Binomial^4

Coefficients, equivalence, complexity, and beyond

Manon Stipulanti

19th Journées Montoises d'Informatique Théorique
Université Côte d'Azur (Nice, France)
September 3, 2024

It's *Nice* to be here!

summary of results from

- ⋆ M. Lejeune, On the k-binomial equivalence of finite words and k-binomial complexity of infinite words, ULiège (Belgium), 2021
- ⋆ M. Rigo, M. S., M. A. Whiteland, Binomial complexities of Parikh-Collinear morphisms, DLT 2022, *Lect. Notes in Comput. Sci.* **13257** (2022), 251–262
- ⋆ M. Rigo, M. S., M. A. Whiteland, Characterizations of families of morphisms and words via binomial complexities, *European J. Comb.* **118** (2024), 103932

Notation

- infinite words in **bold**
- $|w|_a = \#$ letters a in w
- in a word
  factor = subsequence of consecutive letters
  (scattered) subword = subsequence of letters

Example: $|\texttt{reappear}|_\texttt{a} = 2 = |\texttt{reappear}|_\texttt{e}$

factor	subword
reappear	reappear
reappear	reappear
reappear	reappear
reappear	reappear

- length-n factors of $\mathbf{x}$: $\mathrm{Fac}_n(\mathbf{x})$

How to study combinatorial structure?

factor complexity $\mathsf{p}_{\mathbf{x}}\colon \mathbb{N} \to \mathbb{N}, n \mapsto \#\mathrm{Fac}_n(\mathbf{x})$

Example: Fibonacci word $\mathbf{f} = 0100101001\cdots$ (f.p. of $\phi\colon 0 \mapsto 01, 1 \mapsto 0$)

n	$\mathrm{Fac}_n(\mathbf{f})$	$\mathsf{p}_{\mathbf{f}}(n)$
0	ε	1
1	$0, 1$	2
2	$00, 01, 10$	3
3	$001, 010, 100, 101$	4
4	$0010, 0100, 0101, 1001, 1010$	5

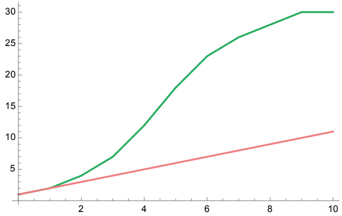
Theorem (Morse–Hedlund 1938)

$\mathbf{x}$ with ℓ distinct letters

$\mathbf{x}$ ultimately periodic

iff $\mathsf{p}_{\mathbf{x}}$ bounded

iff $\exists n \in \mathbb{N}$ s.t. $\mathsf{p}_{\mathbf{x}}(n) < n+\ell-1$



$\mathbf{x}$ Sturmian iff $\mathsf{p}_{\mathbf{x}}(n) = n+1 \quad \forall n$

(binary, aperiodic, minimal factor complexity)

$\mathbf{x} = y\,01\,01\,01\cdots$

$\mathbf{f} = 0100101001001\cdots$

Variations

counting "different enough" factors

- with specific properties

  e.g. palindromes (Droubay–Pirillo 1999)

  privileged (Peltomäki 2013)

- extracted along specific subsequences

  e.g. arithmetical (Avgustinovich–Fon-Der-Flaass–Frid 2000)

  maximal pattern (Kamae–Zamboni 2002)

- with equivalence relations $u \sim v$

  $\mathbb{N} \to \mathbb{N}, n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\sim)$

today 2 equivalence relations

but many more

e.g. 05/09 10:30AM Popoli

+see recent [arXiv:2406.09302] by Allouche–Campbell–Li–Shallit–S.

First variation (Erdős 1958)

- abelian equivalence relation: $u \sim_{\text{ab}} v$ if $|u|_a = |v|_a \ \forall a \in A$

  Example: $\texttt{evil} \sim_{\text{ab}} \texttt{live} \sim_{\text{ab}} \texttt{veil} \sim_{\text{ab}} \texttt{vile}$

- abelian complexity $\mathsf{a}_{\mathbf{x}} : \mathbb{N} \to \mathbb{N}, n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\sim_{\text{ab}})$

