Parametric upper and lower bounds of linear variations of a linear problem's LHS

EURO24: DTU, Denmark

Bardhyl Miftari, Guillaume Derval, Quentin Louveaux, Damien Ernst

July 2024

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Sensitivity analysis for linear changes of the constraint matrix of a linear program

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LIÈGE université

Uncertainty

Linear programming

- In LP optimization
  - Formalize problem in terms of
    - Constraints
    - Objective function
  - Get one optimal solution

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Parameters (Constraints, Objective function)

Uncertainty

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  - Get one optimal solution

Parameters ← Hypothesis

Uncertainty

Linear programming

- In LP optimization
  - Formalize problem in terms of
    - Constraints
    - Objective function
  - Get one optimal solution → Further assessment and analysis

Constraints, Objective function ← Parameters ← Hypothesis

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{aligned} \min \quad & c^t x \\ \text{s.t} \quad & Ax \leq b \end{aligned}$$

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{aligned} \min \quad & c^t x + \lambda c_\lambda^t x \\ \text{s.t} \quad & Ax \leq b \end{aligned}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & Ax \leq b + \lambda b_\lambda \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x + \lambda c_\lambda^t x \\ \text{s.t} & (A + \lambda D)x \leq b + \lambda b_\lambda \end{array}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$
- Modification on the left-hand side $+\lambda D$

Formalization

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} Discussed a LOT in the literature

- Modification on the left-hand side $+\lambda D$

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- Modification on the left-hand side $+\lambda D$ → Not much ?

Note : The left-hand side modification $+\lambda D$ encapsulates the other modifications

Uncertainty

Example

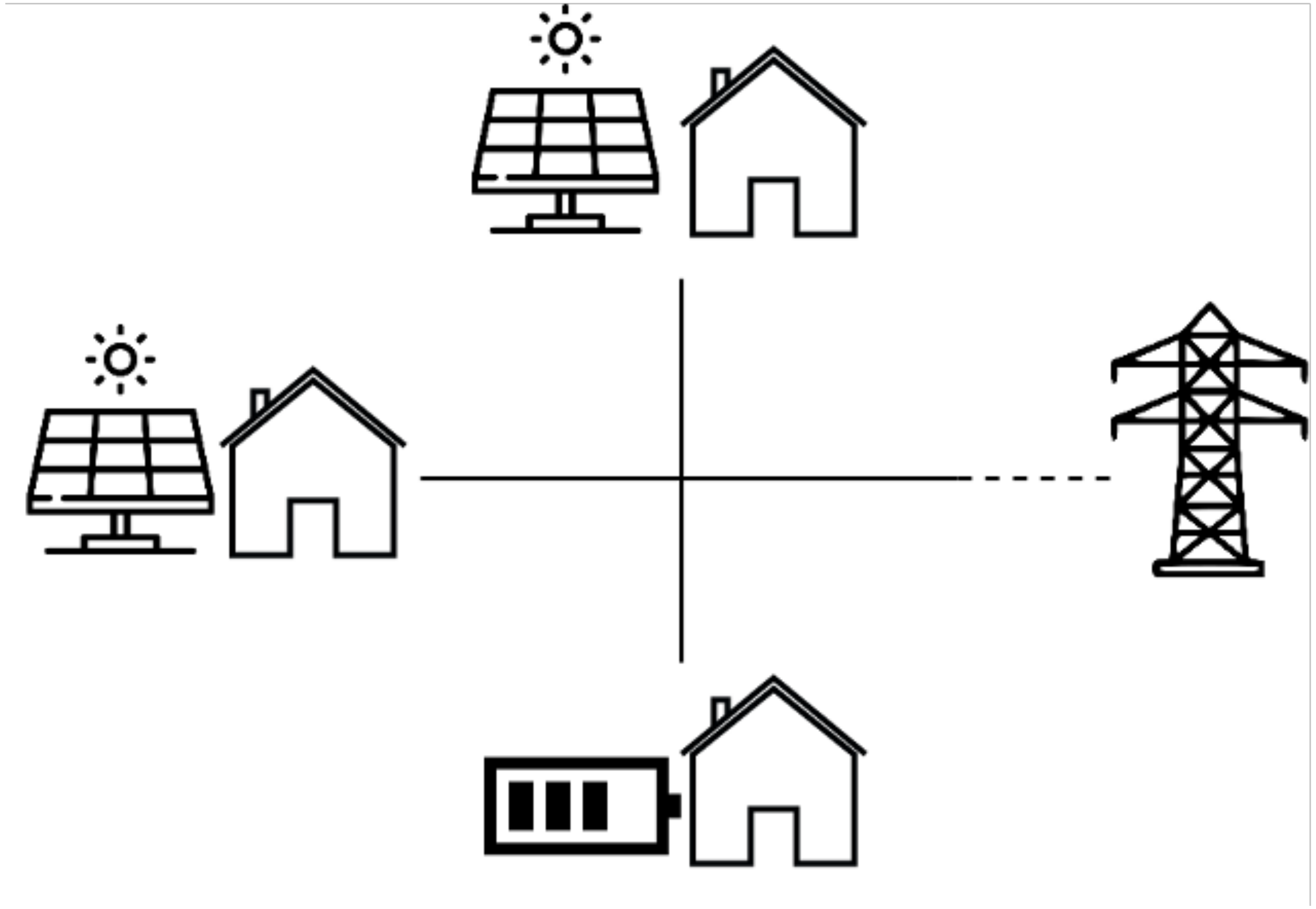


Figure 1: Renewable energy community

Uncertainty

Example

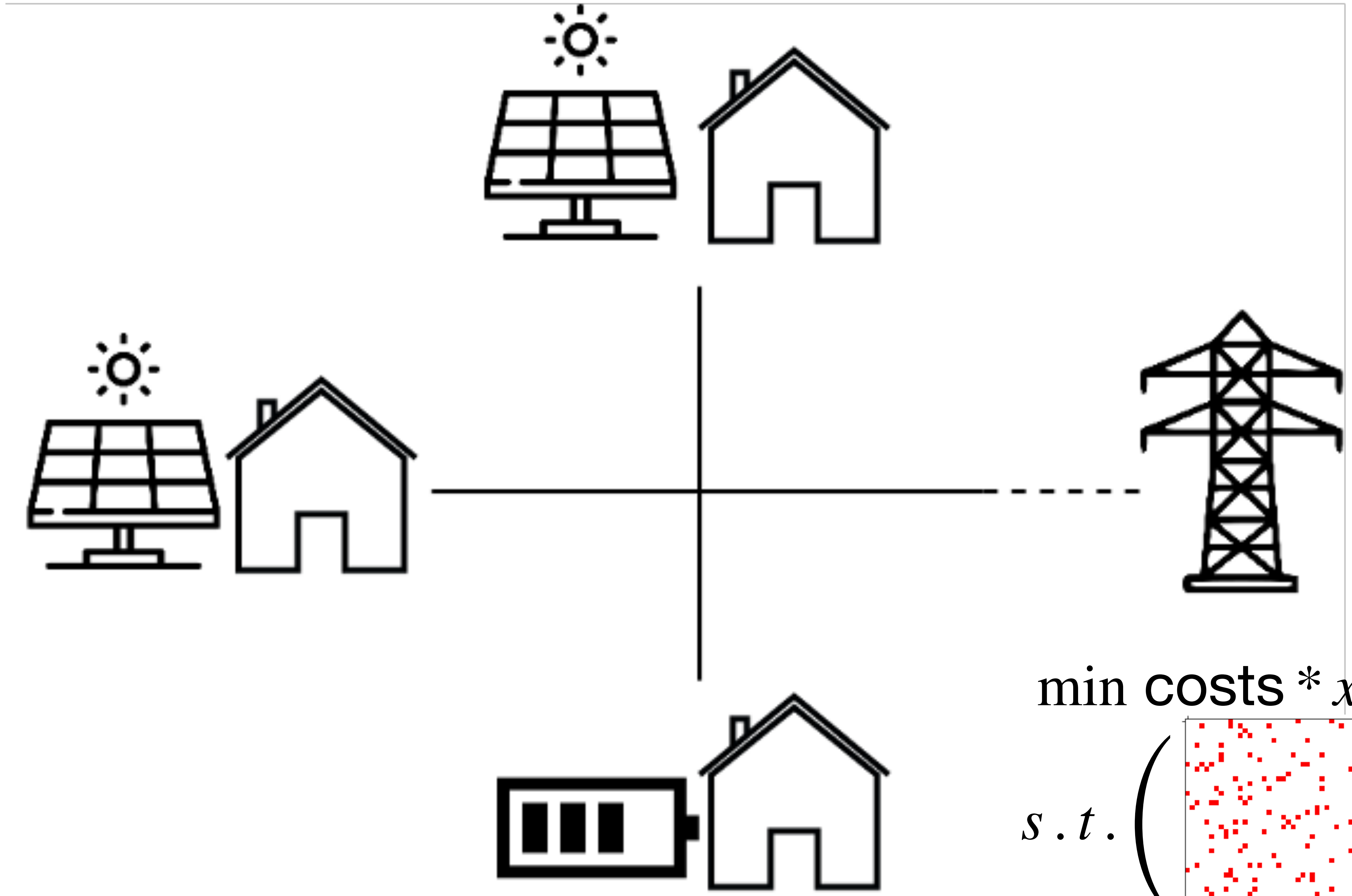
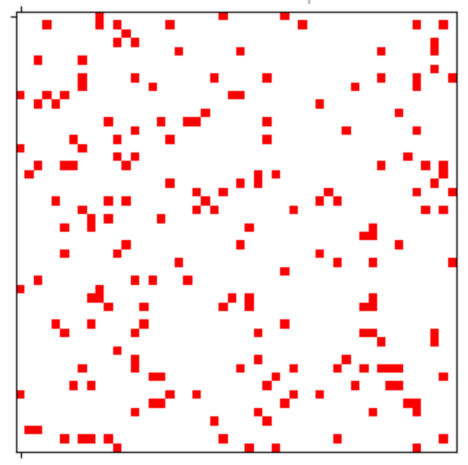


