Efficient exact recomputation of linear modifications of the constraints matrix in Linear Programming

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Uncertainty

Linear programming

- In LP optimization
  - Formalize problem in terms of
    - Constraints
    - Objective function
  - Get one optimal solution

Parameters ← Hypothesis

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Further assessment and analysis

Formalization

Parametric uncertainty

- Generic parametric linear optimization problem:

$$\begin{array}{ll} \min & c^t x \\ \text{s.t} & Ax \le b \end{array}$$

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$$\begin{aligned} \min \quad & c^t x + \lambda c_\lambda^t x \\ \text{s.t} \quad & Ax \leq b \end{aligned}$$

- Modification in the objective coefficients $+\lambda c_\lambda^t x$

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- Modification on the left-hand side $+\lambda D$ → Not much ?

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- Modification in the objective coefficients $+\lambda c_\lambda^t x$ $f(\lambda)$ concave and piecewise linear!
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- Modification on the left-hand side $+\lambda D$ — $f(\lambda)$ piecewise rational function...

Note : The left-hand side modification $+\lambda D$ encapsulates the other modifications

Formalization

Parametric uncertainty

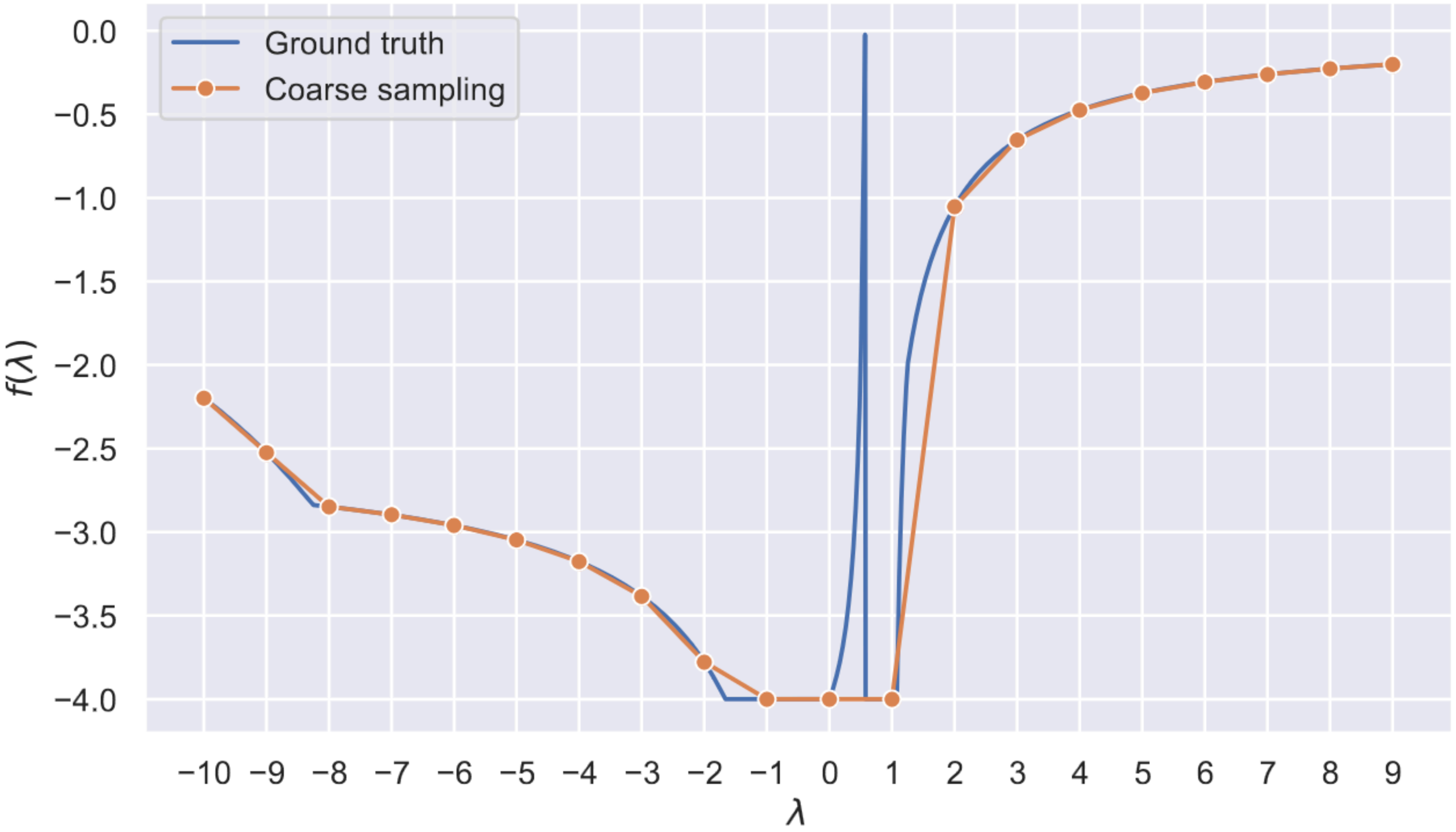
- We focus on this standard form:

$$
\begin{aligned}
f(\lambda) = \min \quad & c^t x \\
s.t. \quad & (A + \lambda D)x = b \\
& x \geq 0
\end{aligned}
$$

- In literature :
  - Usually rely on heavy computations,
  - approximations
  - and/or hypothesis on the matrix D

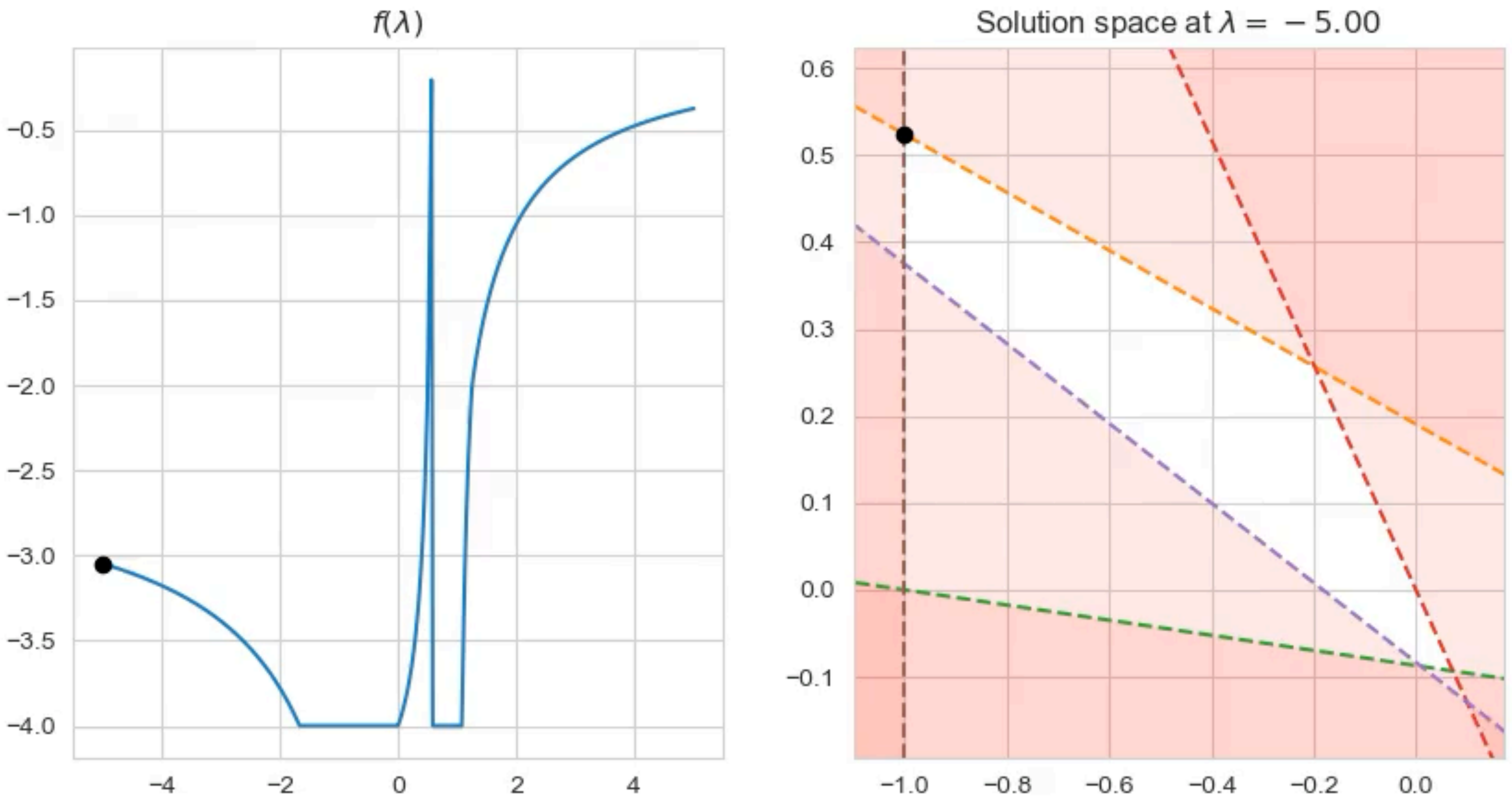
Naive solution

Heavy computations



An observation

Piecewise component <=> optimal basis/face



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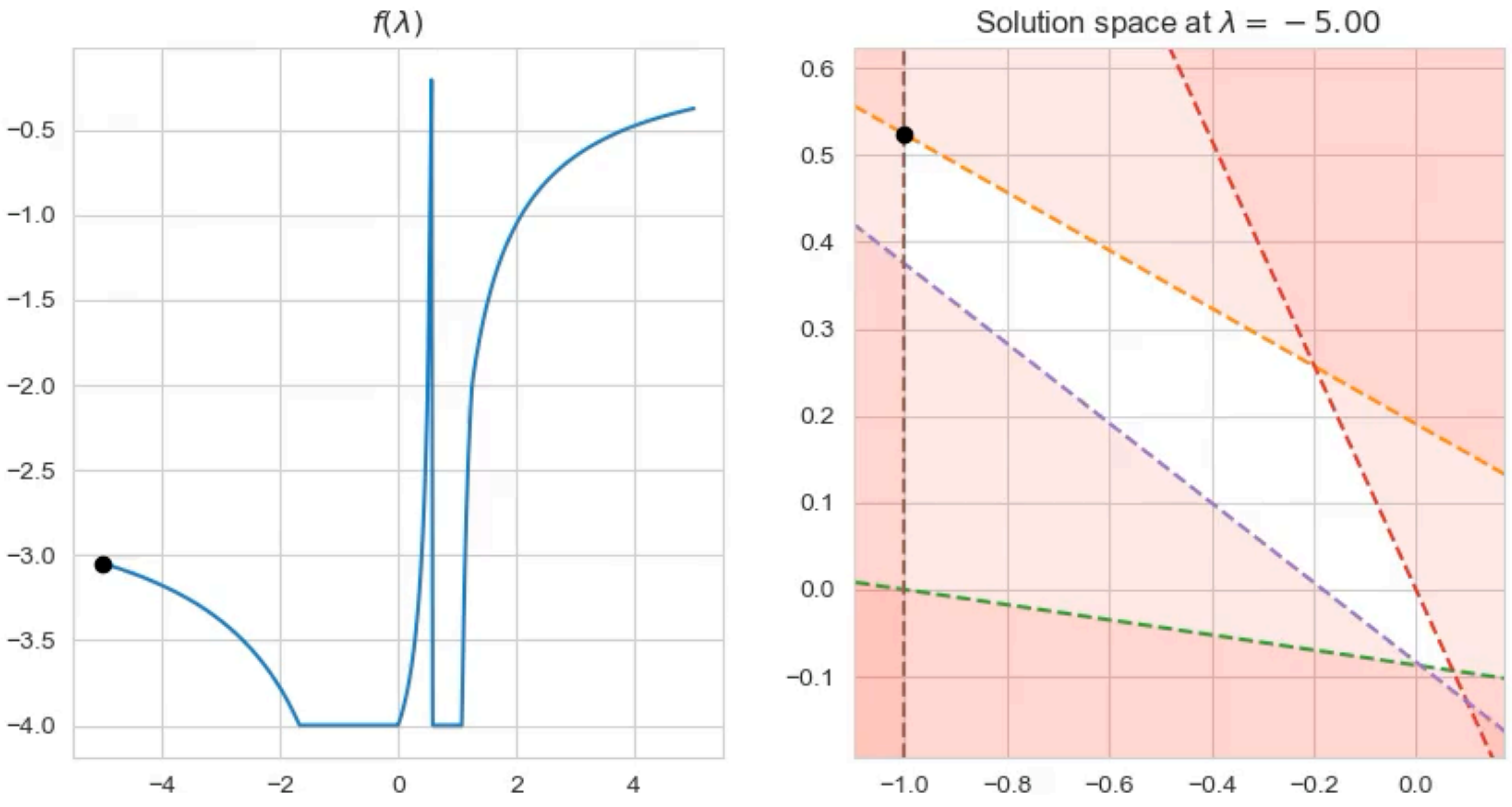

