Méthodes d’ondelettes pour l’étude de processus chaotiques

Séminaire équipe ADA – Calais

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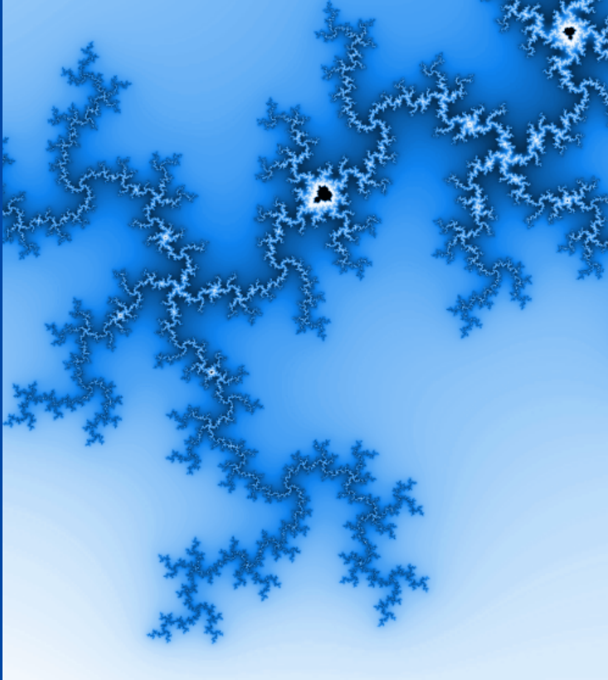
LIÈGE université
Mathématique



Brownian motion

Gaussian process

A stochastic process $\{X_t\}_{t\in T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Gaussian if, for all $d \in \mathbb{N}^*$, $t_1, \ldots, t_d \in T$, the random vector $(X_{t_1}, \ldots, X_{t_d})$ is Gaussian.

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Brownian motion

Any Gaussian process $\{B_t\}_{t\geqslant 0}$ satisfying

- $B_0 = 0$ almost surely;
- **stationary increments:** for all $0 \leqslant s < t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$;
- **independent increments:** for all $d \in \mathbb{N}^*$ and $0 \leqslant t_1 < t_2 < \cdots < t_d$, $B_{t_d} - B_{t_{d-1}}, \ldots, B_{t_2} - B_{t_1}$ are independent.
- for almost every $\omega \in \Omega$, the path $t \mapsto B_t(\omega)$ is continuous.

Brownian motion in the Schauder basis

Consider the triangle function

$$\Lambda(x) := \begin{cases} x & \text{if } 0 \leqslant x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leqslant x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

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and, for all $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j - 1\}$, $\Lambda_{j,k}(x) = 2^{-j/2}\Lambda(2^j x - k)$.

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$$\left\{ B_t := \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \varepsilon_{j,k}\Lambda_{j,k}(t) + \varepsilon t \right\}_{t\in[0,1]} \tag{1}$$

is a Brownian motion on $[0, 1]$.

Two-sided Brownian motion

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Explicit construction

Take $\{B_t^1\}_{t\geqslant 0}$ and $\{B_t^2\}_{t\geqslant 0}$ two independent Brownian motions on $[0,+\infty[$ and define $\{B_t\}_{t\in\mathbb{R}}$ by

$$B_t := \begin{cases} B_t^1 & \text{si } t \geqslant 0 \\ B_{-t}^2 & \text{si } t < 0. \end{cases}$$

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By mean and covariance

Any centred Gaussian process $\{B_t\}_{t\in\mathbb{R}}$ with almost surely continuous path and covariance operator

$$K : \mathbb{R}^2 \to \mathbb{R} : (s,t) \mapsto \frac{1}{2}(|t| + |s| - |t-s|).$$

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$$f = \sum_{j_1,\dots,j_d=1}^{n} a_{j_1,\dots,j_d} \mathbb{1}_{[s_{j_1},t_{j_1})} \otimes \cdots \otimes \mathbb{1}_{[s_{j_d},t_{j_d})}, \tag{2}$$

where, $\otimes$ stands for the tensor product, $a_{j_1,\dots,j_d}$ are such that, for all permutation σ, $a_{\sigma(j_1),\dots,\sigma(j_d)} = a_{j_1,\dots,j_d}$ and $a_{j_1,\dots,j_d} = 0$ as soon as two indices $j_1, \dots, j_d$ are equal and, for all $1 \leqslant \ell \neq \ell' \leqslant d$, $[s_{j_\ell}, t_{j_\ell}) \cap [s_{j_{\ell'}}, t_{j_{\ell'}}) = \varnothing$.

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$$I_d(f) := \sum_{j_1,\ldots,j_d=1}^{n} a_{j_1,\ldots,j_d} (B(t_{j_1}) - B(s_{j_1})) \times \ldots (B(t_{j_d}) - B(s_{j_d})). \tag{3}$$

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It is a random variable in $L^2(\Omega)$.

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- $(I_d(f_j))_j$ is a Cauchy sequence in $L^2(\Omega)$;
- $I_d(f) := \lim_{j\to+\infty} I_d(f_j)$ (in $L^2(\Omega)$).