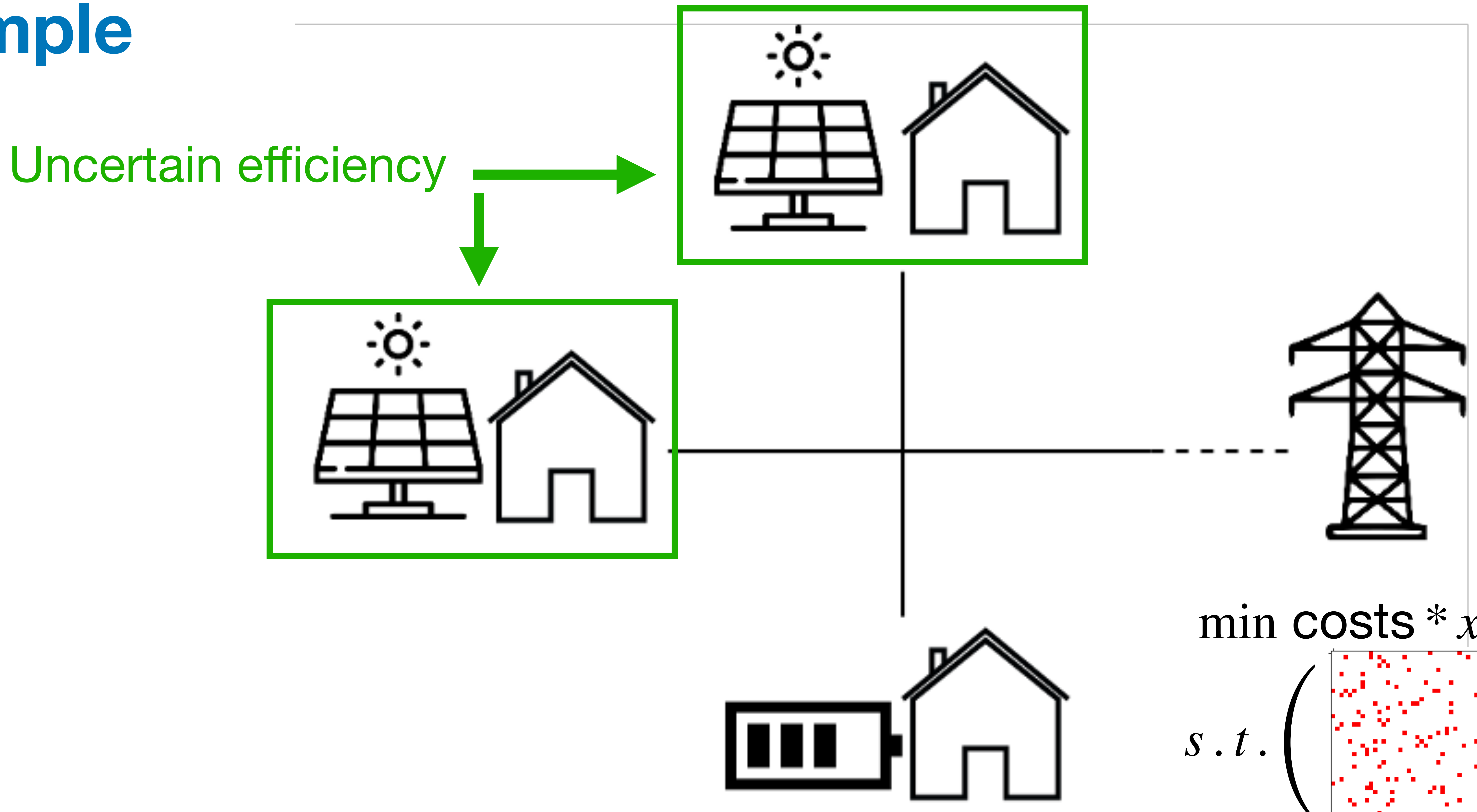
Figure 1: Renewable energy community

$$\min \text{costs} * x$$

$$s.t. \left(\right.$$  $$\left. \right) x \leq \begin{pmatrix} \text{demand} \\ \text{physicial limits} \end{pmatrix}$$

Uncertainty

Example



$$\min \text{costs} * x$$

$$s.t. \left(\cdot \right) x \leq \begin{pmatrix} \text{demand} \\ \text{physicial limits} \end{pmatrix}$$

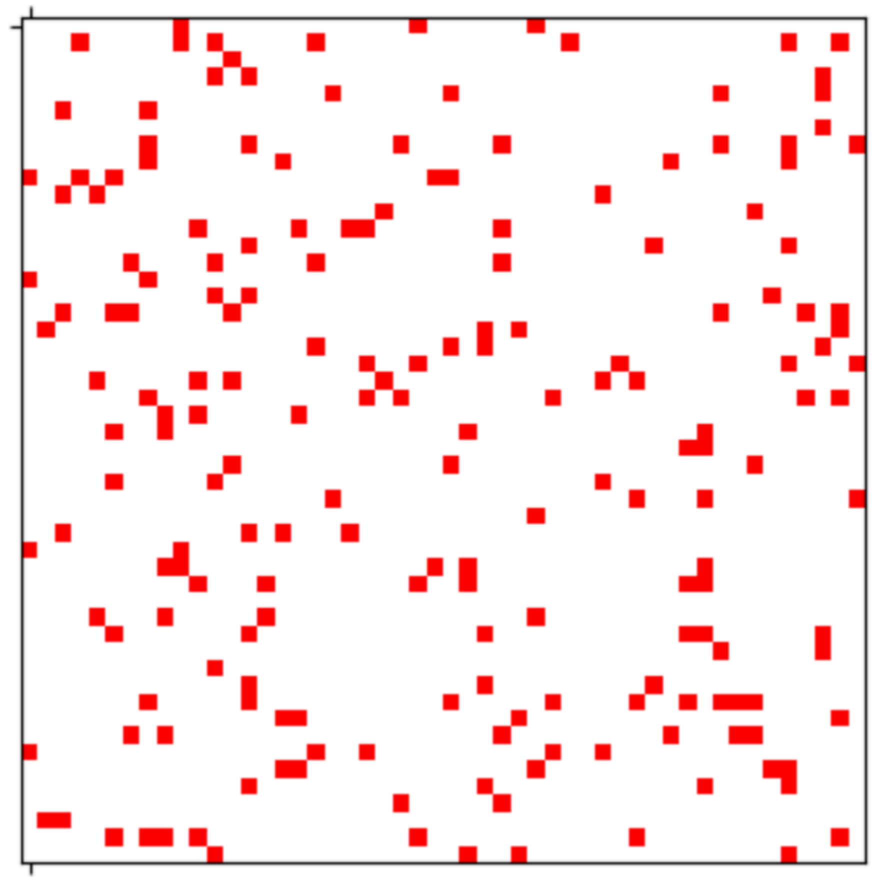
Figure 1: Renewable energy community

Uncertainty

Example

$$\min \text{costs} * x$$

$s.t.$



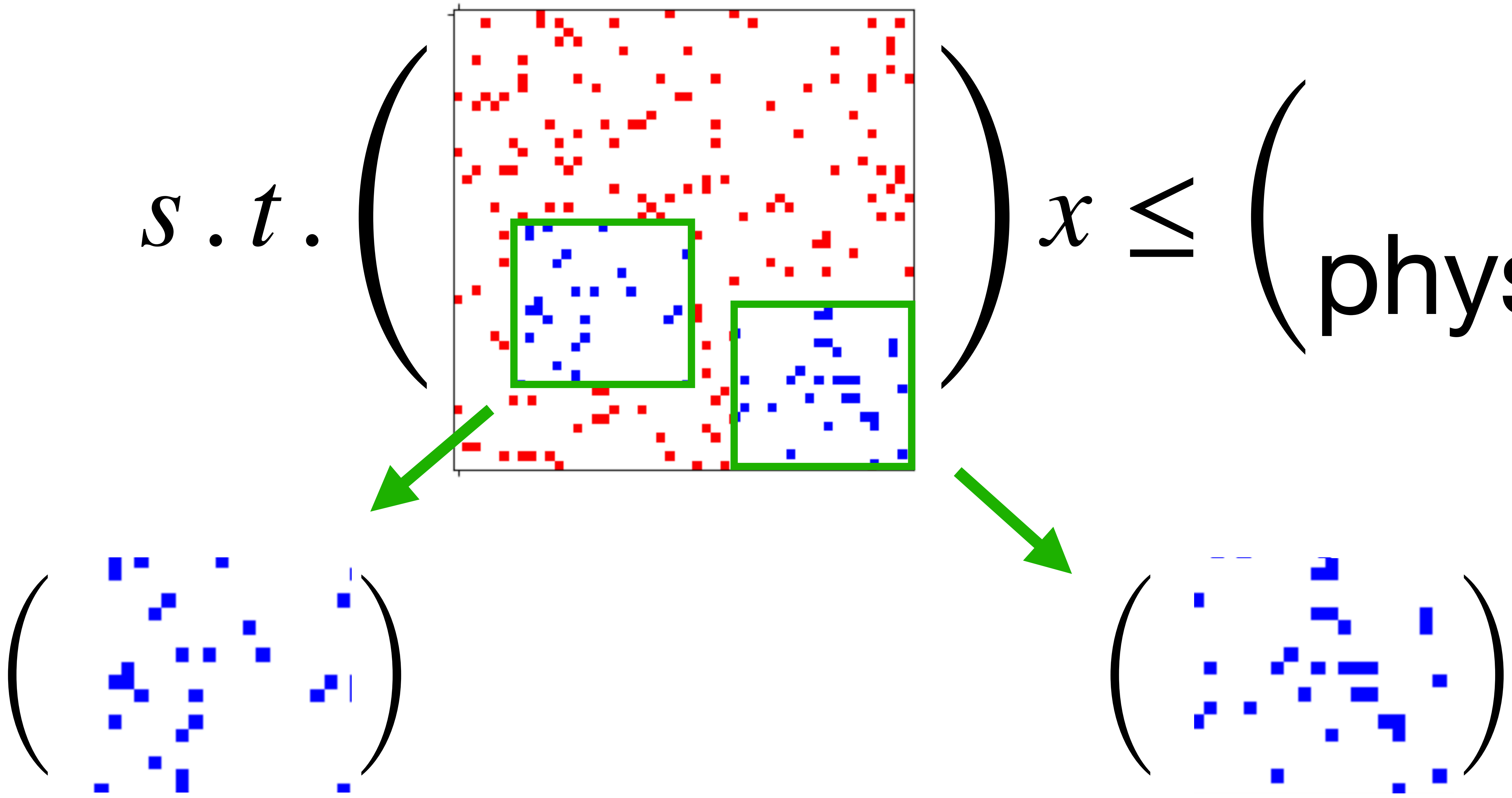
$$x \leq \begin{pmatrix} \text{demand} \\ \text{physicial limits} \end{pmatrix}$$

