A time-variant/invariant equivalence for the transient response of rotor blades in resonance crossing

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ABSTRACT
The transient response of a time varying oscillator under a chirp loading is considered in this paper. It is assumed that the natural frequency of the system changes with time during sweeping. Numerical simulations indicate that the maximum response amplitude and the apparent damping ratio are both higher than in the case without time varying natural frequency. These two specificities of the time varying problem are caused by the prolongated locking of the excitation frequency to the natural frequency. An asymptotic solution of the problem is derived and demonstrates that, at leading order, the time variant system responds like a time invariant system subjected to another sine sweep, having a slightly lower frequency rate. This suggests a simple equivalence. Previous studies of the time invariant problem indicate that a decrease in the sweeping rate, results in increased amplification and apparent damping. This shows that the proposed equivalence is able to capture the main feature of the transient response of the time variant problem.

Keywords: Resonance crossing, Transient, Dynamic amplification, rotating machinery, perturbation

INTRODUCTION
Resonance crossings are important features in the dynamic response of rotor components. They happen when a multiple of the rotation frequency of the rotor momentarily matches a natural frequency (or a family of natural frequencies) of the rotor assembly during the machine run-up or run-down. On the one hand, the transient response due to resonance crossing is a concern for designers as it may induce high-cycle fatigue, e.g. [**?**]. On the other hand, it is often exploited for system identification purpose during validation and monitoring activities, e.g. [**?**]. When rotor components are flexible like for the case of blades in turbomachinery, their geometric stiffness induced by the centrifugal force can be comparable with their elastic stiffness, making the natural frequencies changing with the rotor speed and therefore with time. In these conditions, the rotor components behave as Linear Time-Varying (LTV) systems, e.g. [**?**] . In this paper, we show that it is possible to treat the LTV problem as an equivalent Linear Time Invariant (LTI) system subject to a modified rotor acceleration. This result can be employed, on the design side, to simplify calculations, but it may be even more useful on the identification side as it provides an analytic response to be used in data fitting.

MATHEMATICAL MODEL
We consider a rotor disk of a turbomachinery having a resonance crossing the Engine Order (EO) r at the time t_0 with a vibration mode whose circular frequency is ω_0. In the neighborhood of this resonance crossing, the rotor speed is $\Omega(t)$ and

its acceleration is $\dot{\Omega}$, while the natural frequency is $\omega(t)$ and its time derivative is $\dot{\omega}$. Assuming that the considered mode is isolated, the equation of motion of its generalized coordinate can be written in the form

$$\ddot{q}+2\xi\,\omega(t)\,\dot{q}+\omega^2(t)\,q=\frac{p_0}{m}\,\mathrm{e}^{\frac{\mathrm{i}}{2}r\dot{\Omega}t^2} \tag{1}$$

where m is the generalized mass, ξ the viscous damping ratio, p_0 the force amplitude assumed as a constant and i the imaginary unit. In large rotating machinery, the rotor acceleration $\dot{\Omega}$ and rate of variation of the natural frequency $\dot{\omega}$ change slowly with time (in terms that needs to be specified), thus, to obtain an analytic formulation valid in the neighborhood of a resonance crossing, we assume $\dot{\Omega}=const$ and $\dot{\omega}=const$. Considering a single resonance, this assumption can be extended everywhere, i.e.

$$\Omega(t)=\dot{\Omega}t \quad ; \quad \omega(t)=\omega_0+\dot{\omega}(t-t_0) \tag{2}$$

Assuming the rotor at rest at $t=0$, the time of the resonance crossing t_0 is given as $t_0=\omega_0/r\dot{\Omega}$. For calculation purpose, it is convenient to introduce a shifted time t' such that the resonance crossing appears at $t'=0$. With this substitution Eq. (**??**) becomes

$$\ddot{q}+2\xi\left(\omega_0+\dot{\omega}t'\right)\dot{q}+\left(\omega_0+\dot{\omega}t'\right)^2 q=\frac{p_0}{m}\,\mathrm{e}^{\frac{\mathrm{i}}{2}r\dot{\Omega}t'^2}\,\mathrm{e}^{\mathrm{i}r\dot{\Omega}t_0t'}\,\mathrm{e}^{\frac{\mathrm{i}}{2}r\dot{\Omega}t_0^2} \tag{3}$$

A dimensionless version of the governing equation is obtained by introducing the dimensionless time $\tau=\omega_0t'$ and nondimensional response

$$q\left[t(\tau)\right]=\frac{p_0}{2\varepsilon\,\mathrm{i}\,m\,\omega_0^2}q(\tau) \tag{4}$$

where the small parameter $\varepsilon=\xi$ is introduced for the subsequent perturbation analysis. After substitutions and simplifications, the governing equation yields

$$x''+2\left(1+\kappa\alpha\varepsilon^2\tau\right)\varepsilon x'+\left(1+\kappa\alpha\varepsilon^2\tau\right)^2 x=2\varepsilon\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\tau}\mathrm{e}^{\frac{\mathrm{i}}{2}\kappa\varepsilon^2\tau^2}\mathrm{e}^{\mathrm{i}\Phi_0} \tag{5}$$

where the symbol $\cdot'$ represents the derivative respect to τ, $\Phi_0=\omega_0t_0$ is a constant phase angle, while κ and α are nondimensional parameters representing, respectively, the rotor acceleration and the rate of increment of the natural frequency. The choice of a proper definition for these parameter is not straightforward and is based on the solution of the time-invariant problem studied in [**?**].

$$\kappa=\frac{\dot{\Omega}}{\xi^2\omega_0^2} \quad ; \quad \alpha=\frac{\dot{\omega}}{\dot{\Omega}} \tag{6}$$

According to this definition, the parameter κ indicates whether or not the rotor acceleration is large enough to trigger transient effects (i.e. transient effects are negligible if $\kappa \ll 1$ [**?**]). The parameter α indicates the sensitivity of the natural frequency respect to the rotor speed. In practice, α may be considered as bounded in the range $[0,1)$. The value $\alpha=0$ represents the case of time invariant system, while slightly negative values are possible, but are rare for rotor blades.