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$$I_d(f) := I_d(\widetilde{f}).$$

Important facts

Wiener isometry

If $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^{d'})$, we have

$$\mathbb{E}\left[I_d(f) I_{d'}(g)\right] = \begin{cases} d!\langle f, g\rangle & \text{if } d = d' \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

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Connection with Hermite polynomials

if $f \in L^2(\mathbb{R})$ is such that $\|f\|_{L^2(\mathbb{R})} = 1$, then

$$H_d\left(I_1(f)\right) = I_d(f^{\otimes d}),$$

where H_d is the dth Hermite polynomial$H_d(x) = (-1)^d e^{x^2/2} D^d e^{-x^2/2}$.

Important facts

Product formula

If $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, we have

$$I_m(f)I_n(g) = \sum_{r=0}^{m\wedge n} r!\binom{m}{r}\binom{n}{r} I_{m+n-2r}(f \otimes_r g), \tag{6}$$

where, for all $0 \leqslant r \leqslant m \wedge n$, $f \otimes_r g$ is the $L^2(\mathbb{R}^{m+n-2r})$ function defined, for all $(x_1, \dots, x_{m+n-2r}) \in \mathbb{R}^{m+n-2r}$, through the Lebesgue integral

$$\begin{aligned}&(f \otimes_r g)(x_1, \dots, x_{m+n-2r})\\ &:= \int_{\mathbb{R}^r} f(x_1, \dots, x_{m-r}, s_1, \dots, s_r) g(x_{m-r+1}, \dots, x_{m+n-2r}, s_1, \dots, s_r)\, ds_1 \dots ds_r\,,\end{aligned}$$

Wiener chaos decomposition

Wiener chaos

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Chaos expansion

Every $F \in L^2(\Omega)$ can be expanded as

$$F = \mathbb{E}[F] + \sum_{d=1}^{+\infty} I_d(f_d)$$

for some unique collection of kernels $f_d \in L^2(\mathbb{R}^d)$.

Quantitative CLT

Central Limit Theorem

Let $(X_j)_{j\in\mathbb{N}^*}$ be a family of independent and identically distributed random variables of expectation μ and variance σ^2. We have

$$\frac{\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) - \mu}{\sqrt{\frac{\sigma^2}{n}}} \xrightarrow[d]{} \mathcal{N}(0,1)$$

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Question : How “good” is the approximation provided by the CLT?

Distance between random variables

General form

$$d_{\mathcal{A}}(F,G) = \sup\{|\mathbb{E}[h(F)] - \mathbb{E}[h(G)]| \,:\, h \in \mathcal{A}\}$$

for some appropriate class $\mathcal{A}$ of Borel-measurable complex valued functions.

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Example

- If $\mathcal{A}$ is the set of functions of the form $\mathbb{1}_{(-\infty,z_1]\times\cdots\times(-\infty,z_d]}$, with $z \in \mathbb{R}^d$, we get the so-called Kolmogorov distance d_{Kol};

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- If $\mathcal{A}$ is the set of Lipschitz functions with Lipschitz norm less than or equal to 1, we get the so-called Wasserstein distance d_{W}.

Malliavin calculus in a nutshell

Smooth random variables

The set $\mathcal{S}$ of random variables of the form

$$F = f(I_1(h_1), \ldots, I_1(h_d))$$

with $d \in \mathbb{N}^*$, $h_1, \ldots, h_d \in L^2(\mathbb{R})$ and $f : \mathbb{R}^d \to \mathbb{R}$ a C^∞-function with all partial derivatives having at most polynomial growth.

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$$DF = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(I_1(h_1), \dots, I_1(h_d))h_j$$

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"Sobolev space"

For all $q \in [1, \infty)$, $\mathbb{D}^{1,q}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\| \cdot \|_{1,q} \, : \, F \mapsto \mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|^q_{L^2(\mathbb{R})}].$$

One can extend the definition of D to $\mathbb{D}^{1,q}$.

Quantitative CLT using Stein-Malliavin calculus

Theorem (Nourdin, Peccati, Réveillac - 2010)

Let $m \geqslant 1$ be an integer number and consider a m-dimensional random vector $F = (F_1, \ldots, F_m)$. Assume $F_j \in \mathbb{D}^{1,4}$ for every $j = 1, \ldots m$. Let $C \in M_m(\mathbb{R})$ be a symmetric and positive definite matrix and let $Z \sim N(0, C)$. Then

$$d_W(F, Z) \leqslant C \sqrt{\sum_{j,k=1}^{m} \mathbf{E}\left[(C_{j,k} - \langle DF_j, D(-L)^{-1} F_k \rangle)^2\right]}. \tag{7}$$

Where we use the notation

$$(-L)^{-1}\left(\mathbb{E}[F] + \sum_{d=1}^{+\infty} I_d(f_d)\right) = \sum_{d=1}^{+\infty} \frac{1}{d} I_d(f_d).$$

Multiresolution analysis

Definition

A multiresolution analysis of the Hilbert space $L^2(\mathbb{R}^d)$ is given by a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that