Uncertainty

Example

$$\min \text{costs} * x$$

$$s.t. \left(\; \cdot \; \right) x \le \begin{pmatrix} \text{demand} \\ \text{physicial limits} \end{pmatrix}$$



Formalization

Parametric uncertainty

- Our problem formalization

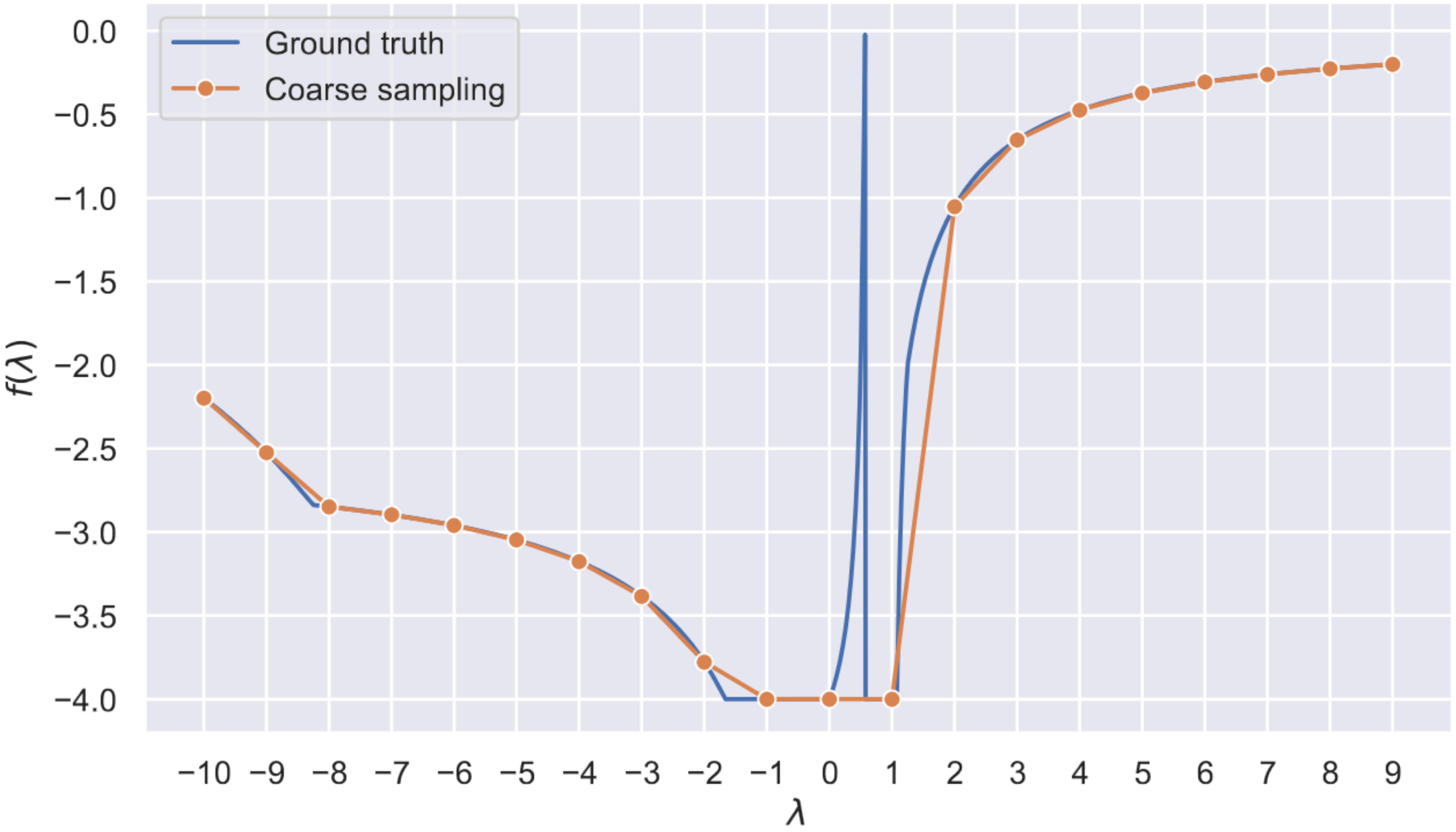
$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \le b_1 \\
& A_2 x + \lambda D x \le b_2
\end{aligned}
$$

For $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

- In literature :
  - Usually rely on heavy computation,
  - approximations
  - and/or hypothesis on the matrix D.
  - Do not extend to bigger problems