Given a basis...

- Assume we solved the problem for $\lambda = 0$ (without loss of generality)
- We obtain a basic optimal solution $x*$, an optimal objective $f(0) = o*$
- And a basis B, such that $x_B^* \geq 0$, $x_N^* = 0$
- For any λ such that the basis remains optimal, we have:

$$f(\lambda) = c_B^t (A_B + \lambda D_B)^{-1} b$$

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Everything depends on this matrix inversion!

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[Zuidwijk] to the rescue

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$$= \frac{1}{\lambda}\left(\prod_i \frac{1 + \lambda\alpha_i}{1 + \lambda\beta_i} - 1\right)$$

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  - Two n-degree polynomials to compute (numerical stability?)
  - To assess if the basis is still optimal: $4n$ eigenvalue problems to compute on an $n \times n$ matrix!
    - (But gives the exact range of λ where the basis is optimal in exchange)

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Today

- Another way of computing $f(\lambda)$
- Another way of computing the range of λs where the basis remains optimal
  - Less expensive...
  - ... but only gives a subset of the real range.

Simpler matrix inversion operations

The idea: simpler inverse operation

- Let us assume we are on a basis B.

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(Factorize A_B)

$(XY)^{-1} = Y^{-1} X^{-1}$

$A_B^{-1} b$ is the $\lambda = 0$ solution

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  - $I + \lambda \Lambda$ is diagonal: inversion trivial in $\mathcal{O}(n)$
  - Problem: most of the time, E_B is not diagonalizable

The idea: simpler inverse operation

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- Second idea: Schur's decomposition
  - $E_B = QUQ^H \implies f(\lambda) = c_B^t Q(I + \lambda U)^{-1} Q^H x_B^*$
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- Potential problem: computing the decomposition
- Better numerical stability?

Ranges of bound validity

Optimality range

- We need three properties for the basis to stay optimal:
  - $A_B + \lambda D_B$ must be invertible/full-rank/non-singular
  - $(A_B + \lambda D_B)^{-1} b \geq 0$
  - Reduced costs ≥ 0

$A_B + \lambda D_B$ must be invertible/full-rank

- We saw that $(A_B + \lambda D_B) = A_B(I + \lambda E_B)$ with A_B already invertible
- Just need to check the eigenvalues α_i of E_B and ensure that $1 + \lambda\alpha_i \neq 0$
- That creates holes in the set of admissible λs
- (That was the easy part)

$$x_B^\lambda = (A_B + \lambda D_B)^{-1} b = (I + \lambda E_B)^{-1} x_B^* \geq 0$$

- Now we have to deal with the inversion
- Schur cannot help us here
- The idea: removing the inversion using Neumann series.

$$\|X\|_\infty < 1 \implies (I - X)^{-1} = \sum_{i=0}^{\infty} X^i$$

- If we apply this on $X = -\lambda E_B$,

$$\|\lambda E_B\|_\infty < 1 \implies x_B^\lambda = \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^*$$