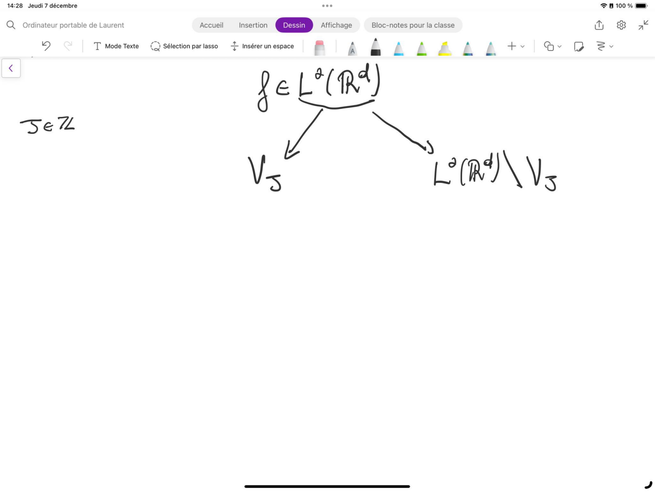
(a) for all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$;

(b) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$;

(c) for all $j \in \mathbb{Z}$, $V_j = \{f(2^j\cdot) \, : \, f \in V_0\}$;

(d) there exists a scaling function $\phi^d \in V_0$ such that the sequence $(\phi^d(\cdot - k))_{k\in\mathbb{Z}^d}$ is an orthogonal basis of V_0.

Approximation strategy



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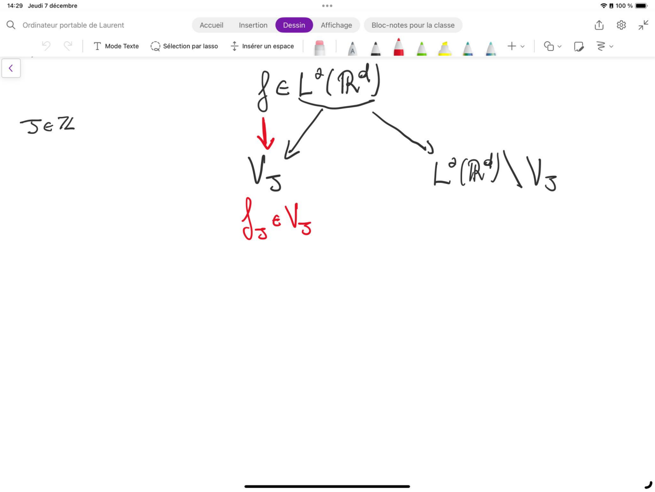

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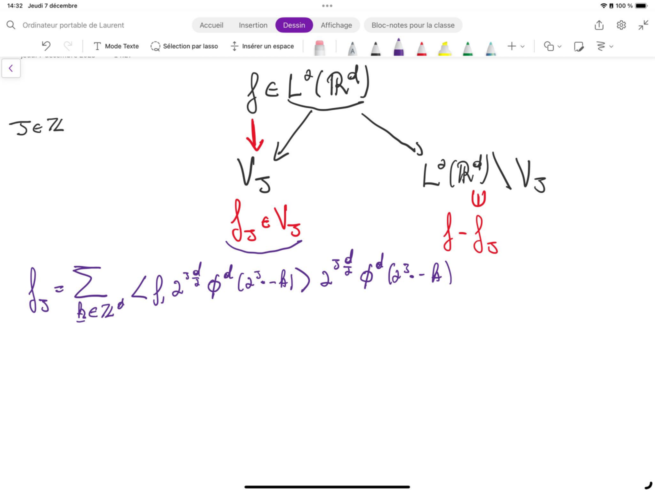

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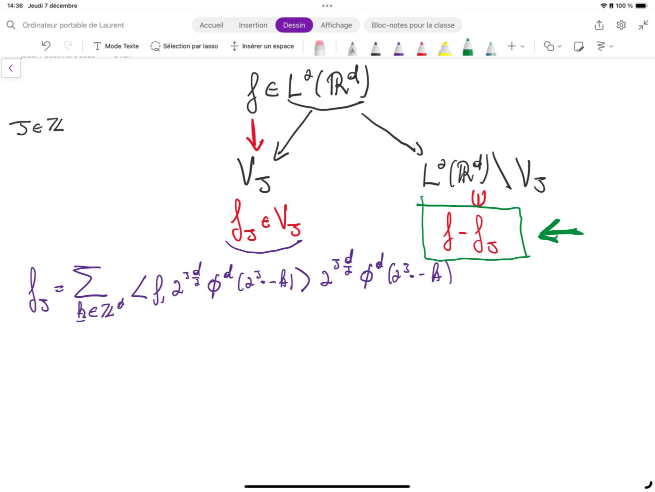

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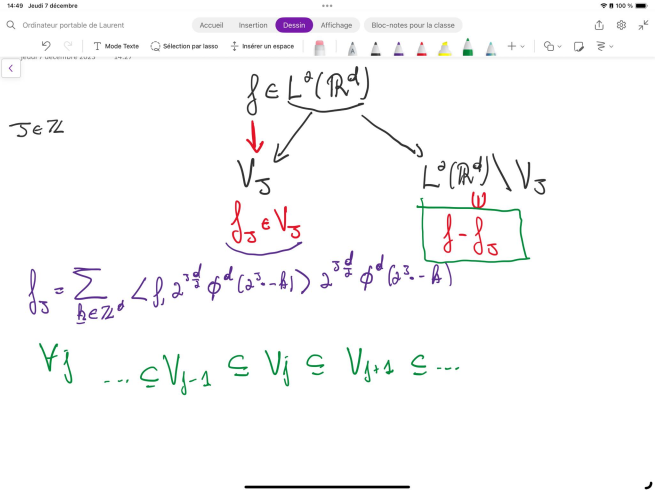