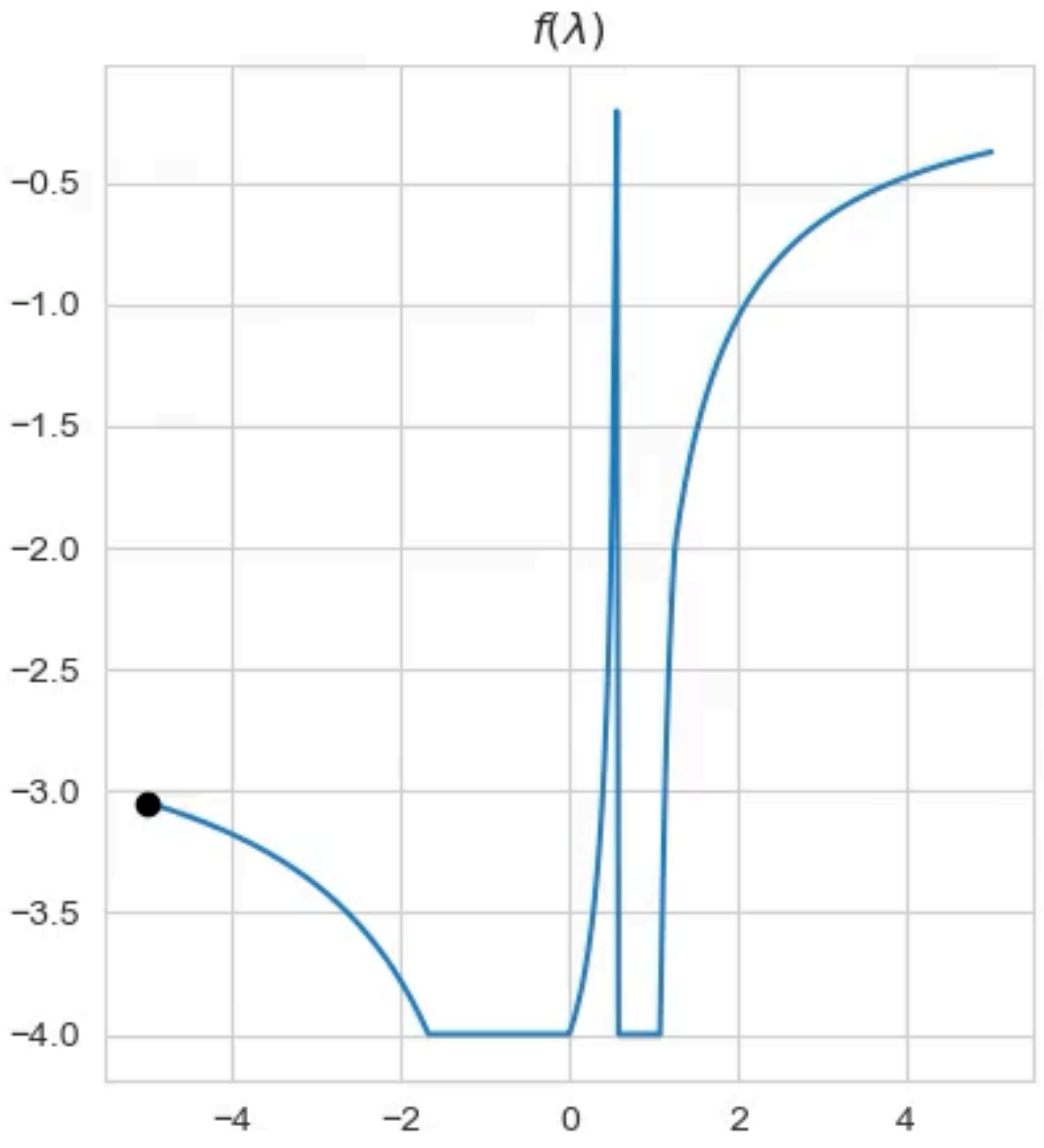
Naive solution

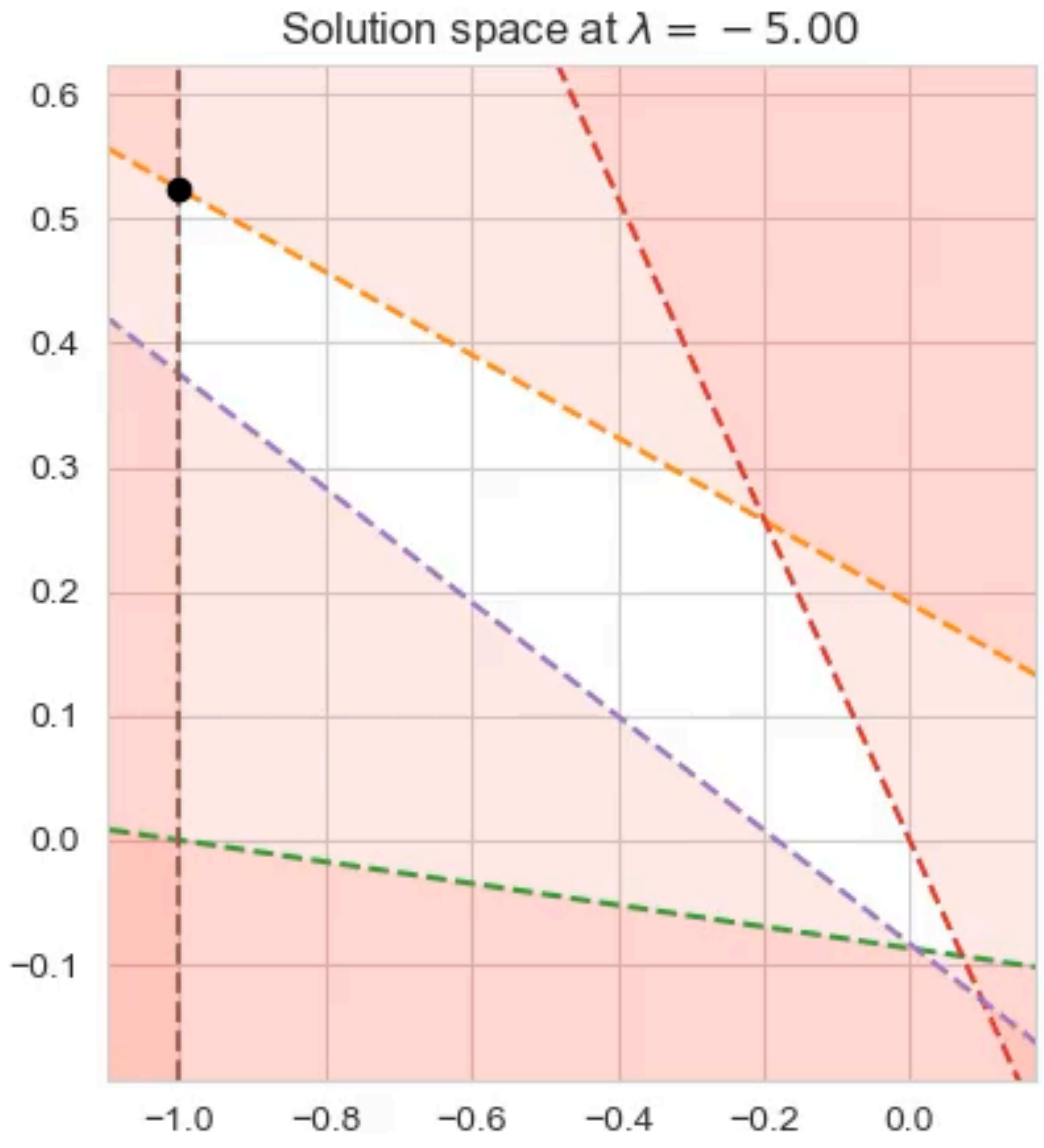
Heavy computations



Naive solution

Heavy computations





Bounds

Idea

$$lb(\underline{\lambda}, \overline{\lambda}) \leq \quad f(\lambda) = \min \quad c^t x \quad \leq ub(\underline{\lambda}, \overline{\lambda})$$

$$s.t. \quad A_1 x \leq b_1$$

$$A_2 x + \lambda D x \leq b_2$$

$$\lambda \in [\underline{\lambda}, \overline{\lambda}]$$

If we find a lower bound (resp. upper bound) to the primal, it becomes an upper (resp. lower) bound on the dual

Constant bounds

Approaches

- Get an upper bound using robust reformulations

$$
\begin{aligned}
f(\lambda) \leq \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda_1 D x \leq b_2 \\
& A_2 x + \lambda_2 D x \leq b_2
\end{aligned}
$$

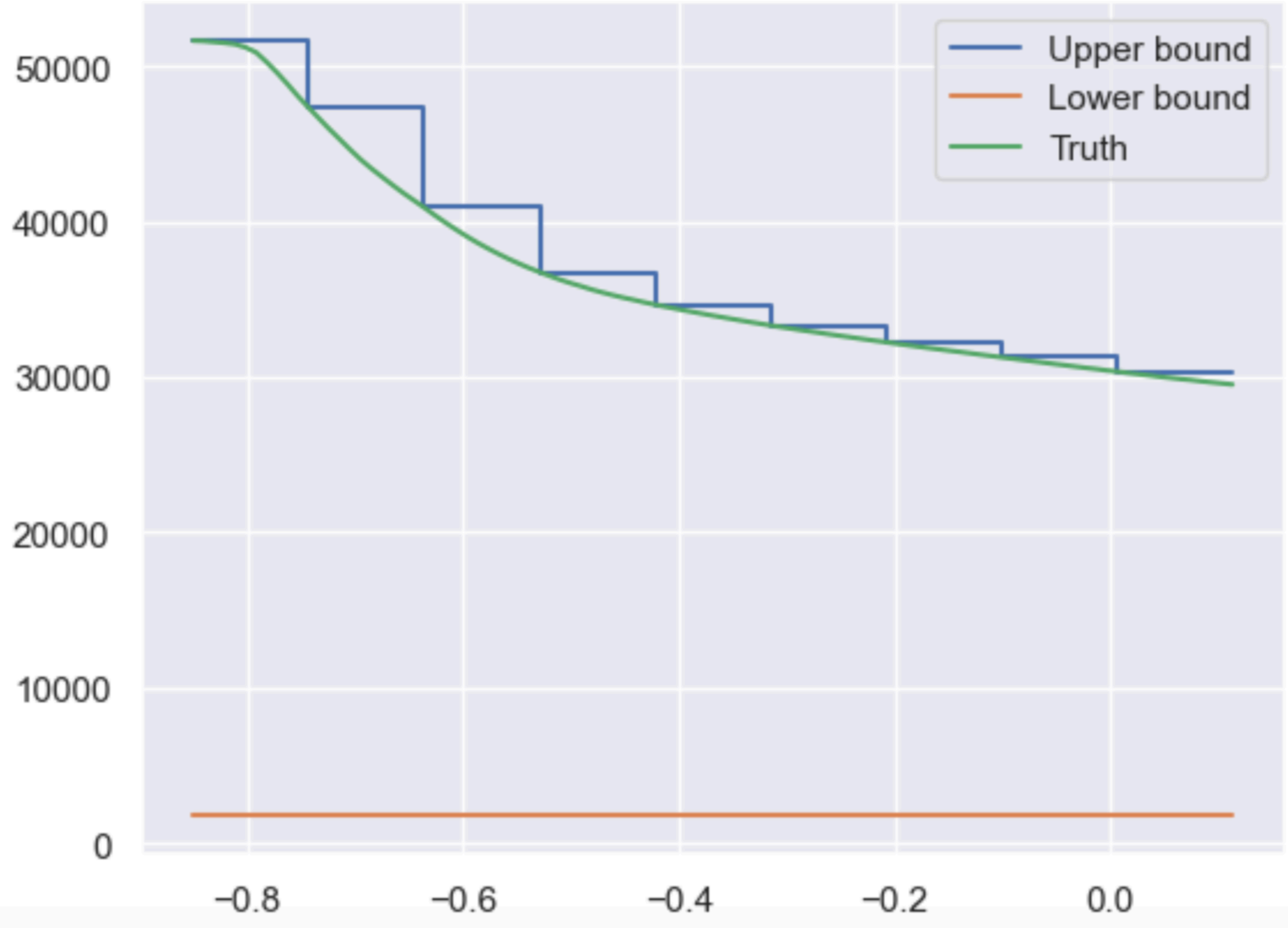
- Get a lower bound using Lagrangian relaxations

$$
\begin{aligned}
f(\lambda) \geq h(\lambda, \alpha) = \min \quad & c^t x + \alpha(A_2 x + \lambda D x - b_2) \\
s.t. \quad & A_1 x \leq b_1
\end{aligned}
$$

$$
\max(\min(f(\lambda_1), h(\alpha_1, \lambda_2)), \min(f(\lambda_2), h(\alpha_2, \lambda_1))) \leq \min f(\lambda)
$$

Constant solutions

Flat bounds



Variable solutions

Robust bounds

- We reformulate all the variables x by a linear function $y + \lambda z$
- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2
\end{aligned}
\quad \leq \quad
\begin{aligned}
\min \quad & c^t(y + \lambda z) \\
s.t. \quad & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (D y + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$

Variable solutions

Robust bounds

- We reformulate all the variables x by a linear function $y + \lambda z$ (we add degrees of freedom)
- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$
f(\lambda) = \min \quad c^t x \quad s.t. \quad A_1 x \leq b_1, \quad A_2 x + \lambda D x \leq b_2
$$

$$
\leq
$$

$$
\begin{aligned}
\min \quad & c^t(y + \lambda z) \\
s.t. \quad & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
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\end{aligned}
$$