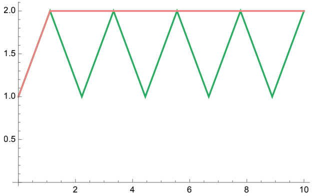
  Example: Fibonacci word $\mathbf{f} = 0100101001\cdots$

n	$\mathrm{Fac}_n(\mathbf{f})$	$\mathsf{p}_{\mathbf{f}}(n)$	$\{\cdot\}_{\sim_{\text{ab}}}$	$\mathsf{a}_{\mathbf{f}}(n)$
0	ε	1	$\{\varepsilon\}$	1
1	$0, 1$	2	$\{0\}$ $\{1\}$	2
2	$00, 01, 10$	3	$\{00\}$ $\{01, 10\}$	2
3	$001, 010, 100, 101$	4	$\{001, 010, 100\}$ $\{101\}$	2

Theorem (Coven–Hedlund 1973)

$\mathbf{x}$ purely periodic iff $\exists n \in \mathbb{N}$ s.t. $\mathsf{a}_{\mathbf{x}}(n) = 1$

$\mathbf{x}$ Sturmian iff $\mathbf{x}$ binary aperiodic
and $\mathsf{a}_{\mathbf{x}}(n) = 2 \ \ \forall n \geq 1$



$\mathbf{x} = 01\,01\,01\cdots$

$\mathbf{f} = 0100101001001\cdots$

+see survey by Fici–Puzynina 2023

Second variation

- binomial coefficient

$$\binom{u}{v} = \#\text{ occurrences of } v \text{ as a subword of } u$$

Example: $\binom{101001}{101} = 6$ 101001 101001 101001 101001 101001 101001

Remark: $\binom{a^n}{a^k} = \binom{n}{k}$

- binomial equivalence relation

$$k \geq 1 \qquad u \sim_k v \text{ if } \binom{u}{x} = \binom{v}{x} \ \forall x \text{ with } |x| \leq k$$

Example: $0110 \sim_2 1001$ but $0110 \not\sim_3 1001$

x	0	1	00	01	10	11	000	001
$\binom{0110}{x}$	2	2	1	2	2	1	0	0
$\binom{1001}{x}$	2	2	1	2	2	1	0	1

Remark: $\sim_1 = \sim_{\mathrm{ab}}$

$\sim_{k+1}$ refines $\sim_k$

Binomial coefficients (notably) appear in...

- Chapter 6 in Lothaire's book   Sakarovitch–Simon 1983
- reconstruction problem

  Given n, what is the smallest k s.t. each length-n word is uniquely determined by all its length-k subwords (with multiplicities)?   Still open but bounds and variations

  Kalashnik 1973   Krasikov–Roditty 1997   Levenshtein 2001   Dudik–Schulman 2003

  Fleischmann–Lejeune–Manea–Nowotka–Rigo 2021   Richomme–Rosenfeld 2023   etc.
- strictly locally testable languages

  regular languages defined by the presence/absence of given subwords

  Simon 1975
- p-group languages

  L language and p prime, $\exists\ G$ p-group and $\alpha\colon A^* \to G$ morphism s.t. $L = \alpha^{-1}(G)$ iff $L =$ finite Boolean combination of $L_{v,r,p} = \left\{u \in A^* \colon \binom{u}{v} \equiv r \pmod{p}\right\}$

  Eilenberg 1976   Renard–Rigo–Whiteland 2024 + 03/09 10:30AM
- Parikh matrices
- generalized Pascal's triangles

  Leroy–Rigo–S. 2016-2017-2018   S. 2019
- gapped binomial coefficients

  Golm–Nahvi–Gabrys–Milenkovic 2022   Rigo–S.–Whiteland 2023

Binomial complexities

Definition (Rigo–Salimov 2015)

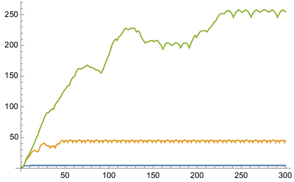
Let $k \geq 1$.
k-binomial complexity of $\mathbf{x}$: $\mathsf{b}_{\mathbf{x}}^{(k)} : \mathbb{N} \to \mathbb{N},\ n \mapsto \#(\mathrm{Fac}_n(\mathbf{x})/\!\sim_k)$