Now we have an infinite sum...

$$\|\lambda E_B\|_\infty < 1 \implies x_B^\lambda = \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^*$$

- Typically what you say here is "let's neglect the terms for $i \geq 3$"
- But that's an approximation for which you lose all guarantees...
- Instead we use (sub-multiplicative) matrix norms!
- Let's say we focus on the two first terms of component j:

$$\begin{aligned} 0 \leq (x_B^\lambda)_j &= e_j \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^* \\ &= e_j (I - \lambda E_B + \sum_{i=2}^{\infty} (-\lambda E_B)^i) x_B^* \end{aligned}$$

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Now we have an infinite sum...

$$0 \leq (x_B^\lambda)_j = e_j x_B^* - \lambda e_j E_B x_B^* + \lambda^2 e_j E_B^2 \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^*$$

- The trick: take the absolute norm of the last part

$$(x_B^\lambda)_j \geq e_j x_B^* - \lambda e_j E_B x_B^* - \|\lambda^2 e_j E_B^2 \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^*\|$$

- Now use sub-multiplicativity, triangular inequality, and geometric series closed-form on the last term:

$$\begin{aligned}\|\lambda^2 e_j E_B^2 \sum_{i=0}^{\infty} (-\lambda E_B)^i x_B^*\| &\leq \|\lambda^2 e_j E_B^2\| \cdot \sum_{i=0}^{\infty} \|(-\lambda E_B)^i\| \cdot \|x_B^*\| \\ &\leq \|\lambda^2 e_j E_B^2\| \cdot \frac{1}{1 - \|\lambda E_B\|} \cdot \|x_B^*\|\end{aligned}$$

Success

$$e_j x_B^* - \lambda e_j E_B x_B^* - \lambda^2 \frac{\|e_j E_B^2\| \cdot \|x_B^*\|}{1 - |\lambda| \cdot \|E_B\|} \geq 0 \implies (x_B^\lambda)_j \geq 0$$

(if $\|\lambda E_B\|_\infty, \|\lambda E_B\| < 1$)

- We thus have n polynomials of degree two for which we need the roots!
- Here we focused on the first 2 terms, giving degree 2 polynoms
  - if you keep the first v terms you get degree v polynoms.
- Far easier than n polynomials of degree n :-)
- But gives a subset of the true range of validity...
- The same tricks can be used to find ranges of optimality (using reduced costs)

What now?

What now?

- We have two "easy" way of computing $f(\lambda)$ given a known-optimal basis B
  - Zuidwijk's eigenvalues-based method
  - Schur decomposition
- We have now a scalable method to get the range of optimality of the basis B
  - At least a subset of it
- We lack a way to recompute the new matrix $E_B^\lambda = (A_B + \lambda D_B)^{-1} D_B$ for a λ a bit further than 0, to get a new range...
- For this, you can use one of the many $\mathcal{O}(n^\omega)$ techniques

Summary

	Eigenvalues	Schur+norms
Initial computation for each new basis	Compute eigenvalues	Compute Schur's decomposition
	$O(n^3)$	$O(n^3)$
For a new λ	Solve the polynomials	Solve the triangular system
	$O(n)$	$O(n^2)$
Validity/optimality range computation	Compute multiple eigenvalue problems	Compute norms and solve n degree-v polynomials
	$O(n^4)$	$O(vn^3+nv^2)$
(For the whole range)	Same: $O(n^4)$	An unknown number of time p, we need to recompute the basis matrix and the range
		$O(pvn^3+pnv^2)$

- Worth if p small.
- Schur decomposition can be updated in $O(n^3)$

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- Schur decomposition can be updated in $O(n^3)$
- You can mix the methods
- Paper not yet available
- Other talk on this problem tomorrow!

Open questions

- How to characterize p?
- What is the physical meaning of $E_B = A_B^{-1} D_B$?
- What is the physical meaning of the closed-form of $f(\lambda)$?
- How well does all of this work in practice?