Two optimization variable robust

Robust bounds

$$
\begin{aligned}
\min \quad & \underline{c^t(y + \lambda z)} \\
s.t. \; & A_1 y + \lambda_1 A_1 z \leq b_1 \\
& A_1 y + \lambda_2 A_1 z \leq b_1 \\
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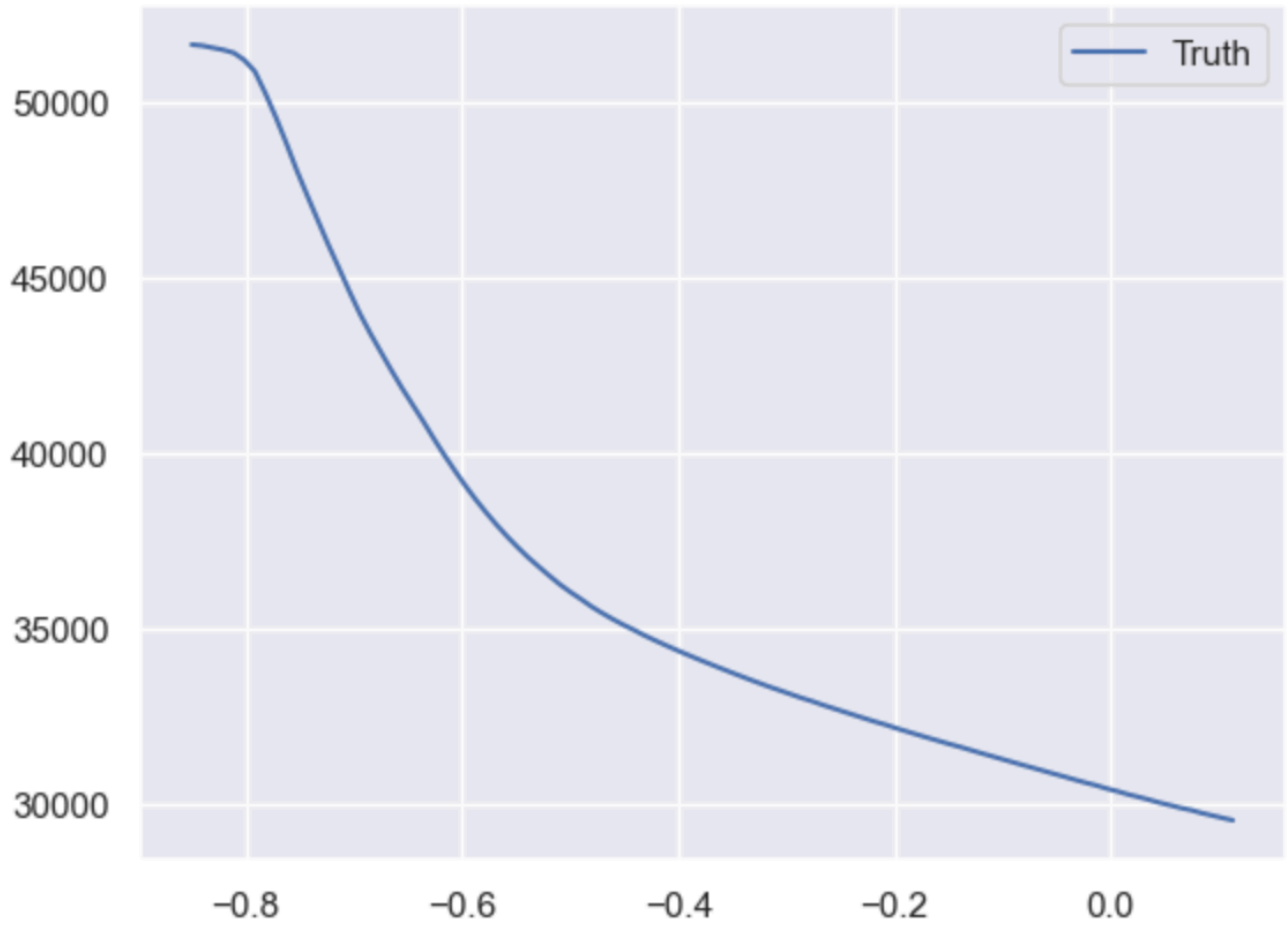
$$



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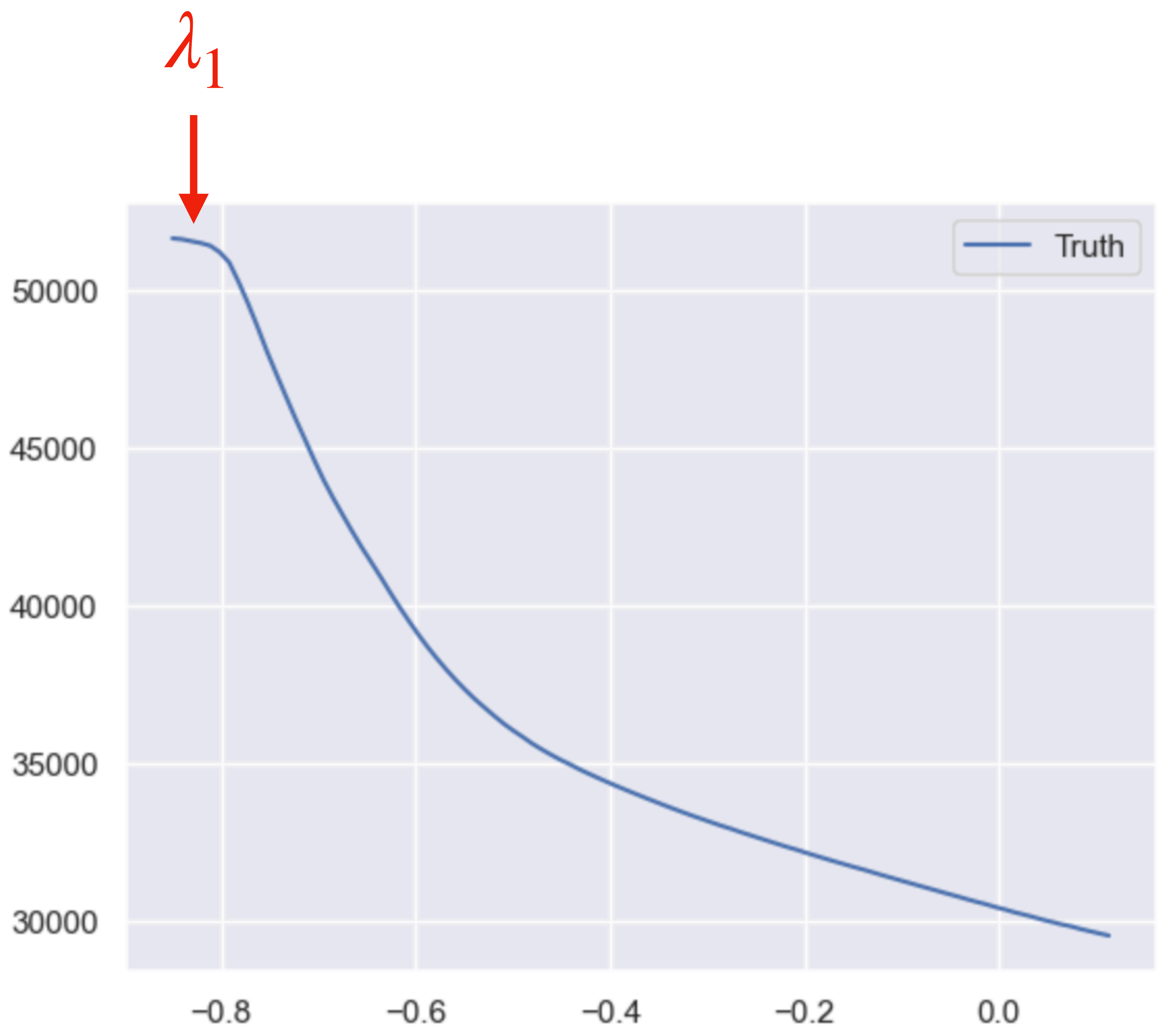

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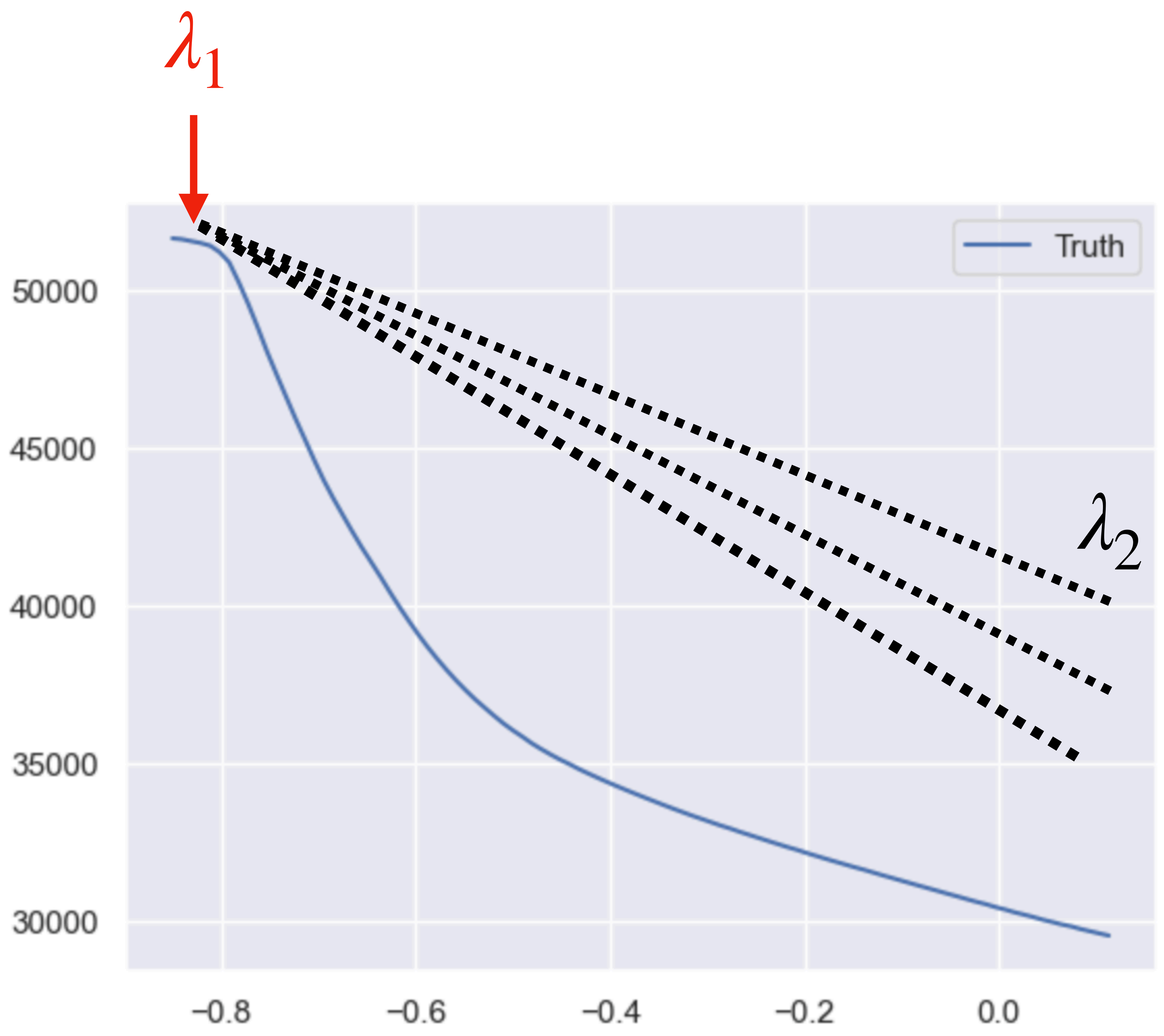
$$