Example: Thue–Morse $\mathbf{t} = 011010011001\cdots$ (f.p. of φ: $0 \mapsto 01, 1 \mapsto 10$)

n	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{a}_{\mathbf{x}}(n) = \mathsf{b}_{\mathbf{t}}^{(1)}(n)$	1	2	3	2	3	2	3	2	3	2	3
$\mathsf{b}_{\mathbf{t}}^{(2)}(n)$	1	2	4	6	9	8	8	8	9	8	8
$\mathsf{p}_{\mathbf{t}}(n)$	1	2	4	6	10	12	16	20	22	24	28

Observation: $\sim_{k+1}$ refines $\sim_k$ so

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$



A little bit of history since 2015

Sturmian word $\mathbf{s}$	$\mathsf{b}_{\mathbf{s}}^{(k)} = \mathsf{p}_{\mathbf{s}}\ \forall k \geq 2$	Rigo–Salimov 2015
Tribonacci word $\mathbf{z}$ (f.p. of $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$)	$\mathsf{b}_{\mathbf{z}}^{(k)} = \mathsf{p}_{\mathbf{z}}\ \forall k \geq 2$	Lejeune–Rosenfeld–Rigo 2020 Lejeune's PhD thesis 2021
Thue–Morse word $\mathbf{t}$ + wider class of words	$\mathsf{b}_{\mathbf{t}}^{(k)}$ bounded $\forall k \geq 1$	Rigo–Salimov 2015
Thue–Morse word $\mathbf{t}$	precise values of $\mathsf{b}_{\mathbf{t}}^{(k)}$ $\forall k \geq 1$	Lejeune–Leroy–Rigo 2020 Lejeune's PhD thesis 2021
generalized TM words	precise values of $\mathsf{b}^{(2)}$	Lü–Chen–Wen–Wu 2021
hypercubic billiard words	??? $\rightsquigarrow$ 05/09 3PM	Vivion 2024

Motivation and goal

complexity	p	a	$\mathsf{b}^{(k)}$
theory general behavior properties etc.	rich	rich	not much is known

What do we want?

- possible behavior of binomial complexities (growth)
- deduce structure of words from their binomial complexities
- understand binomial complexities of large classes of words
- find words attaining lowest complexities

today

- 3 characterizations
- 1 (or 2) question(s)

Sturmian words

Theorem (Rigo–Salimov 2015)

$\mathbf{s}$ Sturmian $\Longrightarrow \forall k \geq 2$, $\mathsf{b}_{\mathbf{s}}^{(k)}(n) = \mathsf{p}_{\mathbf{s}}(n) = n + 1 \quad \forall n$

Corollary (Fici–Puzynina 2023)

TFAE

- $\mathbf{x}$ Sturmian
- $\mathsf{b}_{\mathbf{x}}^{(1)}(n) = 2$ & $\exists k \geq 2$ s.t. $\mathsf{b}_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$
- $\mathsf{b}_{\mathbf{x}}^{(1)}(n) = 2$ & $\forall k \geq 2$ $\mathsf{b}_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

stronger version $\rightsquigarrow$ characterization of Sturmian words

Theorem

$\mathbf{x}$ binary s.t. $\exists k \geq 2$ with $\mathsf{b}_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n \Longrightarrow \mathbf{x}$ Sturmian

Proof by mixing results of Morse–Hedlund (1938) Coven–Hedlund (1973) Richomme-Séébold (2011) Rigo–Salimov (2015)

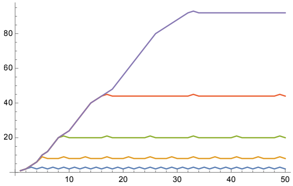
Words with bounded binomial complexities

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$ (f.p. of $\varphi\colon 0 \mapsto 01, 1 \mapsto 10$)

$\mathsf{b}_{\mathbf{t}}^{(k)}$ is bounded + precise values

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1 \;\; \mathsf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathsf{p}_{\mathbf{t}}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$



Observation:

$$\begin{pmatrix} |\varphi(0)|_0 \\ |\varphi(0)|_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} |\varphi(1)|_0 \\ |\varphi(1)|_1 \end{pmatrix}$$

Definition:

Parikh vector of w: $\Psi(w) = (|w|_a)_{a \in A}$

Parikh-constant: $\Psi(f(a)) = \Psi(f(b)) \; \forall a, b \in A$

Theorem (Rigo–Salimov 2015)

A fixed point of a Parikh-constant morphism has bounded $\mathsf{b}^{(k)}$ $(\forall k)$.