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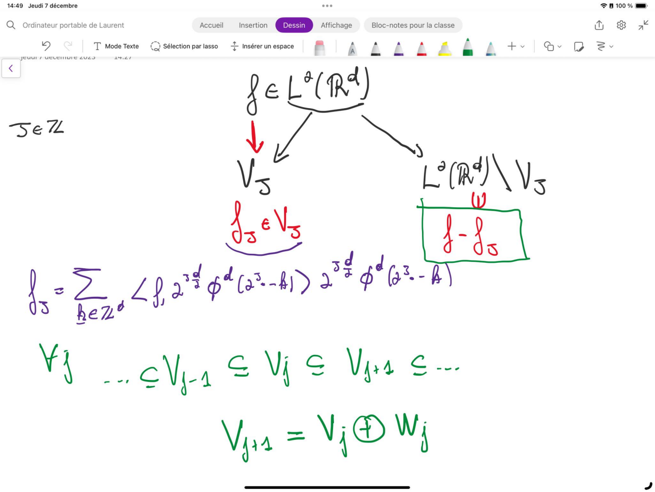

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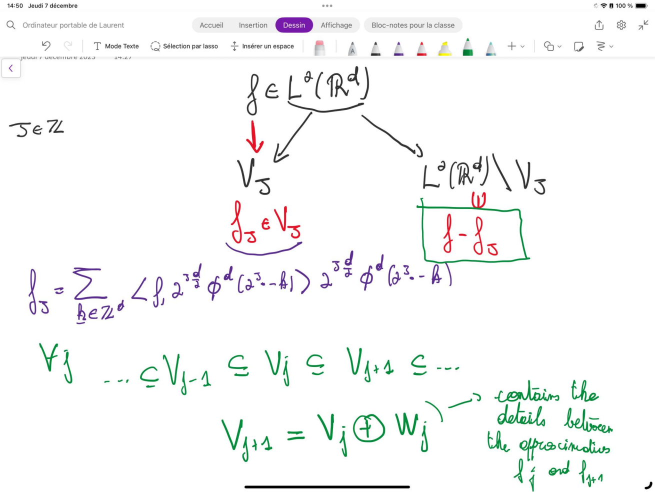

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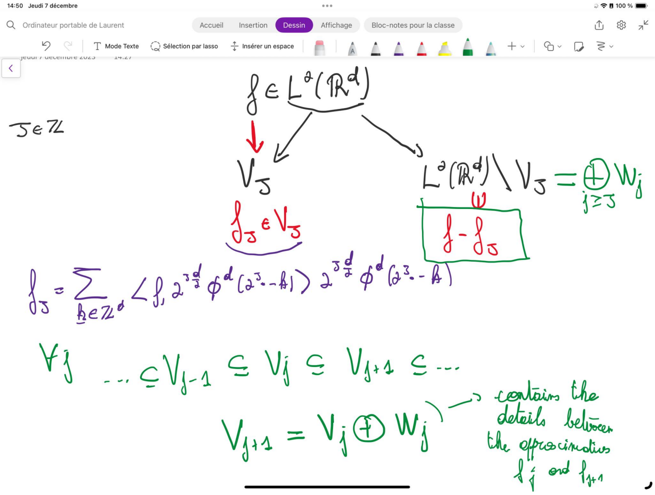

A base for the details spaces

In dimension 1 (Meyer)

There exists a function ψ, called mother wavelet, belonging to W_0^1 and such that, for all $j \in \mathbb{Z}$, the sequence $(2^{j/2}\psi(2^j \cdot -k))_{k\in\mathbb{Z}}$ is an orthonormal basis in W_j^1. Moreover, for all $J \in \mathbb{Z}$, the family

$$\{2^{J/2}\phi^1(2^J x - k) \, : \, k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) \, : \, k \in \mathbb{Z}, \, j \geqslant J\}$$

is a base in $L^2(\mathbb{R})$, called wavelet base.

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In dimension d: tensor product

$$\{\bigotimes_{\ell=1}^{d} \phi_{J,k_\ell} \,:\, k_\ell \in \mathbb{Z}\} \cup \{\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \,:\, k_\ell \in \mathbb{Z},\, \max_{1\leqslant \ell \leqslant d} j_\ell \geqslant J\}$$

Processes in the dth Wiener chaos

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$$\{I_d(K(t,\bullet))\}_{t\geqslant 0}$$

where, for all $t \geqslant 0$, the function $K(t,\bullet) \in L^2(\mathbb{R}^d)$.

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Hermite process

$$K(t,x_1,\ldots,x_d) = c(h,d)\int_0^t \prod_{\ell=1}^d (s-x_\ell)_+^{\frac{h-1}{d}-\frac{1}{2}}\,ds$$

with $h \in \left(\frac{1}{2},1\right)$.

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3. As soon as $d \geqslant 2$, it is a non Gaussian process.