Fixed-slope variable robust

Robust bounds

$$
\begin{aligned}
\min \quad & c^t(y + \lambda z) \\
s.t. \; & A_1 y + \lambda_1 A_1 z \le b_1 \\
& A_1 y + \lambda_2 A_1 z \le b_1 \\
& (A_2 + \lambda_1 D)(y + \lambda_1 z) \le b_2 \\
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \le b_2 \\
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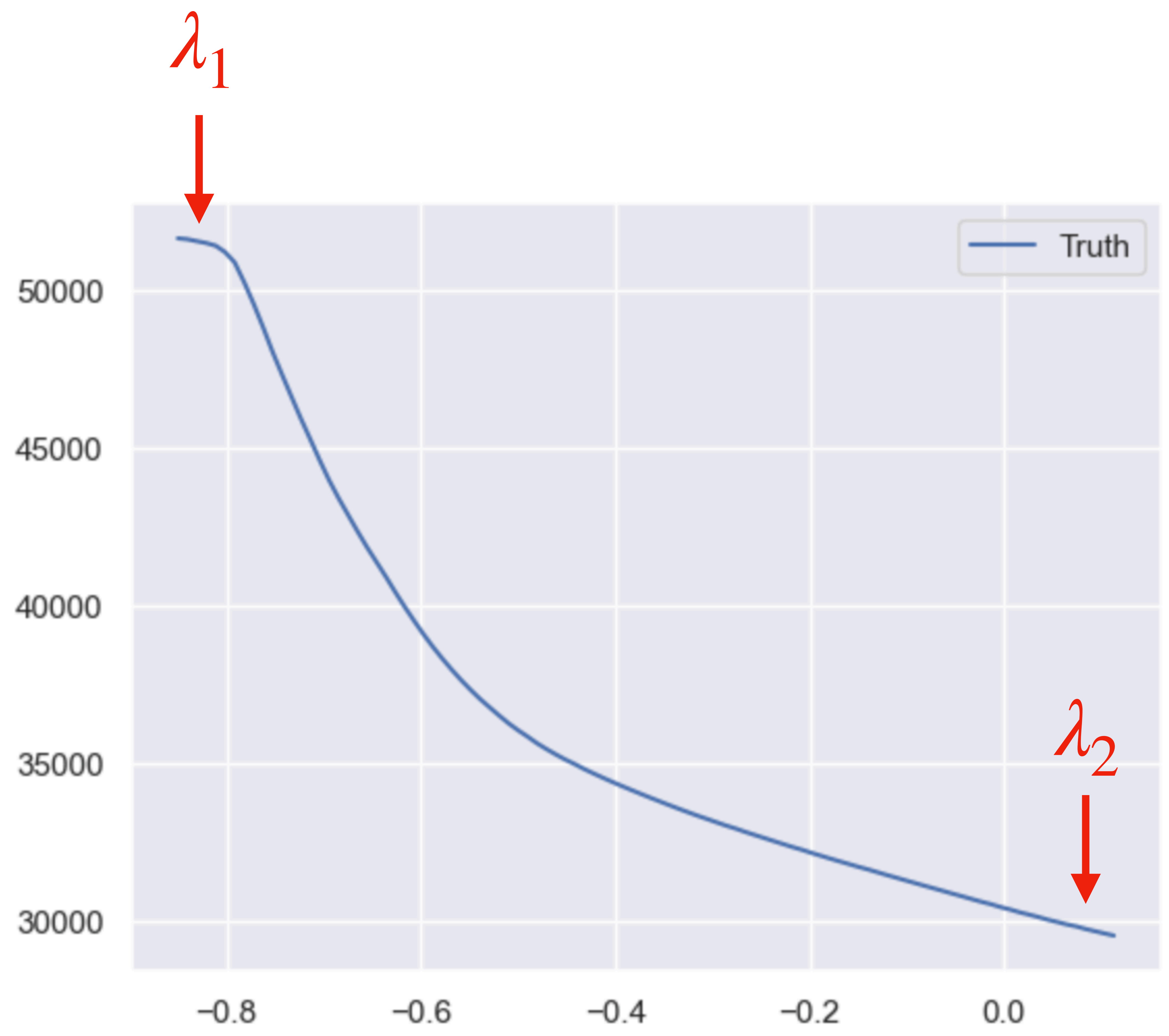
\end{aligned}
$$



Fixed-slope variable robust

Robust bounds

$$
\begin{aligned}
\min \quad & c^t(y+\lambda z) \\
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& A_1 y + \lambda_2 A_1 z \leq b_1 \\
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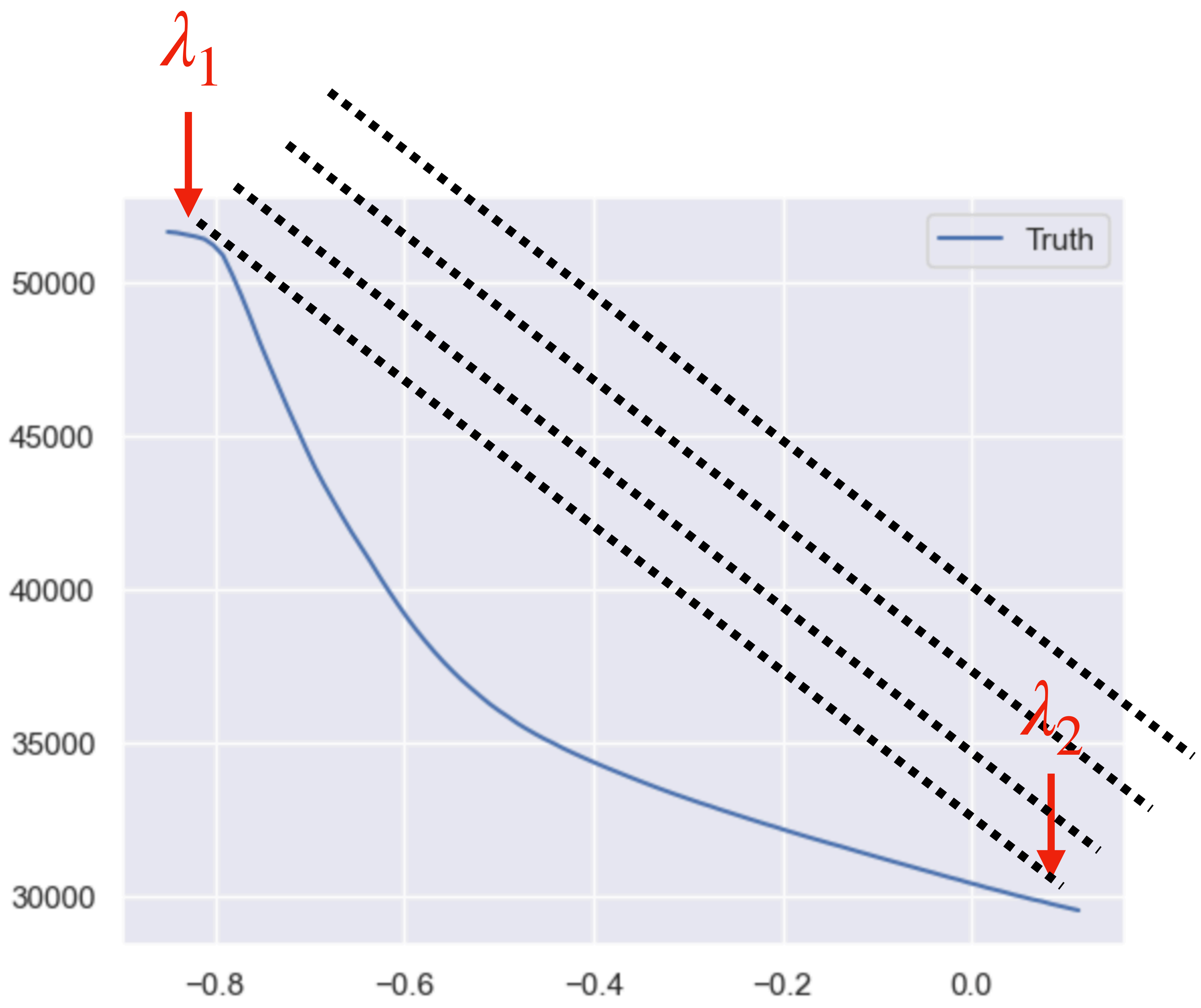
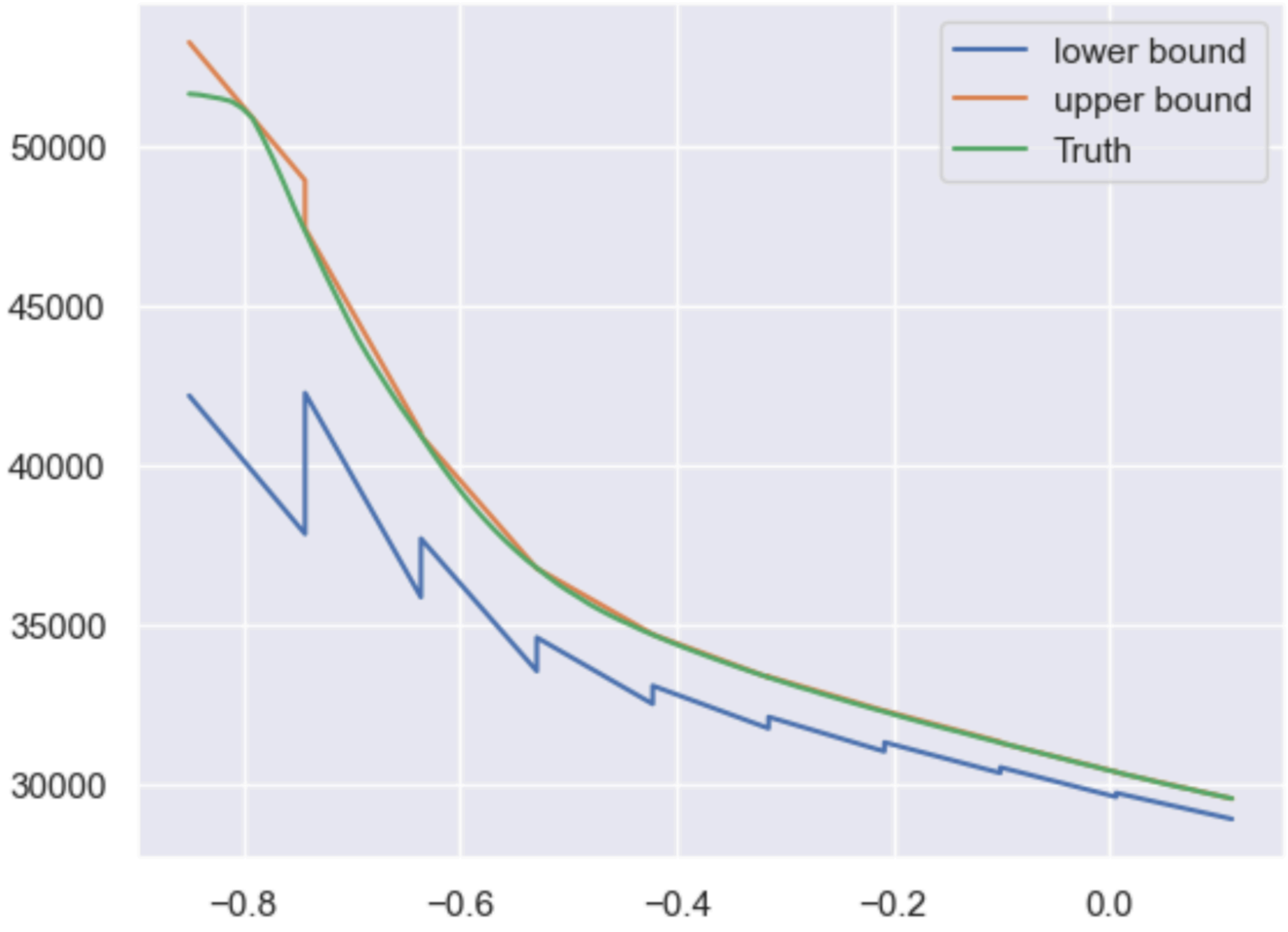
& (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\
& A_2 y + D z \lambda_1 \lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2
\end{aligned}
$$