Parikh-collinear: $\forall a, b \in A$, $\exists r_{a,b} \in \mathbb{Q}$ s.t. $\Psi(f(b)) = r_{a,b}\Psi(f(a))$

Example: $f\colon 0 \mapsto 000111, 1 \mapsto 0110$

$$\Psi(f(0)) = \begin{pmatrix} |f(0)|_0 \\ |f(0)|_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad \Psi(f(1)) = \frac{2}{3}\Psi(f(0))$$
$$\Psi(f(1)) = \begin{pmatrix} |f(1)|_0 \\ |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

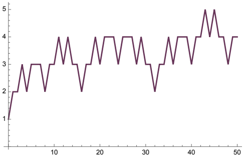
Remark: Parikh-collinear iff rank-1 adjacency matrix

Example: $\begin{pmatrix} |f(0)|_0 & |f(1)|_0 \\ |f(0)|_1 & |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

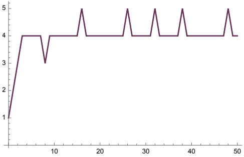
Characterizations in terms of a and $\mathsf{b}^{(k)}$

Theorem (Cassaigne–Richomme–Saari–Zamboni 2011)

f Parikh-collinear
iff f maps all infinite words to words with bounded $\mathsf{b}^{(1)}$



$\mathsf{b}_{\mathbf{x}}^{(1)}$



$\mathsf{b}_{f(\mathbf{x})}^{(1)}$ with $f\colon 0 \mapsto 000111, 1 \mapsto 0110$

generalization

Theorem

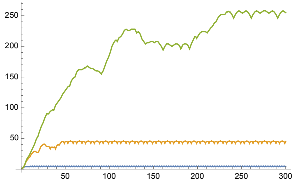
TFAE

- f Parikh-collinear
- $\forall k$ f maps words with bounded $\mathsf{b}^{(k)}$ to words with bounded $\mathsf{b}^{(k+1)}$
- $\exists k$: f maps words with bounded $\mathsf{b}^{(k)}$ to words with bounded $\mathsf{b}^{(k+1)}$

Generalization of Rigo and Salimov

Corollary

A f.p. of a Parikh-collinear morphism has bounded $\mathsf{b}^{(k)}$ $(\forall k)$.



$f: 0 \mapsto 000111, 1 \mapsto 0110$

<u>Remark</u>: no stronger version

$f: 0 \mapsto 0^3 2^3, 1 \mapsto 0^3 1^3 2, 2 \mapsto 2^4 0^6 1^3$ adjacency matrix of rank 2

$\rightsquigarrow$ $f^\omega(0)$ has unbounded $\mathsf{a} = \mathsf{b}^{(1)}$ (Adamczewski 2003)

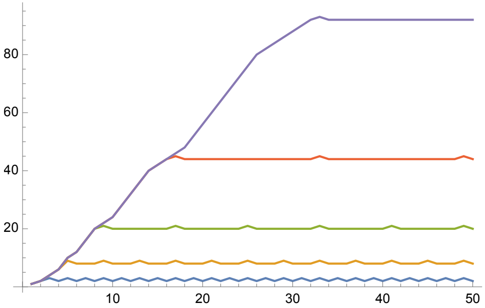
$\rightsquigarrow$ $f^\omega(0)$ has unbounded $\mathsf{b}^{(k)}$ $\forall k$

Words sharing binomial complexities with t

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$ (f.p. of $\varphi\colon 0 \mapsto 01, 1 \mapsto 10$)

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1 \;\; \mathsf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathsf{p}_{\mathbf{t}}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$



Definition: $\mathbf{x}$ has $\mathcal{P}_k$ if $\mathsf{b}_{\mathbf{x}}^{(j)} = \mathsf{b}_{\mathbf{t}}^{(j)}$ $\forall j \in \{1, \ldots, k\}$

Theorem (Richomme–Saari–Zamboni 2011)

$\mathbf{x}$ aperiodic binary has $\mathcal{P}_1$ iff $\exists \mathbf{y}$ s.t. $\mathbf{x} = a\varphi(\mathbf{y})$ with $a \in \{\varepsilon, 0, 1\}$

