Efficient Substructured Domain-Decomposition in Inverse Problems using Krylov Subspace Recycling

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Full Waveform Inversion (in the frequency domain)

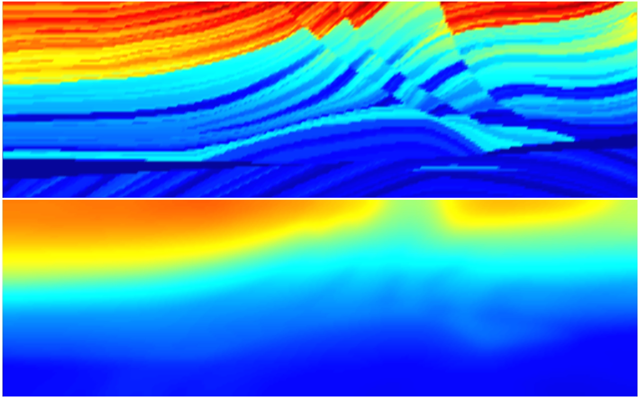

Figure: Slowness squared of the Marmousi model, a common geophysics benchmark for FWI (Target above, initial guess below)

Full Waveform Inversion (in the frequency domain)

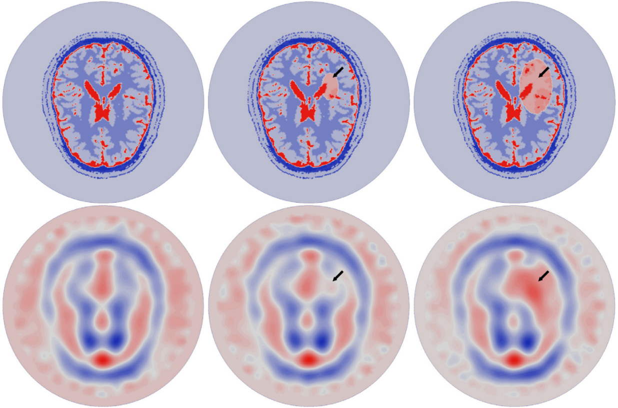

Figure: Imaginary part of permittivity in the brain. *Tournier et al., 2019, Microwave tomographic imaging of cerebrovascular accidents by using high-performance computing*

Full Waveform Inversion (in the frequency domain)

Problem statement: For a model $m(x)$, a wavefield $u(x)$, data d, excitation f and a measurment operator R, find m that minimizes $J(m) = \|Ru(m) - d\|_2^2$ under constraint $A(m)u = f$, where $A(m)$ where A represents the wave equation.

Main cost: solve $A(m)u = f$ for different f and m.

Computing $J(m)$ and $\nabla J(m)$ requires solving 2 wave propagation problems. Then, local optimization (l-BFGS, Newton) is performed.

Domain Decomposition Methods

At large-scale, **direct solvers don't scale** and iterative methods are necessary.
With Domain Decomposition Methods (DDM) we can either:

- Build a preconditioner made of local solves.
- Solve an **interface problem** to glue local solutions together.

Efficient handling of all sources ? Of related models ?

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Substructured DDM

Let g be the vector of interface fields and b the physical sources. The *Additive Schwarz Method* leads to a fixed-point scheme.

Propagated incoming wave

$$g^{n+1} = \tilde{A}g^{n} + b$$

New outgoing wave

Local excitation

Instead, solve with **a Krylov solver** (e.g. GMRES, GCR) $Ag = b$ with $A = I - \tilde{A}$.

Substructured DDM

Some properties of the interface problem:

- Significantly lower number of unknowns than the volume problem
- Eigenvalues are in the unit ball centered on 1.
- One matrix-vector product involves solving each subproblem once

Solving all subproblems is the most computationally expensive part.

Efficient FWI

Possible optimizations:

- Handle multiple sources (10, 100, 1000 ?)
- Recycle information when the model changes ?

