A linear algebra perspective on folding

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1 Main result

Consider the optimization problem

$$\begin{aligned} (\mathcal{P}_{init}):\ & \min c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0, \end{aligned}$$

with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.

Assumption 1 *There exist matrices $P_1, P_2 \in \mathbb{R}^{m \times m'}$, $Q \in \mathbb{R}^{n \times n'}$ and a matrix $A' \in \mathbb{R}^{m' \times n'}$ such that*

(i) $AQ = P_1 A'$

(ii) $P_2^T A = A' Q^T$

(iii) $P_1 P_2^T b = b$

(iv) $QQ^T c = c$

(v) $Q \geq 0$.

Using the notations introduced in Assumption 1, we can define another linear optimization problem

$$\begin{aligned} (\mathcal{P}_{small}):\ & \min c^T Q x' \\ \text{s.t. } & A'x' = P_2^T b \\ & x' \geq 0. \end{aligned}$$

Lemma 1 *If x' is feasible for $(\mathcal{P}_{small})$ then $x = Qx'$ is feasible for $(\mathcal{P}_{init})$.*

Proof: The nonnegativity of x is implied by the fact that $Q \geq 0$ and $x' \geq 0$. We then check that the constraints of $(\mathcal{P}_{init})$. To this end, we replace x by Qx' in the left-hand-sides of the constraints. We obtain successively

$$AQx' = P_1 A' x' = P_1 P_2^T b = b,$$

where the first equality is obtained by Assumption 1(i), the second equality by the feasibility of x' for $(\mathcal{P}_{small})$ and the last equality by Assumption 1(iii). $\square$

We now turn to the dual problems. Let us first write the problems. The dual of $(\mathcal{P}_{init})$ reads

$$\begin{aligned} (\mathcal{D}_{init}):\ & \max b^T u \\ & \text{s.t. } A^T u \leq c \\ & \quad u \text{ free.} \end{aligned}$$

The dual of $(\mathcal{P}_{small})$ reads

$$\begin{aligned} (\mathcal{D}_{small}):\ & \max b^T P_2 u' \\ & \text{s.t. } A'^T u' \leq Q^T c \\ & \quad u' \text{ free.} \end{aligned}$$

Lemma 2 *If u' is feasible for $(\mathcal{D}_{small})$ then $u = P_2 u'$ is dual feasible for $(\mathcal{D}_{init})$.*

Proof: We replace u by $P_2 u'$ in the constraints of $(\mathcal{D}_{init})$ and we obtain

$$A^T P_2 u' = Q A'^T u' \leq Q Q^T c = c,$$

where the first equality is obtained by Assumption 1(ii), the inequality is obtained by the dual feasibility of u' and the fact that $Q \geq 0$ and the last equality by Assumption 1(iv). $\square$

Lemma 3 *The objective value of a feasible solution x' of $(\mathcal{P}_{small})$ is equal to the corresponding objective value of $x = Qx'$.*
The objective value of a dual-feasible solution u' of $(\mathcal{D}_{small})$ is equal to the corresponding objective value of $u = P_2 u'$.

Proof: This follows from the fact that $c^T x = c^T Q x'$ and $b^T u = b^T P_2 u'$. $\square$

We conclude with the following corollary.

Corollary 1 *If (x', u') is an optimal primal-dual pair for $(\mathcal{P}_{small}) - (\mathcal{D}_{small})$ then $(x, u) = (Qx', Pu')$ is an optimal primal-dual pair for $(\mathcal{P}_{init}) - (\mathcal{D}_{init})$.*

2 One way to construct a matrix Q

One way to construct a valid (orthogonal) matrix Q that satisfies Assumption 1(iv) is explained in the following. Assume that we split the vector c in r blocks $c = (c^{(1)} \cdots c^{(r)})$. Furthermore, we assume that that the sign of every element in the same block $c^{(i)}$ is the same.

Lemma 4 *The matrix*

$$Q = \left(\begin{array}{c|c|c} \frac{|c^{(1)}|}{\|c^{(1)}\|} & & \\ \hline & \frac{|c^{(2)}|}{\|c^{(2)}\|} & \\ \hline & & \ddots \end{array}\right)$$

satisfies part (iv) and (v) of Assumption 1. Furthermore $Q^TQ = I$.

Proof: The nonnegativity of Q is straightforward.
We compute

$$QQ^T = \left(\begin{array}{c|c|c} \frac{c^{(1)}(c^{(1)})^T}{\|c^{(1)}\|^2} & & \\ \hline & \frac{c^{(2)}(c^{(2)})^T}{\|c^{(2)}\|^2} & \\ \hline & & \ddots \end{array}\right).$$

Observe that the absolute value can be dropped because all components of $c^{(i)}$ have the same sign. Hence

$$QQ^Tc = \begin{pmatrix} c^{(1)} \frac{(c^{(1)})^T c^{(1)}}{\|c^{(1)}\|^2} \\ c^{(2)} \frac{(c^{(2)})^T c^{(2)}}{\|c^{(2)}\|^2} \\ \vdots \end{pmatrix} = c.$$

Similary, it can be readily verified that $Q^TQ = I$. □

3 Fixing Q

Assumption 2 *We assume that Q is fixed as in Lemma 4 and has the form*

$$Q = \left(\begin{array}{c|c|c} \frac{|c^{(1)}|}{\|c^{(1)}\|} & & \\ \hline & \frac{|c^{(2)}|}{\|c^{(2)}\|} & \\ \hline & & \ddots \end{array}\right).$$

We explore the complexity of finding P_1, P_2 that allow us to satisfy conditions (i)-(iii) of Assumption 1.