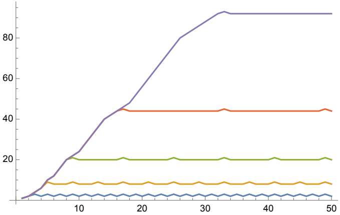
generalization of $\Longleftarrow$

Theorem

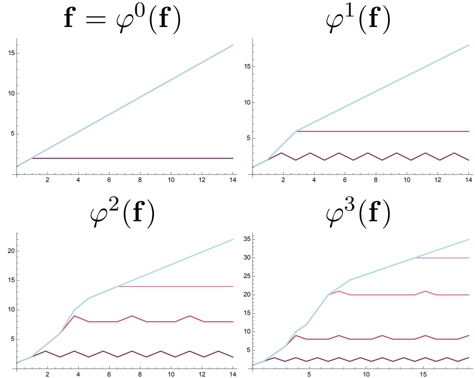
$\mathbf{y}$ aperiodic binary $\Longrightarrow u\varphi^k(\mathbf{y})$ has $\mathcal{P}_k$ ($k \geq 1$, u suffix of $\varphi^k(0)/\varphi^k(1)$)

Example:

Thue–Morse word $\mathbf{t}$



Fibonacci word $\mathbf{f} = 010010100100\cdots$



A partial converse

Definition: $\mathbf{x}$ is recurrent if each factor appears infinitely often

Example:

- recurrent: Thue–Morse, Fibonacci
- not recurrent: $101001000100001\cdots = 1010^2 10^3 10^4 1\cdots$
  e.g. 101 appears only once

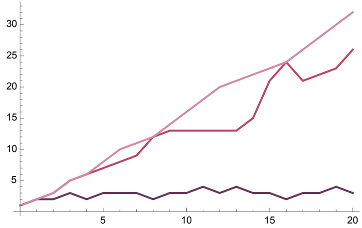
Theorem

$\mathbf{x}$ recurrent binary with $\mathcal{P}_k$ $(k \geq 1)$
$\Longrightarrow \exists \mathbf{y}$ aperiodic s.t. $\mathbf{x} = u\varphi^k(\mathbf{y})$ with u suffix of $\varphi^k(0)/\varphi^k(1)$

Open question: get rid of recurrence

Observation

period-doubling word $\mathbf{pd} = 01000101010001\cdots$ (f.p. of $0 \mapsto 01, 1 \mapsto 00$)



Proposition (Lejeune-Rigo-S. in Lejeune 2021)

$$\mathsf{b}^{(2)}_{\mathbf{pd}}(2^n) = \mathsf{p}_{\mathbf{pd}}(2^n) \quad \forall n \qquad\qquad \mathsf{b}^{(2)}_{\mathbf{pd}}(m) < \mathsf{p}_{\mathbf{pd}}(m) \quad \forall m \neq 2^n$$

Notation: $\mathsf{f}, \mathsf{g} \colon \mathbb{N} \to \mathbb{N}$ functions

$$\mathsf{f} \prec \mathsf{g} \equiv \quad \mathsf{f}(n) \leq \mathsf{g}(n) \;\; \forall n \quad \& \quad \mathsf{f}(n) < \mathsf{g}(n) \text{ for } \infty\text{ly many } n$$

Example: $\mathsf{b}^{(2)}_{\mathbf{pd}} \prec \mathsf{p}_{\mathbf{pd}}$

Question B

binomial complexities are increasingly nested

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$

and $\mathsf{b}_{\mathbf{x}}^{(k)} = \mathsf{p}_{\mathbf{x}}$ ($\forall k \geq 2$) for $\mathbf{x}$ Sturmian, Tribonacci

$\rightsquigarrow$ Can the factor complexity coincide with any binomial complexity?

Question B (Stabilization)

For $k \geq 1$, does there exist $\mathbf{w}_k$ s.t.