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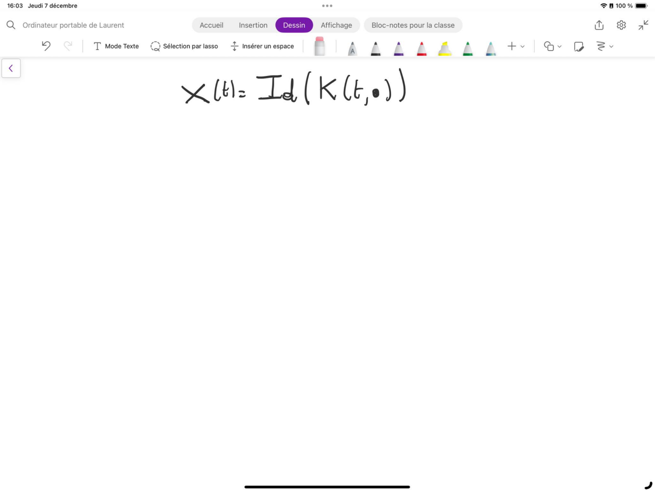
1. First appeared in non-central limit theorems (Breuer, Dobrushin, Major,...).
2. For $d = 1$ (resp. $d = 2$), it corresponds to the Fractional Brownian Motion (resp. Rosenblatt process).
3. As soon as $d \geqslant 2$, it is a non Gaussian process.
4. Enjoyable properties: self-similarity, stationnarity of increments, Hölder regularity,...

Expansion: a general strategy



$$X(t) = I_d\big(K(t,\bullet)\big)$$

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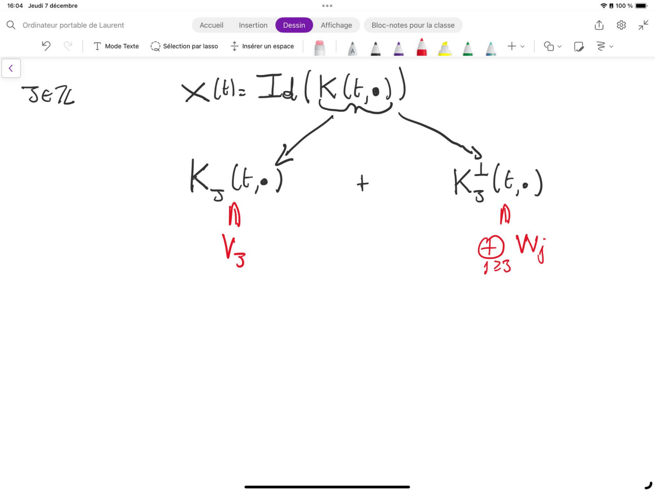

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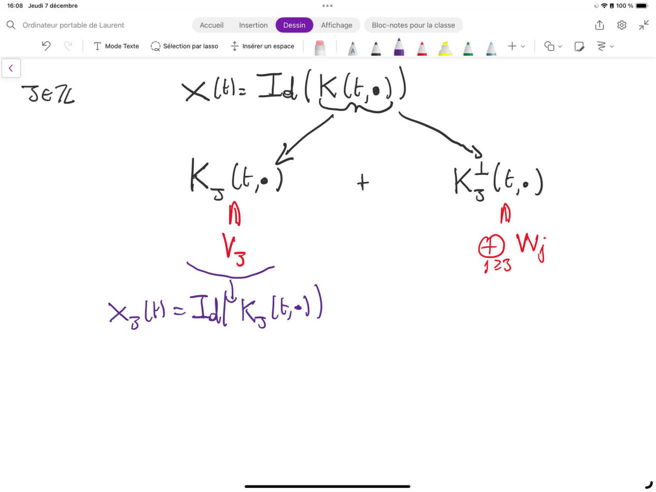

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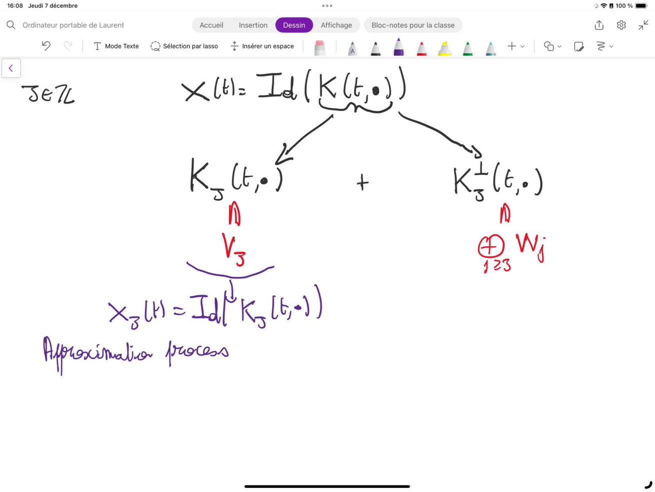

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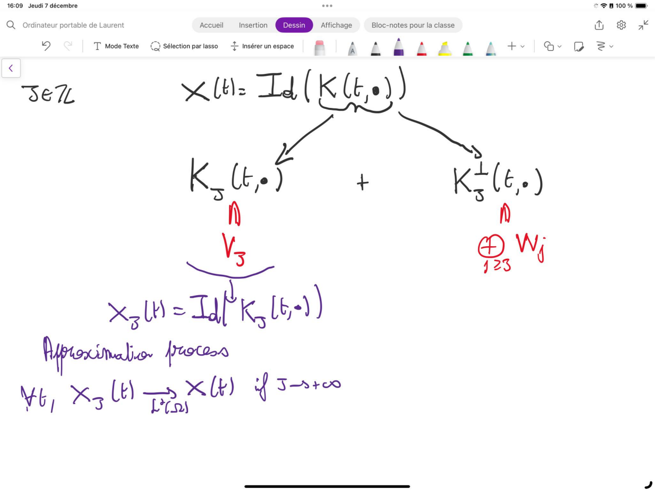