MULTIPLE SCALES SOLUTION OF THE PROBLEM

The time invariant problem ($\alpha=0$) has already been solved with a multiple scales technique [**?**]. It is based on recognizing the existence of a slow timescale $T=\varepsilon\tau$ and a fast timescale τ. It is possible to show that the response reads, at leading order for small ε, $x=A(\varepsilon\tau)\mathrm{e}^{\mathrm{i}\tau}$, where the slow envelope $A(T)$ is the solution of $A'(T)+A(T)=\exp\left(\frac{\mathrm{i}}{2}\kappa T^2\right)$. It is therefore expressed by

$$A(T)=\int_{-\infty}^{T}\mathrm{e}^{\frac{\mathrm{i}}{2}\kappa s^2-(T-s)}\mathrm{d}s=\mathrm{e}^{-T}\mathscr{D}(T;\kappa) \tag{7}$$

with $\mathscr{D}(T;\kappa)=\int_{-\infty}^{T}\mathrm{e}^{\frac{\mathrm{i}}{2}\kappa s^2+s}\mathrm{d}s=\sqrt{\frac{\mathrm{i}\pi}{2\kappa}}\mathrm{e}^{\frac{i}{2\kappa}}\mathrm{erfc}\left[\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\frac{\mathrm{i}-\kappa T}{\sqrt{2\kappa}}\right]$. Although (**??**) does not admit a simple exact analytical solution, the time variant problem can be tackled with the same approach. Following the perturbation technique described in [**?**], it is possible to show that the leading order solution still is $x=A(\varepsilon\tau)\mathrm{e}^{\mathrm{i}\tau}$, but where $A(T)$ now satisfies

$$A'(T)+(1-\mathrm{i}\kappa\alpha T)A(T)=\mathrm{e}^{\frac{\mathrm{i}}{2}\kappa T^2}. \tag{8}$$

This linear (slow) time variant problem admits the solution

$$A(T)=\mathrm{e}^{-T}\mathrm{e}^{\frac{\mathrm{i}}{2}\kappa\alpha T^2}\mathscr{D}(T;\hat{\kappa}). \tag{9}$$

where $\hat{\kappa}=\kappa(1-\alpha)$.

Figure 1: (a) Illustration of the equivalence principle; (b-c) Response of the initial problem (blue), of the same problem but neglecting $\dot{\omega}$ (gray) and of the proposed equivalent problem. Numerical values : $\bar{\omega} = 600$ rad/s, $\dot{\omega} = 0.9$ rad/s^2 $\dot{\Omega} = 2$ rad/s^2, $\xi = 0.1\%$ ($\alpha = 0.45, \kappa = 5.56$).

DISCUSSION AND ILLUSTRATION

Equation (**??**) generalizes (**??**) to cases where $\alpha \neq 0$. The comparison of these two equations shows that the time-varying nature of the system manifests in two ways : (i) a change of phase in the envelope of the response, but this is not noticeable in terms of amplitude, since $\left|e^{\frac{i}{2}\kappa\alpha T^2}\right| = 1$, (ii) more importantly, a change of the second argument in $\mathscr{D}(T;\hat{\kappa})$. But in terms of magnitude of the response envelope, this simple analytical solution demonstrates that, at leading order for small ε, the time variant problem with parameters κ and α has the same solution as a time invariant problem with an equivalent dimensionless rotor acceleration $\kappa(1-\alpha)$. In the time-varying problem, both the natural frequency and the rotor speed increase with time. As a result, they remain longer closer to each other, relatively speaking. This explains why the equivalent rotor acceleration is smaller than in the time varying problem, $\hat{\kappa} \leq \kappa$.

Figure 1 illustrates these findings. Figure 1-a is a simple graphical illustration of the equivalence principle. In particular, it illustrates that the equivalent (LTI) dimensionless rotor acceleration is always smaller than the actual one (LTV). Figure 1-b shows the exact solution of the original problem with the numerical values reported in the figure caption. It compares the exact solution (in blue) to the solution that would have been obtained by neglecting $\dot{\omega}$ (in gray) and to the solution of the equivalent problem with reduced rotor acceleration ($\dot{\Omega}_{\text{equiv}} = \dot{\Omega}(1-\alpha) = 1.1\text{rad/s}^2$). The time shift introduced in Section 1 makes it such that resonance crossing happens around $t_0 = 300$ s for the first two solutions and around $t_0/(1-\alpha)$=545.45s in the equivalent model. Despite this unimportant time shift in the solution, a shifted representation of the solutions in Figure 1-c shows that the proposed equivalence actually yields an almost perfect match with the solution of the initial time variant problem, while the other solution (constant frequency) clearly underestimates the response envelope.

CONCLUSION

With a standard perturbation technique, we could determine the leading order approximation of the response of a rotating blade with varying natural frequency subjected to a sine sweep. We found that the envelope of the response coincides, at leading order , to the envelope of a blade with constant natural frequency but subjected to a chirp with smaller rotor acceleration.

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