Représentation par ondelettes de processus définis via des intégrales de Wiener-Itô multiples

Séminaire de Vannes du LMBA

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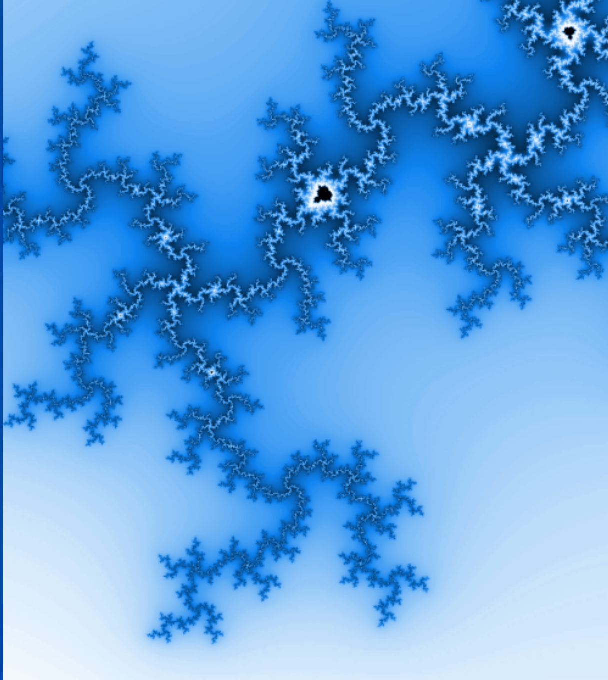
LIÈGE université
Mathématique



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$$f=\sum_{j_1,\ldots,j_d=1}^{n} a_{j_1,\ldots,j_d}\mathbb{1}_{[s_{j_1},t_{j_1})}\otimes\cdots\otimes\mathbb{1}_{[s_{j_d},t_{j_d})}, \tag{1}$$

where, $\otimes$ stands for the tensor product, $a_{j_1,\ldots,j_d}$ are such that, for all permutation σ, $a_{\sigma(j_1),\ldots,\sigma(j_d)}=a_{j_1,\ldots,j_d}$ and $a_{j_1,\ldots,j_d}=0$ as soon as two indices $j_1,\ldots,j_d$ are equal and, for all $1\leq\ell\neq\ell'\leq d$, $[s_{j_\ell},t_{j_\ell})\cap[s_{j_{\ell'}},t_{j_{\ell'}})=\emptyset$.

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$$I_d(f):=\sum_{j_1,\dots,j_d=1}^{n} a_{j_1,\dots,j_d}(B(t_{j_1})-B(s_{j_1}))\times\dots(B(t_{j_d})-B(s_{j_d})). \tag{2}$$

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It is a random variable in $L^2(\Omega)$.

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- $(I_d(f_j))_j$ is a Cauchy sequence in $L^2(\Omega)$;
- $I_d(f) := \lim_{j \to +\infty} I_d(f_j)$ (in $L^2(\Omega)$).

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The dth Wiener chaos is defined as the closed linear subspace of $L^2(\Omega)$ generated by the random variables of the form $I_d(f)$

Important facts

Wiener isometry

If $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^{d'})$, we have

$$\mathbb{E}\left[I_d(f) I_{d'}(g)\right] = \begin{cases} d!\langle f, g\rangle & \text{if } d = d' \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\langle\cdot,\cdot\rangle$ stands for the canonical scalar product in $L^2(\mathbb{R}^d)$.

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Connection with Hermite polynomials

if $f \in L^2(\mathbb{R})$ is such that $\|f\|_{L^2(\mathbb{R})} = 1$, then

$$H_d\left(I_1(f)\right) = I_d(f^{\otimes d}),$$

where H_d is the dth Hermite polynomial $H_d(x) = (-1)^d e^{x^2/2} D^d e^{-x^2/2}$.