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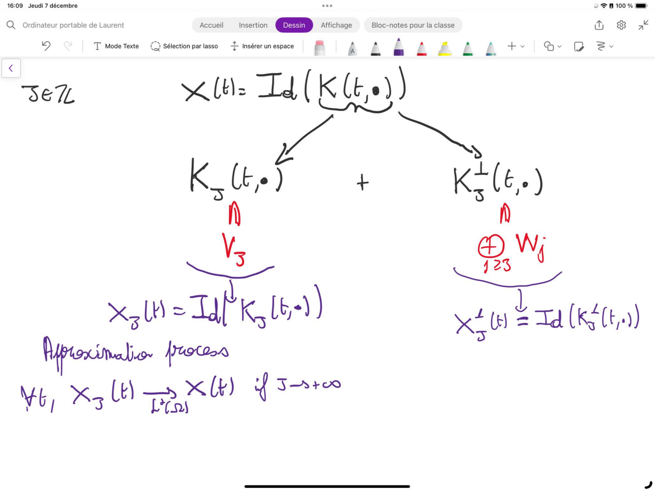

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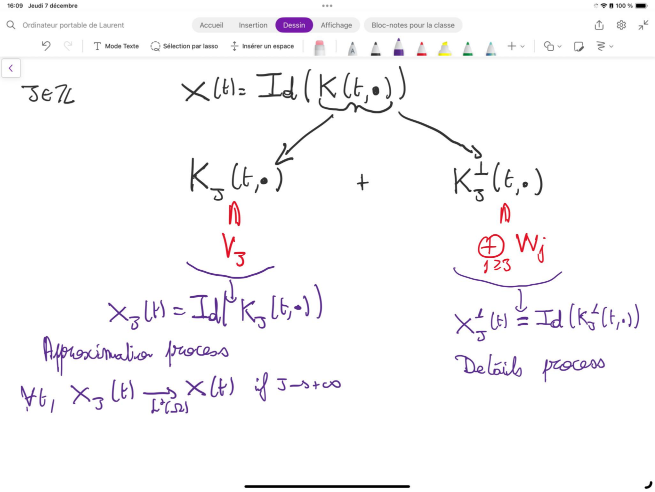

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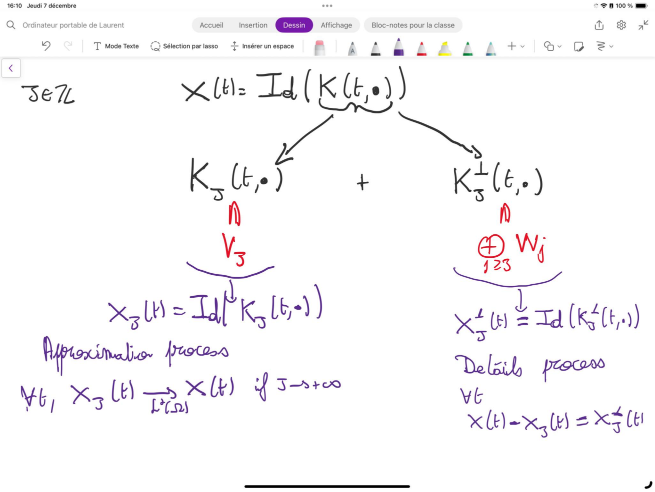

What we get (for "free")

For all $t \geqslant 0$ and $J \in \mathbb{N}$ (fixed)

$$X(t) = X_J(t) + X_J^{\perp}(t)$$

where

$$X_J(t) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle K(t, \bullet), \bigotimes_{\ell=1}^{d} \phi_{J,k_\ell} \rangle I_d \left(\bigotimes_{\ell=1}^{d} \phi_{J,k_\ell} \right)$$

and

$$X_J^{\perp}(t) = \sum_{\substack{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2 \\ \max_{\ell \in [\![1,d]\!]} j_\ell \geqslant J}} \langle K(t, \bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \rangle I_d \left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \right)$$

with convergence of the series in $L^2(\Omega)$.

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- more explicit expressions;
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- rate of convergence:

$$\|X - X_J\|_{I,\infty} = \|X_J^{\perp}\|_{I,\infty}$$

as $J \to +\infty$, for all compact interval $I \subset [0, +\infty)$.

What we do

The product formula and the connection with Hermite polynomials give

A.Ayache, J.Hamonier, L.L.

$$I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\right) = \prod_{\ell=1}^{p} H_{n_\ell}\left(I_1\left(2^{j_\ell/2}\psi(2^{j_\ell}\cdot - k_\ell)\right)\right)$$

where n_ℓ is the multiplicity of (j_ℓ, k_ℓ) in $(\mathbf{j}, \mathbf{k})$.

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while, if we work with the Meyer wavelet basis,

$$\langle K(t,\bullet),\bigotimes_{\ell=1}^{d}\psi_{j_\ell,k_\ell}\rangle=2^{j_1(1-h_1)+\cdots+j_d(1-h_d)}\int_0^t\prod_{\ell=1}^{d}\psi_h(2^{j_\ell}s-k_\ell)\,ds$$

with ψ_h a fractional primitive of ψ.