Variable robust solution

Robust bounds



Lagrangian bound

Variable Lagrangian relaxation

- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$
\begin{aligned}
h(\rho(\lambda), \lambda) \geq & \min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\lambda^{n+1}\rho_n D\mathbf{x} \\
& + \sum_{i=1}^{n} \left(\min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\lambda^{i}(\rho_{i-1} D\mathbf{x} + \rho_i(A_2\mathbf{x} - \mathbf{b_2})) \right) \\
& + \min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\rho_0(A_2\mathbf{x} - \mathbf{b_2}) + \mathbf{c}^t\mathbf{x}
\end{aligned}
$$

- If we fix $n = 0$ -> Linear Lagrangian bound
- If we fix $n = 1$ -> Quadratic Lagrangian bound

Lagrangian bound

Variable Lagrangian relaxation

- Let us consider $\lambda \in [\lambda_1, \lambda_2]$

$$h(\rho(\lambda), \lambda) \geq \min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\lambda^{n+1}\rho_n D\mathbf{x}$$
$$+ \sum_{i=1}^{n} \left(\min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\lambda^{i}(\rho_{i-1}D\mathbf{x} + \rho_i(A_2\mathbf{x} - \mathbf{b_2})) \right)$$
$$+ \min_{\mathbf{x}|A_1\mathbf{x} \leq b_1} -\rho_0(A_2\mathbf{x} - \mathbf{b_2}) + \mathbf{c}^t\mathbf{x}$$

- If we fix $n = 0$ -> Linear Lagrangian bound
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Recall

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{aligned} \min \quad & c^t x + \lambda c_\lambda^t x \\ \text{s.t} \quad & (A + \lambda D)x \leq b + \lambda b_\lambda \end{aligned}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$
- Modification on the right-hand side $+\lambda b_\lambda$

} Discussed a LOT in the literature

- Modification on the left-hand side $+\lambda D$ → Not much ?

Note : The left-hand side modification $+\lambda D$ encapsulates the other modifications

Envelope bound

Robust and Lagrangian

Robust variable

$$\begin{aligned} \min \quad & c^t(y + \lambda z) \\ s.t. \; & A_1 y + \lambda_1 A_1 z \leq b_1 \\ & A_1 y + \lambda_2 A_1 z \leq b_1 \\ & (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \\ & (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\ & A_2 y + Dz\lambda_1\lambda_2 + (Dy + A_2 z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2 \end{aligned}$$

Lagrangian variable

$$\min_{\mathbf{x} | A_1\mathbf{x} \leq b_1} -\lambda^{n+1}\rho_n D\mathbf{x}$$

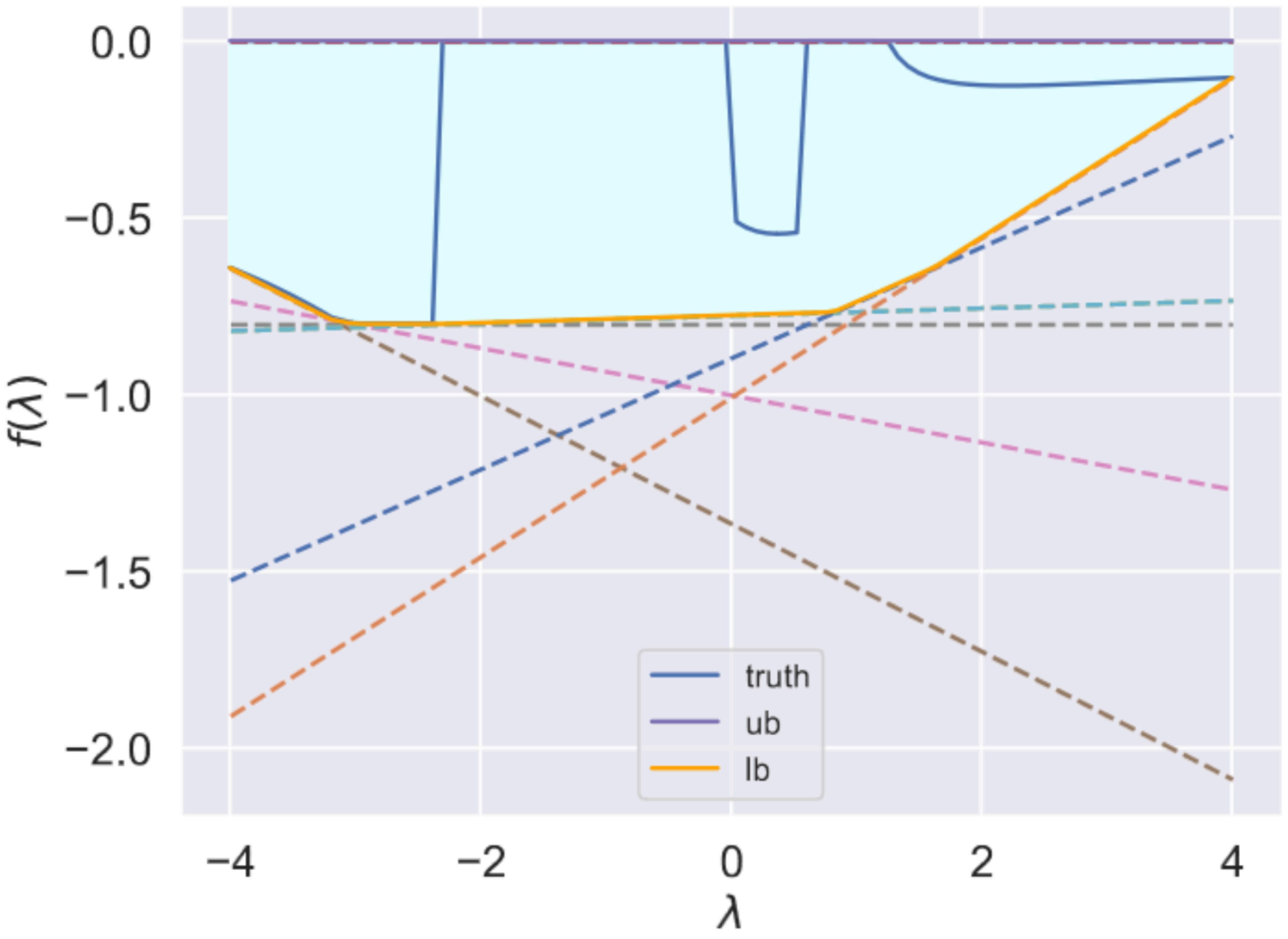
$$+ \sum_{i=1}^{n} \left(\min_{\mathbf{x} | A_1\mathbf{x} \leq b_1} -\lambda^i(\rho_{i-1} D\mathbf{x} + \rho_i(A_2\mathbf{x} - \mathbf{b_2})) \right)$$

$$+ \min_{\mathbf{x} | A_1\mathbf{x} \leq b_1} -\rho_0(A_2\mathbf{x} - \mathbf{b_2}) + \mathbf{c}^t\mathbf{x}$$

- Idea : As the λ term only appears in the objective -> use warm-starting to combine bounds