Important facts

Product formula

If $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, we have

$$I_m(f) I_n(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f \otimes_r g), \tag{5}$$

where, for all $0 \leq r \leq m \wedge n$, $f \otimes_r g$ is the $L^2(\mathbb{R}^{m+n-2r})$ function defined, for all $(x_1, \ldots, x_{m+n-2r}) \in \mathbb{R}^{m+n-2r}$, through the Lebesgue integral

$$\begin{aligned} &(f \otimes_r g)(x_1, \ldots, x_{m+n-2r}) \\ &:= \int_{\mathbb{R}^r} f(x_1, \ldots, x_{m-r}, s_1, \ldots, s_r) g(x_{m-r+1}, \ldots, x_{m+n-2r}, s_1, \ldots, s_r) \, ds_1 \ldots ds_r , \end{aligned}$$

Multiresolution analysis

Definition

A multiresolution analysis of the Hilbert space $L^2(\mathbb{R}^d)$ is given by a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that

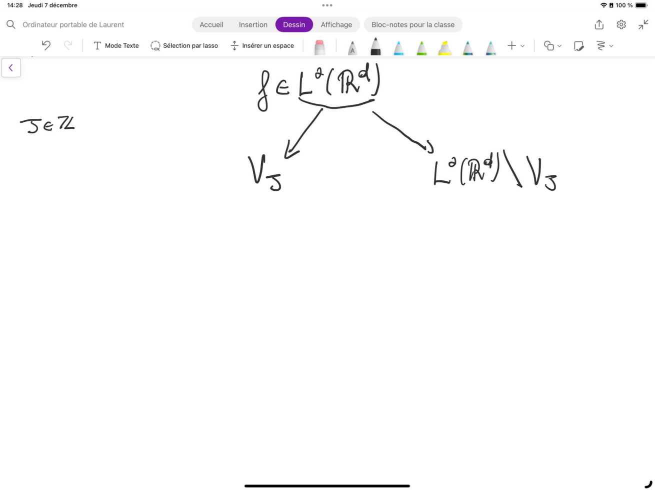
(a) for all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$;

(b) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$;

(c) for all $j \in \mathbb{Z}$, $V_j = \{f(2^j\cdot) : f \in V_0\}$;

(d) there exists a scaling function $\phi^d \in V_0$ such that the sequence $(\phi^d(\cdot - k))_{k\in\mathbb{Z}^d}$ is an orthogonal basis of V_0.

Approximation strategy



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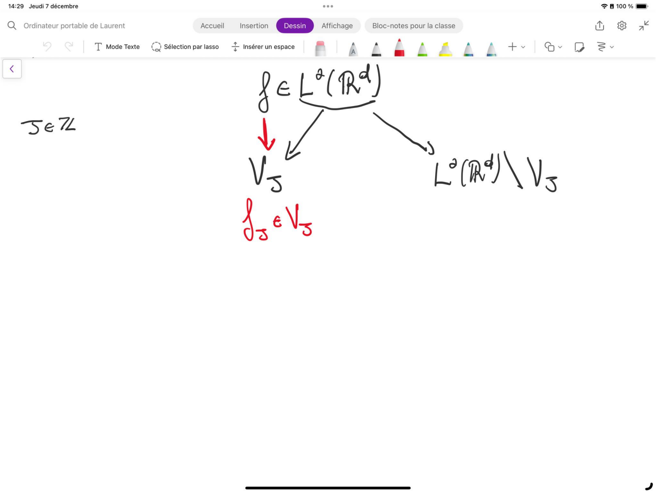

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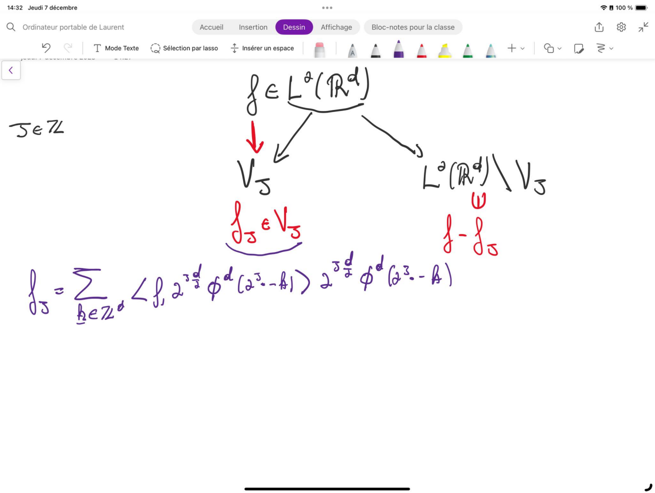