Rate of convergence

Using

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Theorem (A.Ayache, J.Hamonier, L.L.)

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X - X_J\|_{I,\infty} = \|X_{\mathbf{h},J}^{(d,\perp)}\|_{I,\infty} \leqslant C J^{\frac{d}{2}} 2^{-J(h-1/2)}. \tag{8}$$

Explicit expression for the approximation process

Theorem (A.Ayache, J.Hamonier, L.L.)

The approximation process can be expressed, for all $t \in \mathbb{R}_+$, with convergence in $L^2(\Omega)$:

$$X_J(t) = 2^{-J(h-1)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(\frac{h-1}{d}+\frac{1}{2})}(2^J s - k_\ell)\, ds \right) \sigma_{J,\mathbf{k}}^{(\mathbf{h})}, \tag{9}$$

with

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P\in\mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[\epsilon_{J,k_{l_r}}^{(\frac{h-1}{d}+\frac{1}{2})} \epsilon_{J,k_{l'_r}}^{(\frac{h-1}{d}+\frac{1}{2})}] \prod_{s=m+1}^{d-m} \epsilon_{J,k_{l''_r}}^{(\frac{h-1}{d}+\frac{1}{2})}.$$

where $\epsilon_{J,k}^{(\delta)} := \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} I_1(\phi_{J,p-k})$ is the FARIMA sequence associated to $(I_1(\phi_{J,k}))_k$. Moreover this series is also almost surely uniformly convergent in t on each compact interval of $\mathbb{R}_+$.

What for? Numerical simulations !

The simulation process at scale J

$$S_J(t) = 2^{-J(h-1)} \sum_{\mathbf{k}\in\mathcal{J}_J^1(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_{\Delta}^{\left(\frac{h-1}{d}+\frac{1}{2}\right)}(s-k_\ell)\, ds$$

with

$$\mathcal{J}_J^1(t) := \{\mathbf{k} \in (D_J^1(t))^d \; : \; \max_{\ell,\ell'\in[\![1,d]\!]} |k_\ell - k_{\ell'}| \leqslant 2^{\varepsilon J}\},$$

where

$$D_j^1(t) := \{k \in \mathbb{Z} : [k2^{-j} - 2^{-ja}, k2^{-j} + 2^{-ja}] \subseteq [0,t]\},$$

Towards numerical simulation

A.Ayache, J.Hamonier, L.L.

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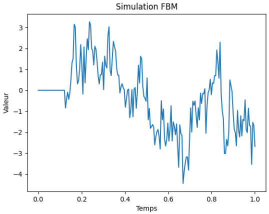
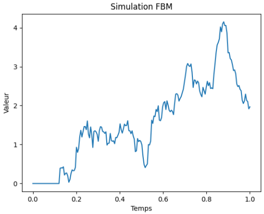
Work in progress

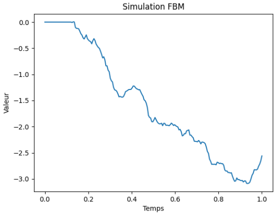
d=1 (Fractional Brownian motion)



$h = 0,2$



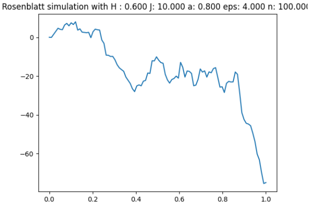
$h = 0,5$



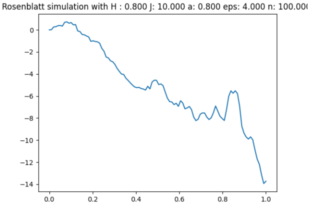
$h = 0,8$

Work in progress

d=2 (Rosenblatt process)



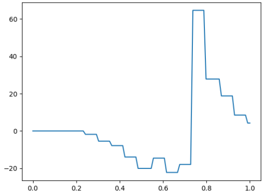
$h = 0,6$



$h = 0,8$

Work in progress

d=3



World premiere ;-)

Second application: precise pointwise regularity

The Khinchin law of the iterated logarithm allows to control the behavior of the Brownian motion B at a given point, in the sense that for every $t \in \mathbb{R}$, it holds

$$\limsup_{r \to 0} \frac{|B(t+r) - B(t)|}{\sqrt{|r| \log\log |r|^{-1}}} = \sqrt{2} \tag{11}$$

on an event of probability one.

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Almost surely, for almost every $t \in \mathbb{R}$,

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This contrasts with the uniform Hölder condition obtained by Paul Lévy: almost surely, one has

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There exists exceptional points, called rapid points, where the law of the iterated logarithm fails.

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Kahane used the expansion of the Brownian motion in the Faber-Schauder system, to propose an easy way to study its regularity and irregularity properties. It allows to recover the law of the iterated logarithm and the estimation of the modulus of continuity of the Brownian motion. Furthermore, Kahane obtained the existence of a third category of points, presenting a slower oscillation:

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and there exist points, called slow points, satisfy the condition

$$\limsup_{r \to 0} \frac{|B(t+r) - B(t)|}{\sqrt{|r|}} < +\infty.$$

Theorem (C. Esser, L.L.)