Example problem

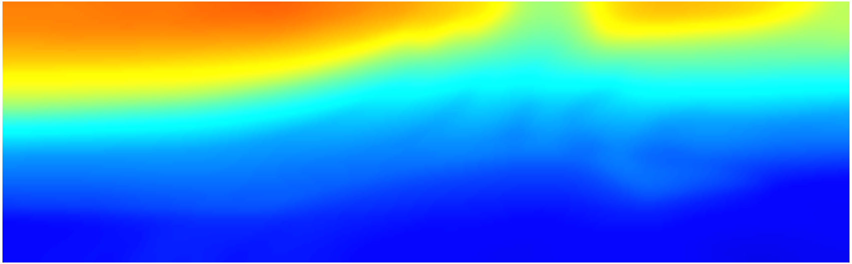
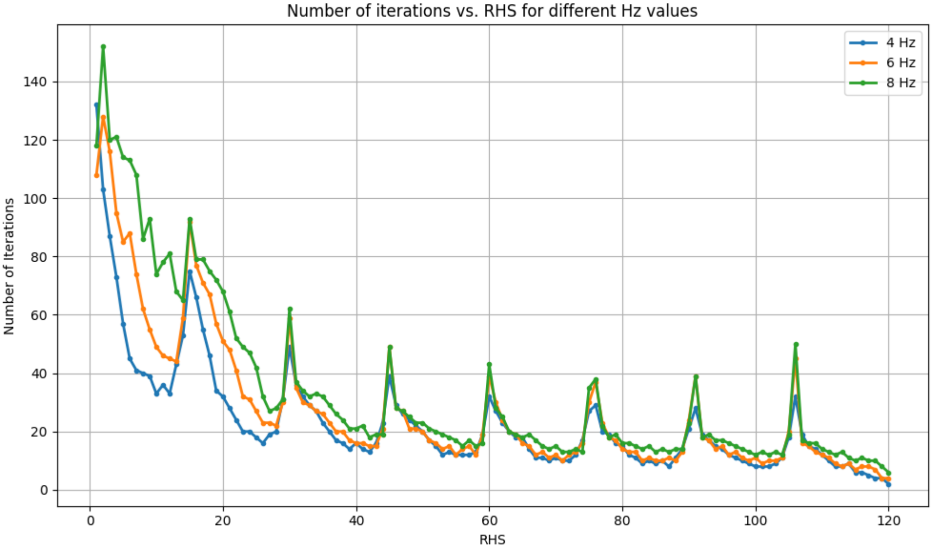


Figure: Test case: 120 sources close to the top. 8x3 subdomains.

Sequential Subspace Recycling

1. Solve $Ag_1 = b_1$ with a solver like GCR.
   - Build an A^*A orthonormal basis of $\mathcal{K}(A, b_1)$: $AU = C, C^*C = I$.
   - Approximation is $g_1 = UC^*b_1$.
2. Reuse this basis **and** expand it to solve $Ag_2 = b_2$.
3. Carry on and accumulate all these search directions.

Results


Number of iterations vs. RHS for different Hz values
4 Hz
6 Hz
8 Hz
Number of Iterations
RHS
0
20
40
60
80
100
120
140
0
20
40
60
80
100
120

Block Krylov Methods

- Solve everything at once, and use the subspace of each RHS in all resolutions.
- Expensive in memory...
- ... But the substructuring makes this bearable!

Influence of block size

Block size	1	10	20	30	60	120
4 Hz	15 580	10 120	7940	6990	5460	3480
6 Hz	12 886	9 930	7700	6810	5580	3600
8 Hz	13 871	11 250	8760	7560	6060	3840

Table: Number of local solves to solve 120 sources on the Marmousi initial model

Results

Table: Number of local solves to solve 120 sources on the Marmousi initial model

	Reference	Recycling	Block	Block deflated
4 Hz	15580	2785	3480	2991
6 Hz	12886	3337	3600	3032
8 Hz	13871	4039	3840	3118

Varying operator

Output of GCR after k steps: directions $U, C \in \mathbb{C}^{n\times k}$ with $AU = C, C^*C = I$.
$\rightarrow UC^*$ is a rank-k approximation of A^{-1}
How to reuse data for a new model ? **Use this approximate inverse as preconditioner** ($+$ correction to make it non-singular)

$$M^{-1} = (I - CC^*) + UC^* = I + (U - C)C^*.$$

Varying operator: Results

Table: Number of local solves to solve 120 sources on 4 similar models.

	Reference	Recycling only	Preconditioner (from 1st)
4 Hz	57 956	12 475	9 496
6 Hz	55 415	16 363	13 759
8 Hz	60 969	20 630	17 440

Truncated Newton methods

Inexact Newton: minimize $J(m)$ by solving with Conjugate Gradients :

$$H\Delta m = -\nabla J.$$

H is the Hessian, and computing Hv for a given v requires 2 additional solves.
The operator is constant but right hand sides are not all available at once
$\rightarrow$More work on the same operator $=$ better recycling

Truncated Newton methods

Perturbed Forward Problem: for a given δA and previously computed u, find δu such that

$$A\delta u = -\delta A u.$$

It's the derivative of the the wavefield with respect to a perturbation.

Truncated Newton methods - Results

For a given model and 5 perturbations: 6 sequences of 120 RHS.

Table: Number of local solves to solve 120 sources and 120×5 perturbations on the Marmousi reconstructed model (3rd iteration)

	Reference	Recycling	Block of 120
4 Hz	94 519	4 246	20 280
6 Hz	78 787	5 778	21 840
8 Hz	87 566	7 362	25 080

Conclusion and future work

- Substructured DDM allows for **greedy subspace recycling**, yielding **fast** convergence.
- Newton methods *could* become more competitive than l-BFGS in this context.

Future work:

- Efficient implementation.
- Further research on changing operator.
- Extension to 3D, more complex equations (Maxwell!).
- Comparison with "DDM as a preconditioner" methods.

References

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2. François-Xavier Roux and A. Barka. *Block Krylov Recycling Algorithms for FETI-2LM Applied to Three-Dimensional Electromagnetic Wave Scattering and Radiation*. IEEE Transactions on Antennas and Propagation, Vol. PP, 1-1, Feb 2017. DOI: 10.1109/TAP.2017.2670541
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