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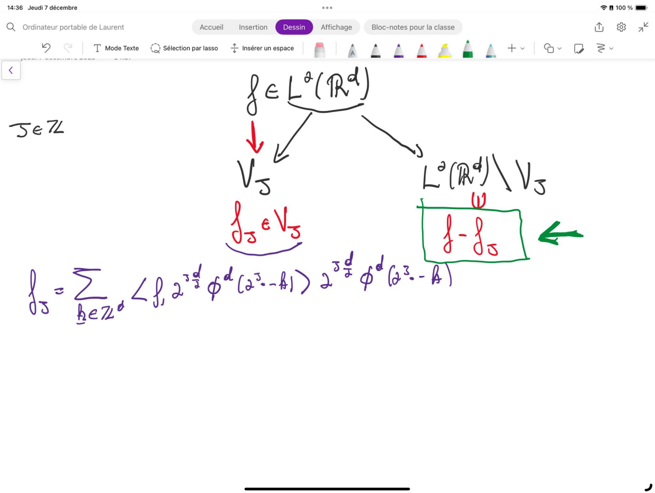

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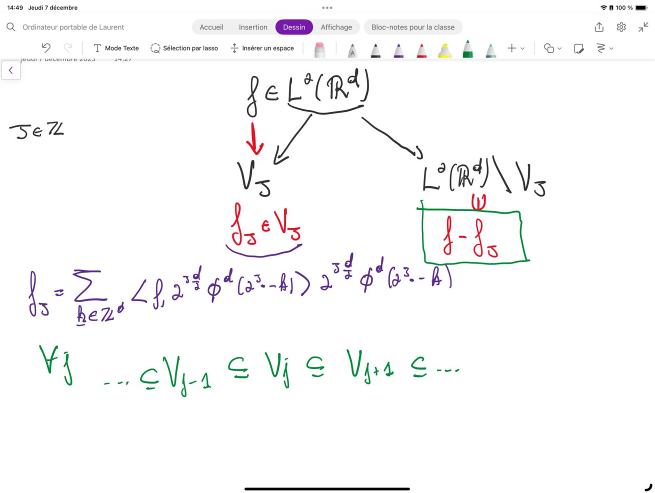

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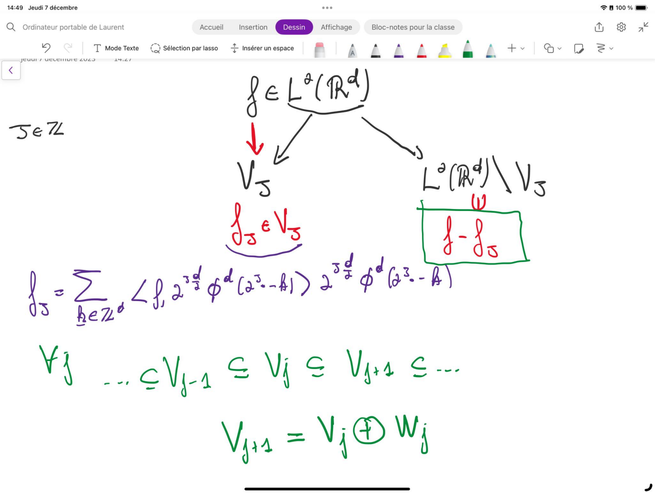

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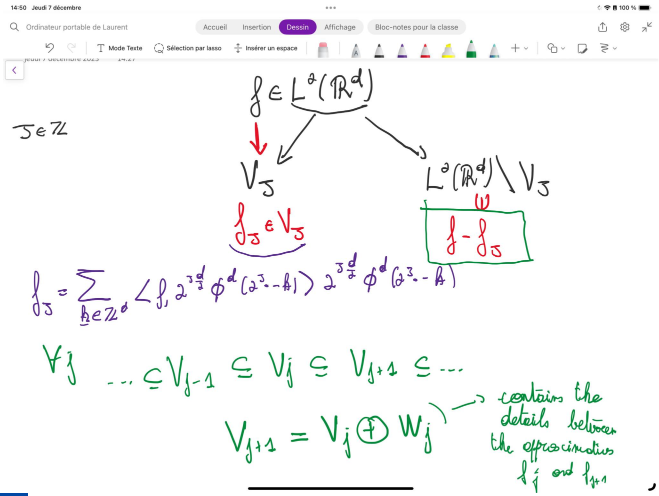