For all $H \in (0,1)$, there exists an event Ω_H of probability 1 satisfying the following assertions for all $\omega \in \Omega_H$ and every non-empty interval I of $\mathbb{R}$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s \to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log\log|t-s|^{-1}}} < +\infty.$$

- There exists a dense set of *rapid* points $t \in I$ such that

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Theorem (L. Daw, L.L.)

For all $H \in (\frac{1}{2}, 1)$ such that $H_1 + H_2 > \frac{3}{2}$, there exists an event Ω_H of probability 1 satisfying the following assertions for all $\omega \in \Omega_H$ and every $I \neq \emptyset$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s \to t} \frac{|R_H(t,\omega) - R_H(s,\omega)|}{|t-s|^H \log\log|t-s|^{-1}} < +\infty.$$

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If the process is h-self-similar, for all $a > 0$, $\{X(at)\}_{t\geqslant 0} =^{(fdd)} a^h\{X(t)\}_{t\geqslant 0}$ and thus

$$c(a,k) =^{(d)} a^{h+\frac{1}{2}} c(1,0)$$

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Up to now, all wavelet based estimators for the Hurst parameter of Hermite processes failed to be asymptotically normal.Undesirable dependencies between coefficients prevent to prove a CLT.

Creating independence

Recalling

$$K(t, x_1, \ldots, x_d) = c(h,d) \int_0^t \prod_{\ell=1}^{d} (s - x_\ell)_+^{\frac{h-1}{d} - \frac{1}{2}} \, ds$$

and writing

$$f_s(x_1, ..., y_q) = c(h,d) \prod_{\ell=1}^{d} (s - x_\ell)_+^{\frac{h-1}{d} - \frac{1}{2}}$$

we get, by Fubini-type argument and using the compact support of the wavelet

$$c(a,k) = I_d \left(\sqrt{a} \int_{\mathbb{R}} \Psi(x) \mathbb{1}^{\otimes d}_{(-\infty, a(k+1))} \int_{ak}^{a(x+k)} f_s ds \, dx \right). \tag{13}$$

Creating independence

Let $M > 0$. We decompose the coefficient $c(a,k)$ as follows

$$c(a,k) = c(\widetilde{a,k,M}) + c(\underbrace{a,k,M})$$

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$$\mathbb{E}\left[c(\overbrace{a,k,M})^2\right] \leqslant C(d,h)a^{2h+1}(M+1)^{\frac{2h-2}{d}}.$$

Procedure to define an estimator

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But : in $c(a,k) =^{(d)} a^{h+\frac{1}{2}} c(1,0)$, $c(1,0)$ depends on h (and Ψ) in a non-trivial-way.

Procedure to define an estimator

1. Choose a scale N and decompose $[0,1]$ in dyadic interval $\Rightarrow 2^N$ wavelet coefficients $c(2^N, k)$.
2. Choose two parameters $0 < \gamma < \beta < 1$,
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In total, we work with the coefficients

$$A_M(\ell, N) = c\left(\frac{1}{M2^N}, \ell M 2^{[N^\beta]}\right). \tag{16}$$

for $\ell \in \mathcal{L}_{N,\gamma}$, $1 \leqslant M \leqslant q$.

Modified wavelet variation

$$V_{N,M} = \frac{1}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{\ell \in \mathcal{L}_{N,\gamma}} \left[\frac{A_M(\ell, N)^2}{\mathbb{E}\left[A_M(\ell, N)^2\right]} - 1 \right]. \tag{17}$$

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$$V_{N,M,1} = \frac{1}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \frac{2^{N(2H+1)} M^{2H+1}}{C_\Psi(h)} \sum_{\ell \in \mathcal{L}_{N,\gamma}} \left(\widetilde{A_M(\ell,N)}^2 - \mathbb{E}\left[\widetilde{A_M(\ell,N)}^2 \right] \right), \tag{18}$$

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Theorem (L.L., C.A. Tudor)

Let $V_{N,M}$ be given by (17). Then the d-dimensional random vector $(V_{N,M}, M = 1, \dots, q)$ converges in distribution, as $N \to \infty$, to a d-dimensional Gaussian vector $N(0,K)$ with

$$d_W\left((V_{N,M}, M = 1, \dots, q), N(0,K) \right) \leqslant C 2^{-\frac{[N^\gamma]}{2}}.$$

Final steps

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3. Perform a log-regression of $(\widehat{S}_{N,M}, M = 1, ..., q)$ to define

$$\widehat{H}_N = -\frac{1}{2} \frac{\left(\sum_{M=1}^q \log \widehat{S}_{N,M} \log M\right) - \left(\sum_{M=1}^q \log \widehat{S}_{N,M}\right)\left(\sum_{M=1}^q \log M\right)}{\left(\sum_{M=1}^q (\log M)^2\right) - \left(\sum_{M=1}^q \log M\right)^2} - \frac{1}{2}$$

Winning strategy!

Theorem (L.L., C.A. Tudor)

The estimator $\widehat{H}_N$ is strongly consistent, i.e.

$$\widehat{H}_N \underset{N\to\infty}{\longrightarrow} h \text{ almost surely.} \tag{20}$$

Moreover,

$$\sqrt{|\mathcal{L}_{N,\gamma}|}(h - \widehat{H}_N) \underset{N\to\infty}{\overset{(d)}{\longrightarrow}} N(0,\sigma^2), \tag{21}$$

with σ^2 computable.