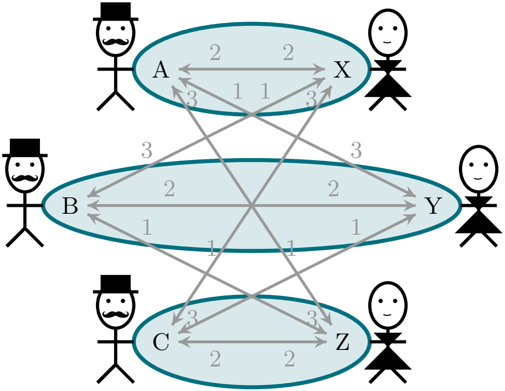

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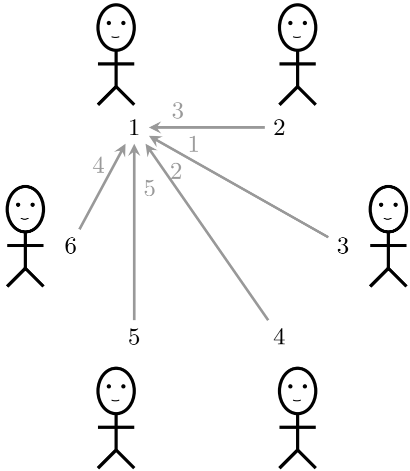
Stable Roommate Problem (SRP)

- Generalization of SMP to non-bipartite model
- Each individual ranks all the others in order of preference.



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2:	6	5	4	1	3
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4:	5	2	3	6	1
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- Aim : to find a stable matching

SMP vs SRP

	SMP	**SRP**
Graphs	Complete bipartite	Complete

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$\exists$ stable matching?	Always	Not always

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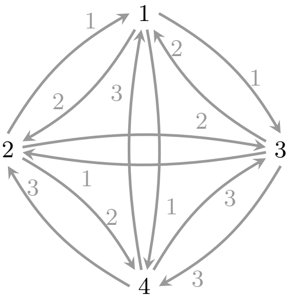
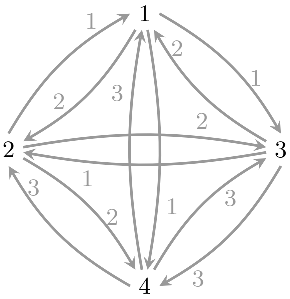


No stable matching

SMP vs SRP

	SMP	**SRP**
Graphs	Complete bipartite	Complete
∃ stable matching?	Always	Not always
Find one (if any)	Polynomial	Polynomial
	Gale-Shapley (1962)	Irving (1985)



No stable matching

Stability in Kidney Exchange Programs

Patient with a serious kidney disease may resort to:

- Dialysis
- Transplant from a deceased donor
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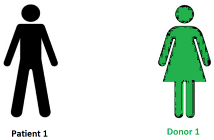

Stability in Kidney Exchange Programs

Patient with a serious kidney disease may resort to:

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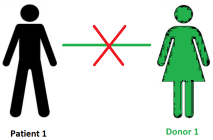
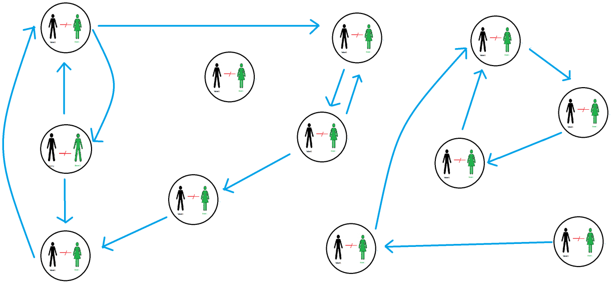
Patient might not be compatible with the donor: e.g.,

- Blood incompatibility
- Tissue type incompatibility



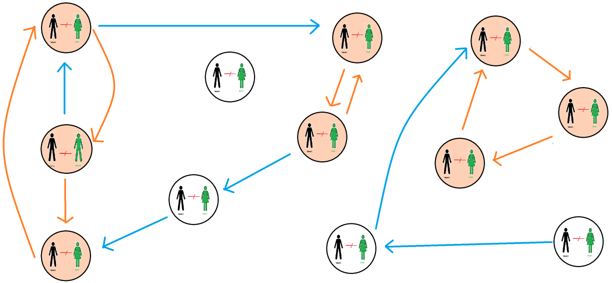
Compatibility graph



G=(V,A) where:

- $V = \{1, ..., n\}$ set of vertices, consisting of all patient-donor pairs.
- A, the set of arcs, designating compatibilities between the vertices. Two vertices i and j are connected by arc (i, j) if the donor in pair i is compatible with the patient in pair j.