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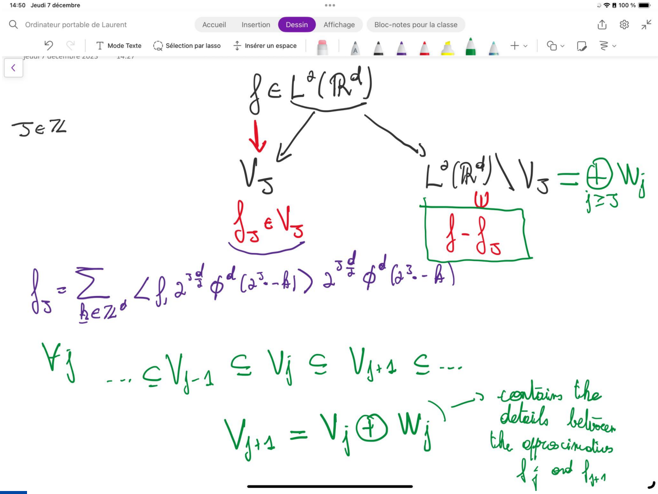

A base for the details spaces

In dimension 1 (Meyer)

There exists a function ψ, called mother wavelet, belonging to W_0^1 and such that, for all $j \in \mathbb{Z}$, the sequence $(2^{j/2}\psi(2^j \cdot - k))_{k\in\mathbb{Z}}$ is an orthonormal basis in W_j^1. Moreover, for all $J \in \mathbb{Z}$, the family

$$\{2^{J/2}\phi^1(2^J x - k) : k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z},\, j \geq J\}$$

is a base in $L^2(\mathbb{R})$, called wavelet base.

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In higher dimension: tensor products

Usually, we use, for $J \in \mathbb{Z}$,

$$\{\bigotimes_{\ell=1}^{d} \phi_{J,k_\ell} : k_\ell \in \mathbb{Z}\} \cup \{\bigotimes_{\ell=1}^{d} \psi^{(\ell)}_{j,k_\ell} : k_\ell \in \mathbb{Z},\, j \geq J, \exists \ell \text{ s.t.} \psi^{(\ell)} = \psi\}$$

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Better in our context

$$\{\bigotimes_{\ell=1}^{d} \phi_{J,k_\ell} : k_\ell \in \mathbb{Z}\} \cup \{\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} : k_\ell \in \mathbb{Z},\ \max_{1\leq \ell\leq d} j_\ell \geq J\}$$

Processes in the dth Wiener chaos

General definition

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Hermite process

$$K(t,x_1,\ldots,x_d) = c(h,d)\int_0^t \prod_{\ell=1}^d (s-x_\ell)_+^{\frac{h-1}{d}-\frac{1}{2}}\,ds$$

whith $h \in \left(\frac{1}{2},1\right)$.

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Generalized Hermite processes (Bai, Taqqu)

$$K(t,x_1,\ldots,x_d) = \frac{1}{\prod_{\ell=1}^d \Gamma(h_\ell - 1/2)} \int_0^t \prod_{j=1}^d (s-x_\ell)_+^{h_\ell - 3/2}\, ds$$

with $h_1,\ldots,h_d \in \left(\frac{1}{2},1\right)$ such that $\sum_{\ell=1}^d h_\ell > d - \frac{1}{2}$.