Envelope bound

Robust envelope



Benchmarking

Protocol

- Generate a dataset of problems
    - 4 toy problems
    - 3 energy real-life problems
    - 191 modified problems from Netlib
- We consider the naive solution of generating 100 points
- We cut the range of variation in 1, 5, 10 equal parts
- Benchmark the problems in terms of
    - Available : percentage points where a bound exists
    - Error : using normalized RMSE +1
    - Timing : time to compute the bound

Benchmarking

Average availability

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	0.02	0.02	0.02	0.03	0.04	0.04
Lagrangian flat	0.11	0.24	0.29	0.07	0.15	0.18
Lagrangian line	0.05	0.08	0.08	0.01	0.02	0.02
Lagrangian quadratic	0.04	0.06	0.06	0.00	0.00	0.00
Robust concave envelope	0.42	0.73	0.81	0.29	0.57	0.64
Robust fixed slope pairwise	0.41	0.73	0.80	0.28	0.56	0.63
Robust flat	0.19	0.32	0.35	0.24	0.42	0.46
Robust line left	0.40	0.67	0.75	0.28	0.53	0.62
Robust line right	0.40	0.69	0.76	0.27	0.54	0.61
Robust yzflat	0.42	0.73	0.81	0.29	0.57	0.65

Benchmarking

Average availability

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	0.02		0.02	0.03		0.04
Lagrangian flat	0.11		0.29	0.07		0.18
Lagrangian line	0.05		0.08	0.01		0.02
Lagrangian quadratic	0.04		0.06	0.00		0.00
Robust concave envelope	0.42		0.81	0.29		0.64
Robust fixed slope pairwise	0.41		0.80	0.28		0.63
Robust flat	0.19		0.35	0.24		0.46
Robust line left	0.40		0.75	0.28		0.62
Robust line right	0.40		0.76	0.27		0.61
Robust yzflat	0.42		0.81	0.29		0.65

Benchmarking

Average availability

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	0.02	0.02	0.02	0.03	0.04	0.04
Lagrangian flat	0.11	0.24	0.29	0.07	0.15	0.18
Lagrangian line	0.05	0.08	0.08	0.01	0.02	0.02
Lagrangian quadratic	0.04	0.06	0.06	0.00	0.00	0.00
Robust concave envelope	0.42	0.73	0.81	0.29	0.57	0.64
Robust fixed slope pairwise	0.41	0.73	0.80	0.28	0.56	0.63
Robust flat	0.19	0.32	0.35	0.24	0.42	0.46
Robust line left	0.40	0.67	0.75	0.28	0.53	0.62
Robust line right	0.40	0.69	0.76	0.27	0.54	0.61
Robust yzflat	0.42	0.73	0.81	0.29	0.57	0.65

Benchmarking

Median error

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	1.00	1.22	1.27	1.00	1.20	1.20
Lagrangian flat	4.79	1.67	1.28	1.78	1.15	1.07
Lagrangian line	3.99	3.12	3.74	2.40	2.25	2.41
Lagrangian quadratic	5.95	5.79	6.50	1.93	1.54	1.54
Robust concave envelope	1.89	1.16	1.07	1.36	1.02	1.01
Robust fixed slope pairwise	2.20	1.21	1.10	1.54	1.04	1.01
Robust flat	2.94	2.88	2.88	1.75	1.16	1.08
Robust line left	5.47	1.58	1.26	1.64	1.07	1.02
Robust line right	4.35	1.53	1.24	1.71	1.06	1.02
Robust yzflat	2.04	1.25	1.12	1.62	1.12	1.06

Benchmarking

Median error

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	1.00	1.22	1.27	1.00	1.20	1.20
Lagrangian flat	4.79	1.67	1.28	1.78	1.15	1.07
Lagrangian line	3.99	3.12	3.74	2.40	2.25	2.41
Lagrangian quadratic	5.95	5.79	6.50	1.93	1.54	1.54
Robust concave envelope	1.89	1.16	1.07	1.36	1.02	1.01
Robust fixed slope pairwise	2.20	1.21	1.10	1.54	1.04	1.01
Robust flat	2.94	2.88	2.88	1.75	1.16	1.08
Robust line left	5.47	1.58	1.26	1.64	1.07	1.02
Robust line right	4.35	1.53	1.24	1.71	1.06	1.02
Robust yzflat	2.04	1.25	1.12	1.62	1.12	1.06

Benchmarking

Median timing

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	0.10	0.27	0.43	0.11	0.30	0.51
Lagrangian flat	0.08	0.20	0.33	0.11	0.26	0.44
Lagrangian line	0.12	0.22	0.31	0.16	0.29	0.43
Lagrangian quadratic	0.14	0.25	0.35	0.18	0.32	0.45
Robust concave envelope	0.89	3.89	7.30	0.31	1.56	2.61
Robust fixed slope pairwise	0.17	0.58	1.06	0.15	0.51	0.93
Robust flat	0.02	0.07	0.12	0.02	0.06	0.11
Robust line left	0.10	0.46	0.91	0.10	0.49	0.91
Robust line right	0.11	0.48	0.92	0.10	0.51	0.94
Robust yzflat	0.08	0.36	0.69	0.09	0.37	0.69

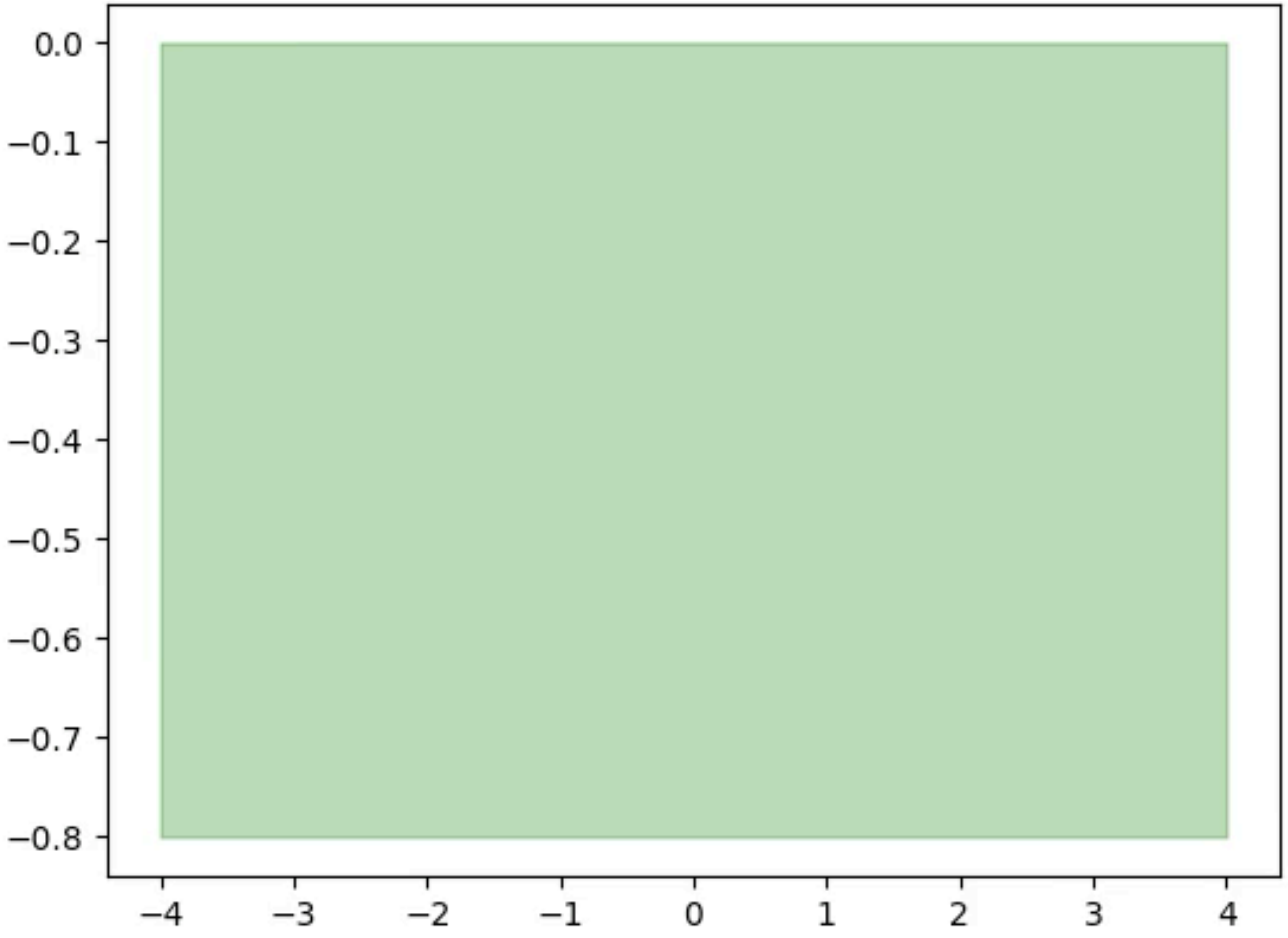
Benchmarking

Median timing

	Lower bounds			Upper bounds		
N	1	5	10	1	5	10
Lagrangian envelope	0.10	0.27	0.43	0.11	0.30	0.51
Lagrangian flat	0.08	0.20	0.33	0.11	0.26	0.44
Lagrangian line	0.12	0.22	0.31	0.16	0.29	0.43
Lagrangian quadratic	0.14	0.25	0.35	0.18	0.32	0.45
Robust concave envelope	0.89	3.89	7.30	0.31	1.56	2.61
Robust fixed slope pairwise	0.17	0.58	1.06	0.15	0.51	0.93
Robust flat	0.02	0.07	0.12	0.02	0.06	0.11
Robust line left	0.10	0.46	0.91	0.10	0.49	0.91
Robust line right	0.11	0.48	0.92	0.10	0.51	0.94
Robust yzflat	0.08	0.36	0.69	0.09	0.37	0.69

Iterative algorithm

Combining the bounds



Summary

Bounds for parametric linear problems

- We presented a novel approach to deal with modifications in the matrix coefficients
- These bounds provide guarantees on the behavior
- No outliers
- Benchmarked on a dataset
  - Showed the efficiency of the approach