Possible exchanges



Definition

An exchange is a set of disjoint cycles in the directed graph such that every cycle length does not exceed a given limit K.

- Aim: to **maximize the number of patients transplanted**
- When $K = 2$, an exchange is a matching.

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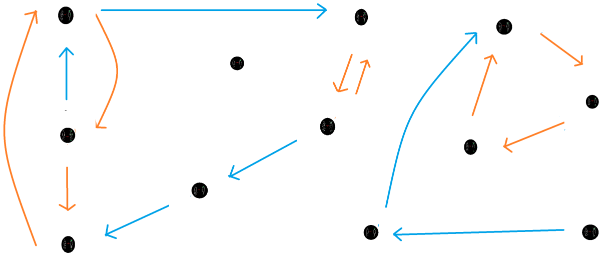

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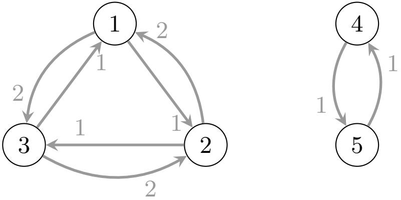

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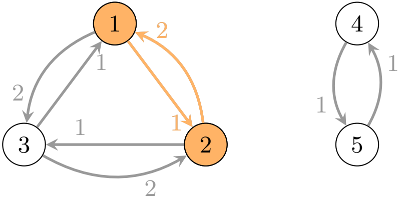

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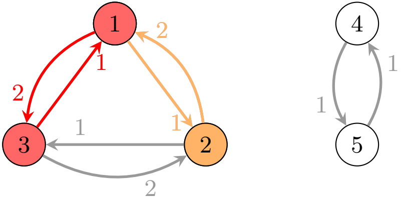

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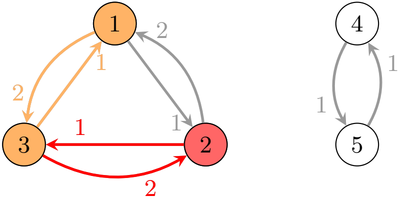

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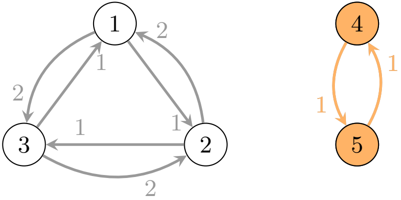

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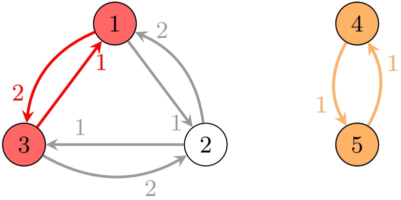

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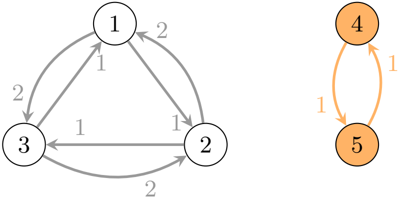
Locally Stable Exchange (LSE)



Definition (Baratto–Crama–Pedroso–Viana, accepted)

Given a directed graph $G = (V, A)$,

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SE vs LSE

A stable matching is maximum if it has the largest possible size among all stable matchings. And similarly for maximum locally stable matchings.

- **SE** problem: What is the maximum size of a stable matching?
  ($K = 2$, 72 don't have a stable matching out of 600 tested - 12%)

- **LSE** problem: What is the maximum size of a non-empty locally stable matching?
  ($K = 2$, 1 out of 600 tested has a solution of cardinality zero - 0.2%)

  - For 50 instances with $V \approx 200$,
    45 out of 50 have a stable exchange;
    50 out of 50 have a locally stable exchange > 0;
    45 instances max. stable exchange = max. locally stable exchange

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Work in Progress:

- Computing a maximum locally stable exch. for $K \geq 3$ is NP-hard.

- Computing a maximum $\underbrace{\text{locally stable exch. for } K = 2}_{\text{locally stable roommate problem}}$ is polynomial.