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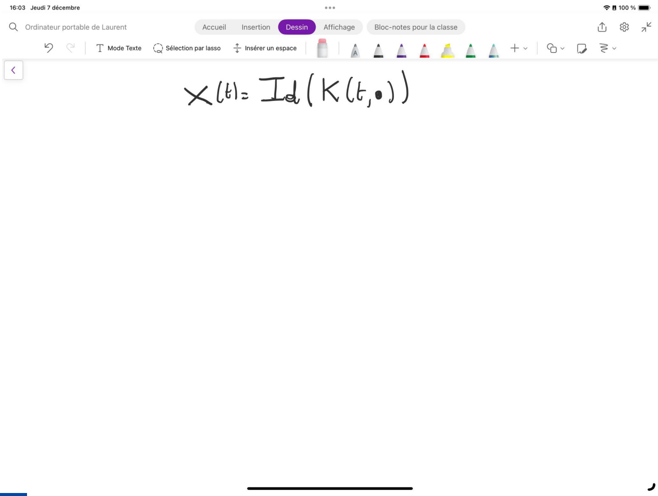
2. For $d = 1$ (resp. $d = 2$), it corresponds to the Fractional Brownian Motion (resp. Rosenblatt process).
3. As soon as $d \geq 2$, it is a non Gaussian process.
4. Enjoyable properties: self-similarity, stationnarity of increments, Hölder regularity,...

Expansion strategy



$$X(t) = I_d(K(t,\bullet))$$

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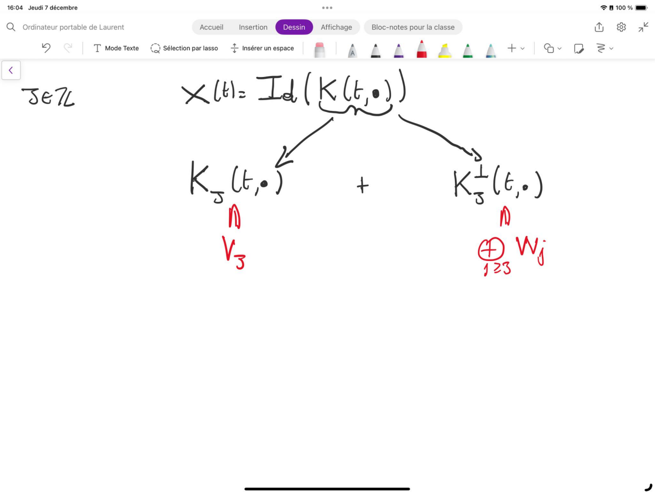

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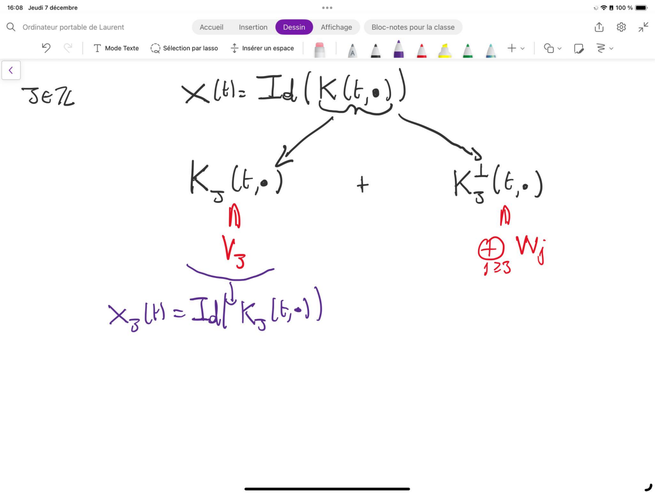

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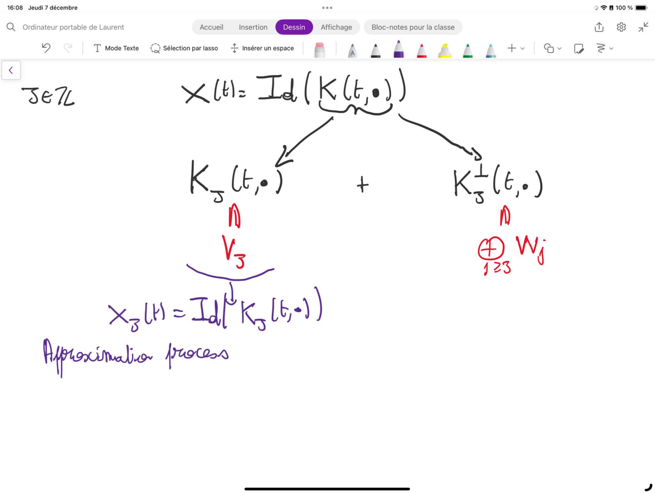