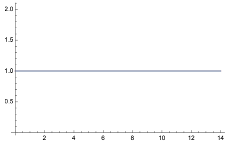
$$\mathsf{b}_{\mathbf{w}_k}^{(1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(2)} \prec \cdots \prec \mathsf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(k)} = \mathsf{p}_{\mathbf{w}_k}?$$

inspired by Lejeune's PhD thesis (2021)

First (naive) answer: periodic words

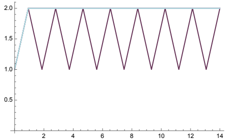
Examples:

$\mathbf{w} = 0\,0\,0\cdots$



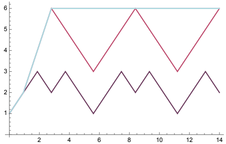
$\mathsf{b}_{\mathbf{w}}^{(1)} = \mathsf{p}_{\mathbf{w}}$

$\mathbf{w} = 01\,01\,01\cdots$



$\mathsf{b}_{\mathbf{w}}^{(1)} \prec \mathsf{b}_{\mathbf{w}}^{(2)} = \mathsf{p}_{\mathbf{w}}$

$\mathbf{w} = 011001\,011001\cdots$



$\mathsf{b}_{\mathbf{w}}^{(1)} \prec \mathsf{b}_{\mathbf{w}}^{(2)} \prec \mathsf{b}_{\mathbf{w}}^{(3)} = \mathsf{p}_{\mathbf{w}}$

but these words have

- bounded complexities
- rather simple structure

more "interesting" words?

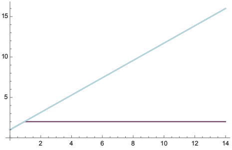
A second answer

Theorem

$k \geq 1 \quad \mathbf{s}$ Sturmian $\Longrightarrow \mathsf{b}^{(1)}_{\varphi^k(\mathbf{s})} \prec \mathsf{b}^{(2)}_{\varphi^k(\mathbf{s})} \prec \cdots \prec \mathsf{b}^{(k+1)}_{\varphi^k(\mathbf{s})} \prec \mathsf{b}^{(k+2)}_{\varphi^k(\mathbf{s})} = \mathsf{p}_{\varphi^k(\mathbf{s})}$

Example: Fibonacci word $\mathbf{f} = 010010100100\cdots$

$\mathbf{f} = \varphi^0(\mathbf{f})$

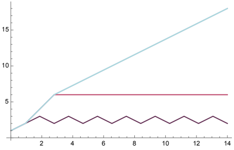


$\mathsf{b}^{(2)} = \mathsf{p}$

True for Sturmian

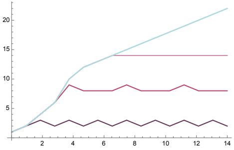
(Rigo–Salimov 2015)

$\varphi^1(\mathbf{f})$



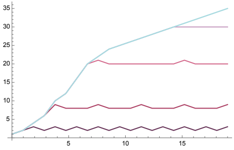
$\mathsf{b}^{(3)} = \mathsf{p}$

$\varphi^2(\mathbf{f})$



$\mathsf{b}^{(4)} = \mathsf{p}$

$\varphi^3(\mathbf{f})$



$\mathsf{b}^{(5)} = \mathsf{p}$

Exact values

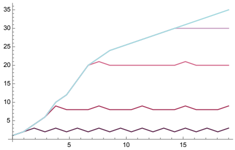
Corollary

$k \geq 1 \qquad \mathbf{s}$ Sturmian

$$\forall 1 \leq j \leq k \qquad \mathsf{b}^{(j)}_{\varphi^k(\mathbf{s})}(n) = \mathsf{b}^{(j)}_{\mathbf{t}}(n) = \begin{cases} \mathsf{p}_{\mathbf{t}}(n) & \text{if } n \leq 2^j - 1 \\ 3 \cdot 2^j - 3 & \text{if } n \equiv 0 \pmod{2^j} \text{ and } n \geq 2^j \\ 3 \cdot 2^j - 4 & \text{otherwise} \end{cases}$$

$$\mathsf{b}^{(k+1)}_{\varphi^k(\mathbf{s})}(2^k n + r) = \begin{cases} \mathsf{p}_{\mathbf{t}}(r) & \text{if } n = 0 \text{ and } 0 \leq r < 2^k \\ 3 \cdot 2^k - 2 & \text{if } n = 1 \text{ and } r = 0 \\ 3 \cdot 2^k + r - 1 & \text{if } n = 1 \text{ and } r > 0 \\ 2^{k+2} - 2 & \text{otherwise} \end{cases}$$

$$\forall j \geq k + 2 \qquad \mathsf{b}^{(j)}_{\varphi^k(\mathbf{s})}(n) = \mathsf{p}_{\varphi^k(\mathbf{s})}(n) = \begin{cases} \mathsf{p}_{\mathbf{t}}(n) & \text{if } n \leq 2^k \\ n + 2^{k+1} - 1 & \text{otherwise (Frid 1999)} \end{cases}$$