Locally Stable Roommate Problem (LSRP)

Proposition

If M is a stable matching and M' is a locally stable matching, then $V(M') \subseteq V(M)$ and $|M'| \leq |M|$.

Proposition

If a graph has a stable matching, then

1. all its stable matchings cover the same set of vertices, and
2. all its stable matchings are maximum locally stable.

A locally stable matching is maximal if it is not included (edge-wise) in any other locally stable matching.

Proposition

All maximal locally stable matchings cover the same set of vertices and hence, they have the same size.

Irving's algorithm - Phase 1

Idea:

- Successive deletion of entries in the preference lists

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procedure Phase1($T$ : table of preference lists)



end procedure
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free vs semi-engaged

x proposes to y

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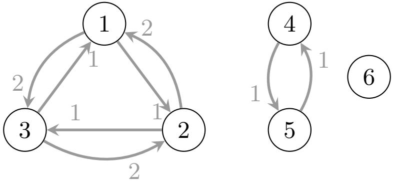

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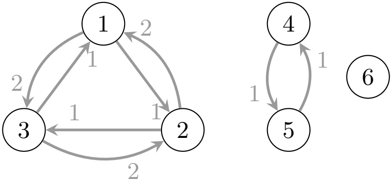

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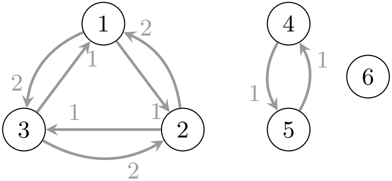

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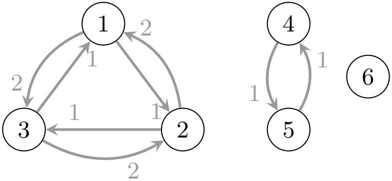

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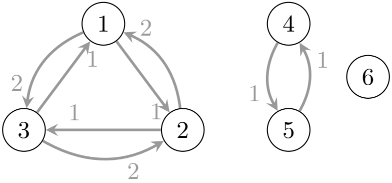

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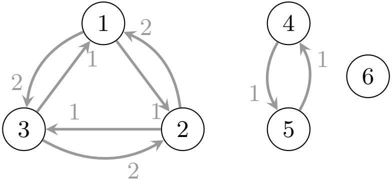

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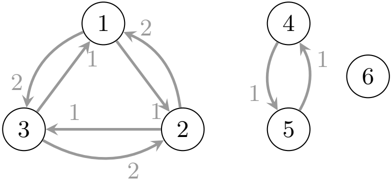

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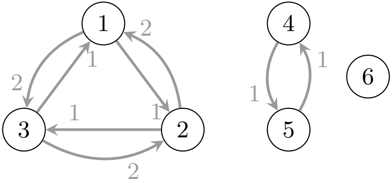

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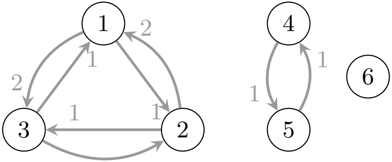

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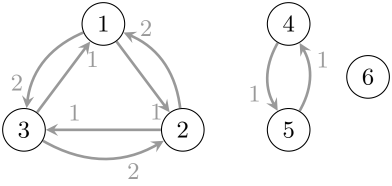

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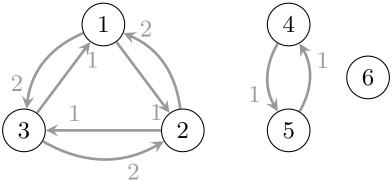

1:	$\textcircled{3}$	$\boxed{2}$	~~4~~	~~5~~	~~6~~
2:	$\textcircled{1}$	$\boxed{3}$	~~4~~	~~5~~	~~6~~
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4:	$\textcircled{5}$	~~1~~	~~2~~	~~3~~	6
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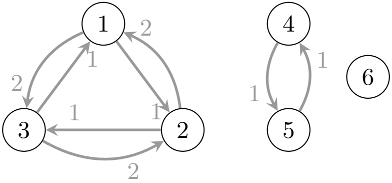

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Irving's algorithm - Phase 1 - comments

- For a given instance of the problem, all possible executions of phase 1 of the algorithm yield the same reduced preference table (Gusfield and Irving 1989)
- If, at any stage of phase 1, x proposes to y, then in a locally stable matching M
  1. x cannot have a better partner than y;
  2. if $y \in V(M)$, y cannot have a worse partner than x.
- If T is the phase-1 table for a roommate instance, then
  1. $y = \mathrm{first}_T(x)$ if and only if $x = \mathrm{last}_T(y)$;
  2. if the edge $\{x, y\}$ is absent from T then x and y cannot be partners in a locally stable matching.