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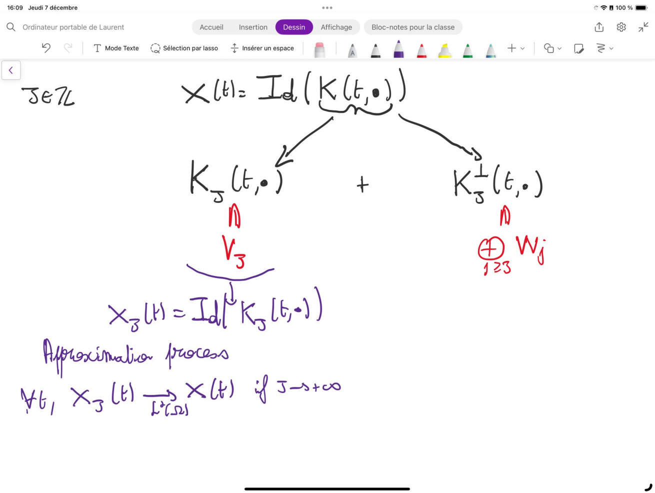

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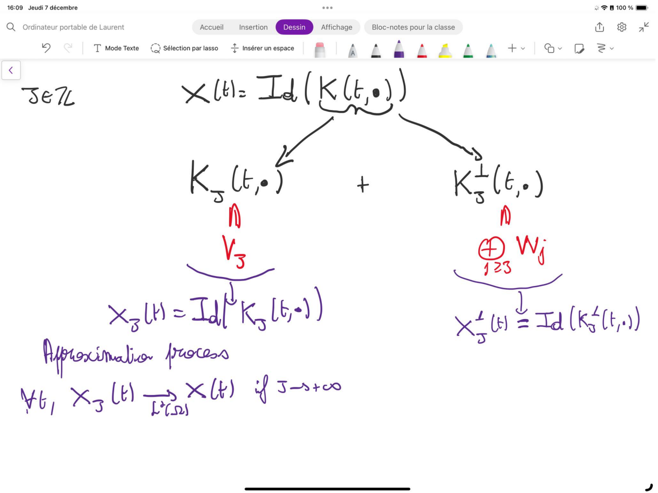

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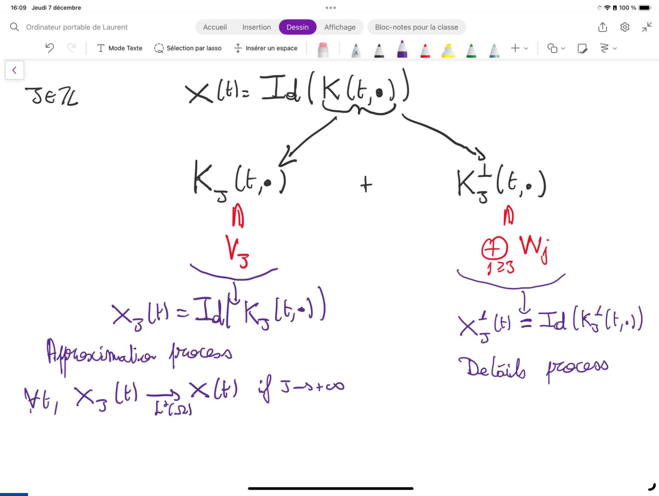

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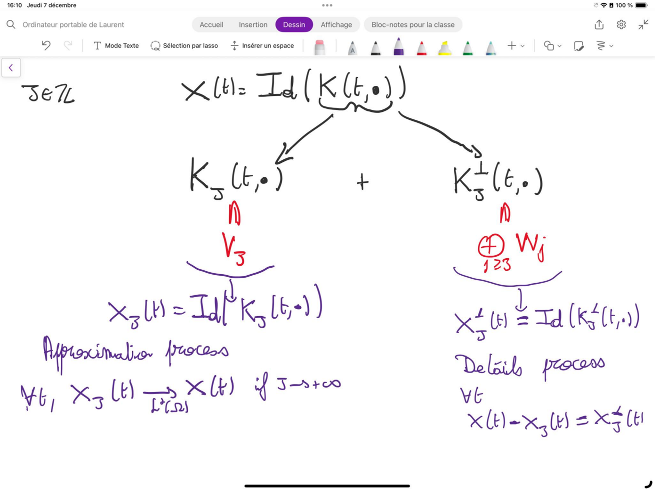

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- Simulation ?