… but bounded $\mathsf{b}^{(1)}, \ldots, \mathsf{b}^{(k+1)}$

Unbounded complexities

Question C

For $k \geq 1$, does there exist $\mathbf{w}_k$ s.t. $\mathsf{b}^{(1)}_{\mathbf{w}_k}$ is unbounded and

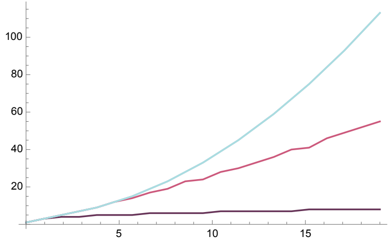
$$\mathsf{b}^{(1)}_{\mathbf{w}_k} \prec \mathsf{b}^{(2)}_{\mathbf{w}_k} \prec \cdots \prec \mathsf{b}^{(k-1)}_{\mathbf{w}_k} \prec \mathsf{b}^{(k)}_{\mathbf{w}_k} = \mathsf{p}_{\mathbf{w}_k}?$$

Answer for $k = 3$:

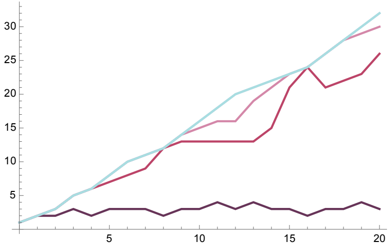
$\mathbf{h} = 0112122122212222122 \cdots$

(f.p. of $0 \mapsto 01, 1 \mapsto 12, 2 \mapsto 2$)

- $\mathsf{b}^{(1)}_{\mathbf{h}}$ unbounded
- $\mathsf{b}^{(1)}_{\mathbf{h}} \prec \mathsf{b}^{(2)}_{\mathbf{h}} \prec \mathsf{b}^{(3)}_{\mathbf{h}} = \mathsf{p}_{\mathbf{h}}$



Conjecture for $k = 4$: period-doubling word $\mathbf{pd} = 01000101010001\cdots$



- $\mathsf{b}^{(1)}_{\mathbf{pd}}$ unbounded (Karhumäki–Saarela–Zamboni 2017)
- $\mathsf{b}^{(1)}_{\mathbf{pd}} \prec \mathsf{b}^{(2)}_{\mathbf{pd}} \prec \mathsf{p}_{\mathbf{pd}}$ (Lejeune-Rigo-S. in Lejeune 2021)
- computer experiments: $\mathsf{b}^{(3)}_{\mathbf{pd}} \prec \mathsf{p}_{\mathbf{pd}}$ and $\mathsf{b}^{(4)}_{\mathbf{pd}} = \mathsf{p}_{\mathbf{pd}}$

Open question: what about larger values of k?

If time permits: Question A

binomial complexities are increasingly nested

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$

Can the non-equality happen ∞ly many times?

Question A

Does there exist $\mathbf{w}$ s.t. $\mathsf{b}_{\mathbf{w}}^{(k)} \prec \mathsf{b}_{\mathbf{w}}^{(k+1)}$ $\forall k \geq 1$?

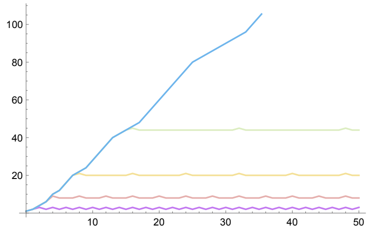
inspired by Lejeune's PhD thesis

A festival of answers: opening act

Thue–Morse word $\mathbf{t}$

- bounded $\mathsf{b}_{\mathbf{t}}^{(k)}$

(Lejeune–Leroy–Rigo 2020)



binary Champernowne word

$\mathbf{c} = 0\,1\,10\,11\,100\,101\,110\,111 \cdots$

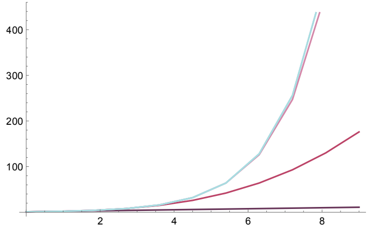
(concatenate all binary representations)