Irving's algorithm - Phase 1

Suppose that T is the phase-1 table for a roommates instance. Then

- if all persons have lists of size 1, there is a stable matching (and therefore a locally stable one) of size n.
- if one person has an empty list, there is no stable matching, but there could be a non-empty locally stable matching.
- if all lists have at least 1 element and some at least 2, then there could exist a non-empty locally stable matching and/or a stable one.

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3:	$\textcircled{2}$	$\boxed{1}$	$\cancel{4}$	$\cancel{5}$	$\cancel{6}$
4:	$\textcircled{\boxed{5}}$	$\cancel{1}$	$\cancel{2}$	$\cancel{3}$	$\cancel{6}$
5:	$\textcircled{\boxed{4}}$	$\cancel{1}$	$\cancel{2}$	$\cancel{3}$	$\cancel{6}$
6:	$\cancel{1}$	$\cancel{2}$	$\cancel{3}$	$\cancel{4}$	$\cancel{5}$

Irving's algorithm - Phase 2

Idea: Further reduction of the table T using rotations.

Definition (Irving 1985)

For a given table T, a sequence

$$\rho = (x_0, y_0), \ldots, (x_{r-1}, y_{r-1})$$

such that $y_i = \mathrm{first}_T(x_i), y_{i+1} = \mathrm{second}_T(x_i)$ for all i (taken modulo r), is called a *rotation* exposed in T.

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(1,3),(3,2),(2,1)

- If T is a table in which some list contains at least two entries, then there is at least one rotation exposed in T.

Irving's algorithm - Phase 2

How to use it ?

Conjecture

Let $\rho = (x_0, y_0), \dots, (x_{r-1}, y_{r-1})$ be a rotation exposed in a table T.

1. In any locally stable matching embedded in T, either x_i and y_i are partners for all values of i or for no value of i.
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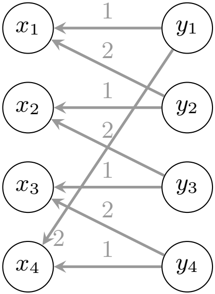

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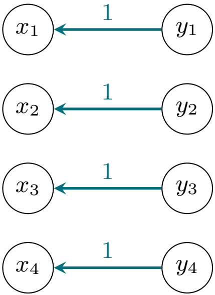

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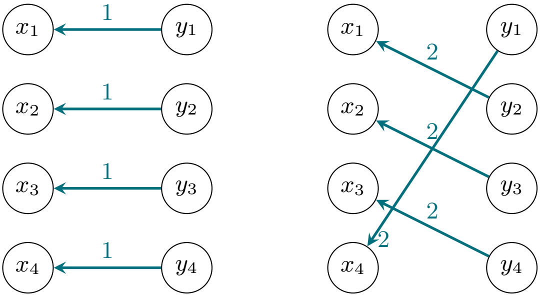

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In the example, (1,3),(3,2),(2,1) is a rotation:

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Conjecture

At the end of Phase 2,

- if the table contains only empty lists, there is no stable matching and only the trivial empty locally stable matching.
- if the table contains only empty lists and lists with one element, there is no stable matching and the nonempty lists specify a maximum locally stable matching.
- if the table contains only lists with one element, they specify a stable matching which is also a maximum locally stable matching.

Future work

When graphs are incomplete, we think the same reasoning applies.

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When ties are allowed in the preference lists,

- SMP : Deciding whether a bipartite graph has a stable marriage is polynomial.
- SRP : Deciding whether a graph has a stable matching is NP-complete.
- LSRP : What is the complexity of deciding whether a graph has a nonempty **locally** stable matching?

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- LSRP : What is the complexity of deciding whether a graph has a nonempty **locally** stable matching?

Only a partial answer for now.

Proposition

When ties are allowed in the preference lists, deciding whether a graph has a locally stable **perfect** matching is NP-complete.