What was known up to now?

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Ayache-Esmili (2020)

Alternative wavelet-type expansion of (generalized) Rosenblatt process.

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2. Even in the case of the Rosenblatt process, the rate of convergence for the approximation was unknown before Ayache-Esmili (2020).

Game changer

$$\text{for all } t \geq 0 \quad K_J^{\perp}(t,\bullet) = \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max\limits_{\ell\in[[1,d]]} j_\ell \geq J}} \langle K(t,\bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\rangle \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}$$

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for all $t \geq 0$ $$X_J^{\perp}(t) = \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ \max_{\ell\in[\![1,d]\!]} j_\ell \geq J}} \langle K(t,\bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \rangle I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \right)$$ in $L^2(\Omega)$.

Explicit expression of the details process

$$\langle K(t,\bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \rangle = 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \int_0^t \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds$$

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A.Ayache, J.Hamonier, L.L.

$$I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\right) = \prod_{\ell=1}^{p} H_{n_\ell}\left(I_1\left(2^{j_\ell/2}\psi(2^{j_\ell}\cdot - k_\ell)\right)\right)$$

where n_ℓ is the multiplicity of (j_ℓ,k_ℓ) in $(\mathbf{j},\mathbf{k})$.

Rate of convergence

A.Ayache, J.Hamonier, L.L.

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X - X_J\|_{I,\infty} = \|X_{\mathbf{h},J}^{(d,\perp)}\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{6}$$

Tools for the approximation process

Fractional scaling function

$$\widehat{\Phi}_{\Delta}^{(\delta)}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi) \ \forall \xi \neq 0 \text{ and } \widehat{\Phi}_{\Delta}^{(\delta)}(0) = 1$$

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$$\widehat{\Phi}^{(\delta)}_{\Delta}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi)\ \forall \xi \neq 0 \text{ and } \widehat{\Phi}^{(\delta)}_{\Delta}(0) = 1$$

With a bit of Fourier analysis tools and tricks, we get to

$$K^{(d)}_{\mathbf{h},J}(t,\bullet) = 2^{-J(h_1+\cdots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \beta^{(\mathbf{h})}_{\mathbf{k}} 2^{J\frac{d}{2}} \bigotimes_{\ell=1}^{d} \Phi^{-(h_\ell-1/2)}_{J,k_\ell}$$

where

$$\beta^{(\mathbf{h})}_{\mathbf{k}} := \int_0^t \prod_{\ell=1}^{d} \Phi^{(h_\ell-1/2)}_{\Delta}(2^J s - k_\ell)\, ds \text{ and } \widehat{\Phi}^{-(\delta)}(\xi) = (1-e^{i\xi})^{-\delta}\widehat{\phi}(\xi)$$

A “new” random part

We need to compute

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} := 2^{J\frac{d}{2}} I_d\left(\bigotimes_{\ell=1}^{d} \Phi_{J,k_\ell}^{-(h_\ell-1/2)}\right)$$

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Lemma (A. Ayache, J. Hamonier, L.L.)

For all $\delta \in (0, \frac{1}{2})$, we have

$$\Phi^{-(\delta)} = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} \phi(\cdot + p) \tag{7}$$

with convergence in $L^2(\mathbb{R})$, where

$$\gamma_p^{(\delta)} := \frac{\delta\,\Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}.$$

A “new” random part

We need to compute

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} := 2^{J\frac{d}{2}} I_d\left(\bigotimes_{\ell=1}^{d} \Phi_{J,k_\ell}^{-(h_\ell-1/2)}\right)$$

$$= \sum_{\mathbf{p}\in\mathbb{N}_0^d} \left(\prod_{\ell=1}^{d} \gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(2^{J\frac{d}{2}} I_d\left(\bigotimes_{\ell=1}^{d} \phi_{J,p_\ell-k_\ell}\right)\right)$$

$$= \sum_{\mathbf{p}\in\mathbb{N}_0^d} \left(\prod_{\ell=1}^{d} \gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(\prod_{\ell=1}^{n} H_{n_\ell}\left(I_1\left(\bigotimes_{\ell=1}^{d} \phi_{J,\widetilde{p_\ell-k_\ell}}\right)\right)\right)$$