Lemma 5 *If Assumption 2 is satisfied, then the columns of Q are eigenvectors of QQ^T with eigenvalue 1 and all other eigenvalues are 0.*

Proof: This follows from the fact that $QQ^TQ = Q$ and the fact that QQ^T is of rank n'. □

If $Q^TQ = I$, and if we multiply condition (ii) of 1 by Q, we observe that A' must take the form $A' = P_2^T AQ$. However, assuming $A' = P_2^T AQ$ does not necessarily imply that condition (ii) of Assumption 1 is satisfied. The following lemmas show under which conditions this holds true.

Lemma 6 *We consider (a) the linear space* $L \subseteq \mathbb{R}^{m+n'}$ *defined by* $L := \ker\left(\left[A^T \mid -Q\right]\right)$*, (b) its projection* $\text{proj}_{\mathbb{R}^m} L \subseteq \mathbb{R}^m$ *be on the first* m *components, (c) a basis* $\{\beta_1, \dots \beta_k\} \subset \mathbb{R}^m$ *of this projection, where* $k = \dim(\text{proj}_{\mathbb{R}^m} L \subseteq \mathbb{R}^m)$ *and (d) the matrix* $B := \left[\ \beta_1 \mid \cdots \mid \beta_k\ \right] \in \mathbb{R}^{m \times k}$*. Under Assumption 2, for all* $U \in \mathbb{R}^{k \times m'}$*, letting* $P_2 = BU$ *implies that Assumption 1(ii) is met.*

Proof: We want to prove that assuming $P_2 = BU$ implies that $P_2^T A = P_2^T AQQ^T$. By (c), (d) and the construction of $P_2 = BU$, every column p of P_2 satisfies $p \in \text{proj}_{\mathbb{R}^m} L$. Using (b), we know that for every column $p \in \mathbb{R}^m$ of P_2, there exists a vector $v \in \mathbb{R}^{n'}$ such that $\begin{pmatrix} p \\ v \end{pmatrix} \in L$. As a result, by (a), we have $\left(\left[A^T \mid -Q\right]\right) \begin{pmatrix} p \\ v \end{pmatrix} = 0$. Writing this for every column p of P_2, we get that there exists a matrix $V \in \mathbb{R}^{n' \times m'}$ such that $\left(\left[A^T \mid -Q\right]\right) \begin{pmatrix} P_2 \\ V \end{pmatrix} = 0$, i.e. $A^T P_2 = QV$. So, regardless of the values in V, the columns of $A^T P_2$ belong to the column space of Q. By Lemma 5, they belong to the eigenspace of QQ^T for the eigenvalue 1, hence $QQ^T A^T P_2 = A^T P_2$ which is exactly the transpose of what we want to prove. □

In the following, we want to find P_1 such that conditions (i) and (iii) of Assumption 1 are met. To present the next results, we need a few definitions.

Definition 1 *For* $A \in \mathbb{R}^{m \times n}$ *and* $Q \in \mathbb{R}^{n \times n'}$*,* $b \in \mathbb{R}^m$*, we define:*

- $B \in \mathbb{R}^{m \times k}$ *to be the projection on the first* m *coordinates of a basis of* $\ker([A^T \ \ -Q])$*,*
- $M \in \mathbb{R}^{m \times (m-l)}$ *be an orthogonal basis of the column space of* $(AQ \mid b)$*,*
- $C \in \mathbb{R}^{m \times l}$ *to be a basis of the orthogonal space of the column space of* $(AQ \mid b)$*.*

Lemma 7 *Under Assumption 2, assuming* $P_2 = BU$*, conditions (i) and (iii) of Assumption 1 are equivalent to finding matrices* $P \in \mathbb{R}^{m \times k}$ *and* $V \in \mathbb{R}^{l \times m}$ *such that*

$$PB^T = MM^T + V^T C^T. \tag{1}$$

Proof: Using Lemma 6 and Assumption 2, we know that $A' = P_2 A^T Q$. If we rewrite conditions (i) and (iii) using $A' = P_2^T AQ$, we obtain respectively $P_1 P_2^T AQ = AQ$ and $P_1 P_2^T b = b$ which can be reinterpreted as the fact that the columns of AQ and b must be eigenvectors of $P_1 P_2^T$ with respect to the eigenvalue 1. Since $P_2 = BU$, it means that $P_1 P_2^T$ can be written $P_1 U^T B^T$ which in turn must have the columns of AQ and b as eigenvectors with eigenvalue 1. This can be rewritten as

$$P_1 U^T B^T = MM^T + V^T C^T,$$

where $V \in \mathbb{R}^{l \times m}$. □

Observe that (1) is a linear system with $m(k+l)$ unknowns and m^2 equations. Finding a solution with P with a lower rank allows us to find a folded problem with a lower size.