- unbounded $\mathsf{b}_{\mathbf{c}}^{(1)}$
- not morphic

($\tau(f^\omega(a))$ with τ letter-to-letter)

- not uniformly recurrent

(recurrent with bounded gaps)



A festival of answers: more "structured" words

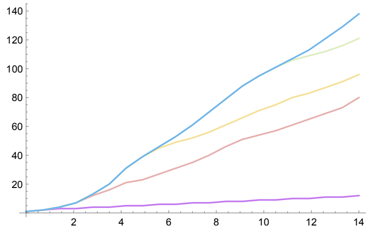
$\mathbf{v} = \tau(f^\omega(a))$

$f\colon a \mapsto a0\alpha, 0 \mapsto 01, 1 \mapsto 10, \alpha \mapsto \alpha^2$

$f^\omega(a) := \lim_{n\to\infty} f^n(a)$

$\tau\colon a \mapsto \varepsilon, 0 \mapsto 0, 1 \mapsto 1, \alpha \mapsto 1$

- unbounded $\mathsf{b}_{\mathbf{v}}^{(1)}$
- binary
- morphic
- not uniformly recurrent



Grillenberger's word

$\mathbf{w} = 0100010101100111\cdots$

- unbounded $\mathsf{b}_{\mathbf{w}}^{(1)}$
- binary
- uniformly recurrent

Construction (Grillenberger 1973)

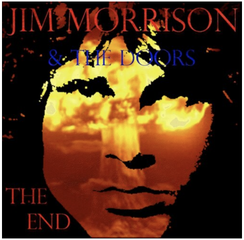
Start: $D_0 = \{0,1\}$ with $0 < 1$

Induction:
w_n = concatenate words in D_n in lex. order
$D_{n+1} = w_n D_n^2$

End: uniformly recurrent $\mathbf{w} = \lim_{n\to\infty} w_n$

n	D_n	w_n
0	$\{0,1\}$	0 1
1	$01\{0,1\}^2$	0100 0101 0110 0111

topological entropy: $\lim_n \frac{\log \mathsf{p}(n)}{n}$



- Characterization 1

  $\mathbf{x}$ Sturmian iff $\exists k \geq 2$ s.t $\mathsf{b}_{\mathbf{x}}^{(k)}(n) = n + 1 \quad \forall n$

- Characterization 2

  f Parikh-collinear iff (bounded $\mathsf{b}_{\mathbf{x}}^{(k)} \implies$ bounded $\mathsf{b}_{\mathbf{x}}^{(k+1)}$)

- Characterization 3

  $\mathbf{x}$ aperiodic, recurrent, binary has $\mathcal{P}_k$ $(k \geq 1)$ iff $\mathbf{x} = u\varphi^k(\mathbf{y})$ with $\mathbf{y}$ aperiodic

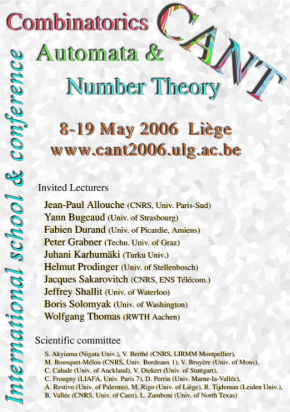
- Question A

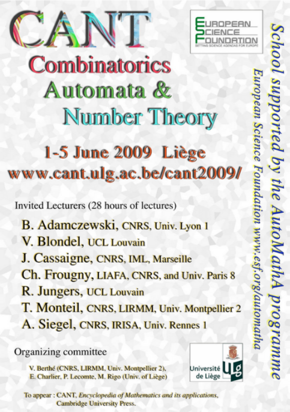
  $\exists \mathbf{w}$ s.t. $\mathsf{b}_{\mathbf{w}}^{(k)} \prec \mathsf{b}_{\mathbf{w}}^{(k+1)}$ $\forall k \geq 1$?

- Questions B-C

  For $k \geq 1$, $\exists \mathbf{w}_k$ s.t. ($\mathsf{b}_{\mathbf{w}_k}^{(1)}$ unbounded and) $\mathsf{b}_{\mathbf{w}_k}^{(1)} \prec \cdots \prec \mathsf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathsf{b}_{\mathbf{w}_k}^{(k)} = \mathsf{p}_{\mathbf{w}_k}$?

Announcement





New CANT school at CIRM 29 September – 3 October 2025

More info soon