A “new” random part

We need to compute

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} := 2^{J\frac{d}{2}} I_d\left(\bigotimes_{\ell=1}^{d} \Phi_{J,k_\ell}^{-(h_\ell-1/2)}\right)$$

FARIMA (autogressive fractionally integrated moving average)

Let $\{Z_j\}_{j\in\mathbb{Z}}$ be a sequence of i.i.d. centred Gaussian random variables and $\delta \in (-\frac{1}{2}, \frac{1}{2})$. The Gaussian FARIMA $(0,\delta,0)$, denoted by $\{Z_j^{(\delta)}\}_{j\in\mathbb{Z}}$, is defined, for all $j \in \mathbb{Z}$ as

$$Z_j^{(\delta)} := \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} Z_{j-p} \qquad (7)$$

Random part with FARIMA

Hermite polynomials and partitions

The dth Hermite polynomials can be written as

$$H_d(x) = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m a_m^{(d)} x^{d-2m},$$

where $a_m^{(d)}$ is the number of partitions of $\{1,\ldots,d\}$ with m (non ordered) pairs and $d-2m$ singletons.

Random part with FARIMA

$\mathscr{P}_m^{(d)}$ is the set of partitions of $\{1,\ldots,d\}$ with m (non ordered) pairs and $d-2m$ singletons.

(A. Ayache, J. Hamonier, L.L.)

For all $J \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, we have

$$2^{J\frac{d}{2}} I_d\left(\bigotimes_{\ell=1}^{d} \phi_{J,k_\ell}\right) = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathscr{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[I_1(\phi_{J,k_{l_r}}) I_1(\phi_{J,k_{l'_r}})] \prod_{s=m+1}^{d-m} I_1(\phi_{J,k_{l''_s}})$$

Explicit expression for the approximation process

In total, we have

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathscr{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[\epsilon_{J,k_{l_r}}^{(h_{l_r}-1/2)} \epsilon_{J,k_{l'_r}}^{(h_{l'_r}-1/2)}] \prod_{s=m+1}^{d-m} \epsilon_{J,k_{l''_r}}^{(h_{l''_r}-1/2)}.$$

where $\epsilon_{J,k}^{(\delta)} := \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} I_1(\phi_{J,p-k})$ is the FARIMA sequence associated to $(I_1(\phi_{J,k}))_k$.

Explicit expression for the approximation process

A.Ayache, J.Hamonier, L.L.

The approximation process can be expressed, for all $t \in \mathbb{R}_+$, as:

$$X_J(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \right) \sigma_{J,\mathbf{k}}^{(\mathbf{h})}, \tag{8}$$

where the series is convergent in $L^2(\Omega)$. Moreover this series is also almost surely uniformly convergent in t on each compact interval of $\mathbb{R}_+$.

Towards numerical simulation

In the proof of the last Theorem, we notice that the rate of convergence is mainly governed by the terms

$$2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\right) \int_0^t \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds$$

for which there is $\ell \in [[1,d]]$ and

$$k_\ell \in D_j^1(t) := \{k \in \mathbb{Z} : [k2^{-j}-2^{-ja}, k2^{-j}+2^{-ja}] \subseteq [0,t]\},$$

with $\frac{1}{2} < a < 1$.

Towards numerical simulation

Definition

$$\mathscr{J}_J^1(t) := \{\mathbf{k} \in (D_J^1(t))^d : \max_{\ell,\ell' \in [[1,d]]} |k_\ell - k_{\ell'}| \leq 2^{\varepsilon J}\},$$

Towards numerical simulation

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The simulation process at scale J

$$S_J(t) = 2^{-J(h_1+\ldots+h_d-d+1)} \sum_{\mathbf{k} \in \mathscr{J}_J^1(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(s-k_\ell)\, ds.$$

Towards numerical simulation

A.Ayache, J.Hamonier, L.L.

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X - S_J\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{9}$$