Wavelet representations for stochastic processes defined by multiple Wiener-Itô integrals

Young Scholar Day of the Belgian Mathematical Society

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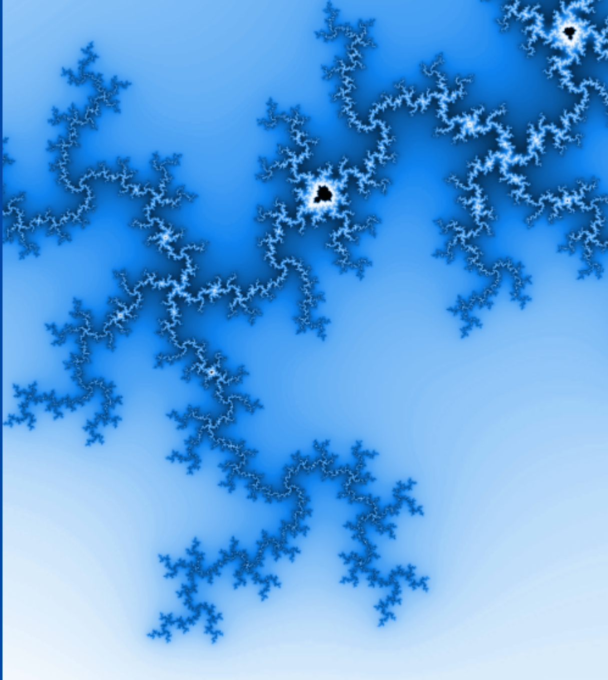
LIÈGE université
Mathématique



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$$f=\sum_{j_1,\dots,j_d=1}^{n} a_{j_1,\dots,j_d}\mathbb{1}_{[s_{j_1},t_{j_1})}\otimes\cdots\otimes\mathbb{1}_{[s_{j_d},t_{j_d})}, \qquad (1)$$

where, $\otimes$ stands for the tensor product, $a_{j_1,\dots,j_d}$ are such that, for all permutation σ, $a_{\sigma(j_1),\dots,\sigma(j_d)}=a_{j_1,\dots,j_d}$ and $a_{j_1,\dots,j_d}=0$ as soon as two indices $j_1,\dots,j_d$ are equal and, for all $1\le\ell\neq\ell'\le d$, $[s_{j_\ell},t_{j_\ell})\cap[s_{j_{\ell'}},t_{j_{\ell'}})=\emptyset$.

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$$I_d(f):=\sum_{j_1,\dots,j_d=1}^{n} a_{j_1,\dots,j_d}(B(t_{j_1})-B(s_{j_1}))\times\dots(B(t_{j_d})-B(s_{j_d})). \tag{2}$$

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It is a random variable in $L^2(\Omega)$.

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- $(I_d(f_j))_j$ is a Cauchy sequence in $L^2(\Omega)$;
- $I_d(f) := \lim_{j\to+\infty} I_d(f_j)$ (in $L^2(\Omega)$).

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The dth Wiener chaos is defined as the closed linear subspace of $L^2(\Omega)$ generated by the random variables of the form $I_d(f)$

Important facts

Wiener isometry

If $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^{d'})$, we have

$$\mathbb{E}\left[I_d(f) I_{d'}(g)\right] = \begin{cases} d!\langle f, g\rangle & \text{if } d = d' \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical scalar product in $L^2(\mathbb{R}^d)$.

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Connection with Hermite polynomials

if $f \in L^2(\mathbb{R})$ is such that $\|f\|_{L^2(\mathbb{R})} = 1$, then

$$H_d\left(I_1(f)\right) = I_d(f^{\otimes d}),$$

where H_d is the dth Hermite polynomial $H_d(x) = (-1)^d e^{x^2/2} D^d e^{-x^2/2}$.

Important facts

Product formula

If $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, we have

$$I_m(f) I_n(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f \otimes_r g), \tag{5}$$

where, for all $0 \leq r \leq m \wedge n$, $f \otimes_r g$ is the $L^2(\mathbb{R}^{m+n-2r})$ function defined, for all $(x_1, \dots, x_{m+n-2r}) \in \mathbb{R}^{m+n-2r}$, through the Lebesgue integral

$$\begin{aligned} &(f \otimes_r g)(x_1, \dots, x_{m+n-2r}) \\ &:= \int_{\mathbb{R}^r} f(x_1, \dots, x_{m-r}, s_1, \dots, s_r) g(x_{m-r+1}, \dots, x_{m+n-2r}, s_1, \dots, s_r) \, ds_1 \dots ds_r, \end{aligned}$$

Multiresolution analysis

Definition

A multiresolution analysis of the Hilbert space $L^2(\mathbb{R}^d)$ is given by a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that

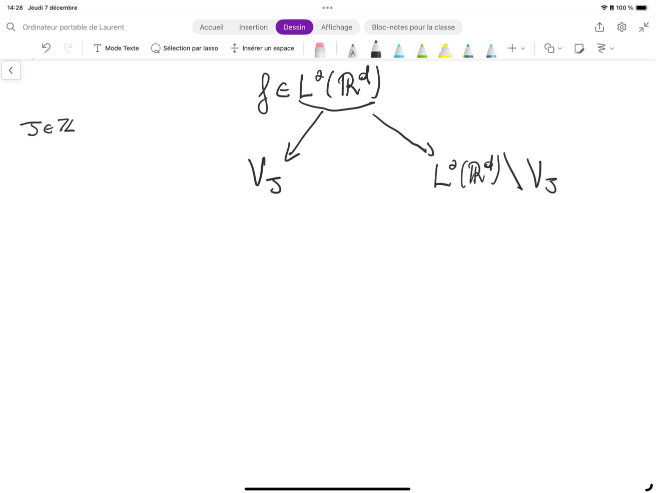
(a) for all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$;

(b) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$;

(c) for all $j \in \mathbb{Z}$, $V_j = \{f(2^j \cdot) : f \in V_0\}$;

(d) there exists a scaling function $\phi^d \in V_0$ such that the sequence $(\phi^d(\cdot - k))_{k\in\mathbb{Z}^d}$ is an orthogonal basis of V_0.

Approximation strategy



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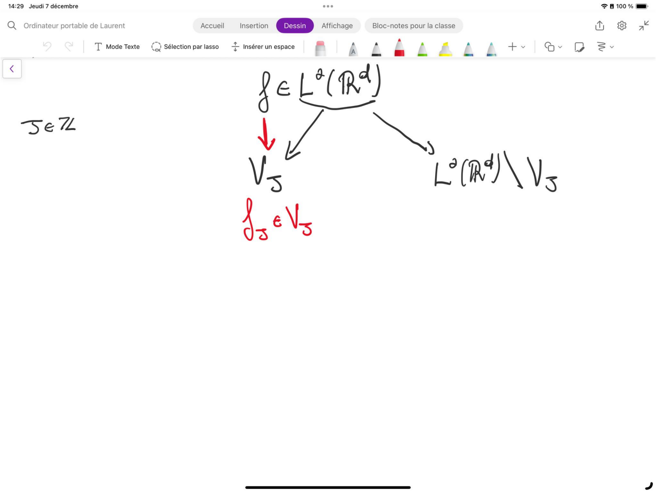

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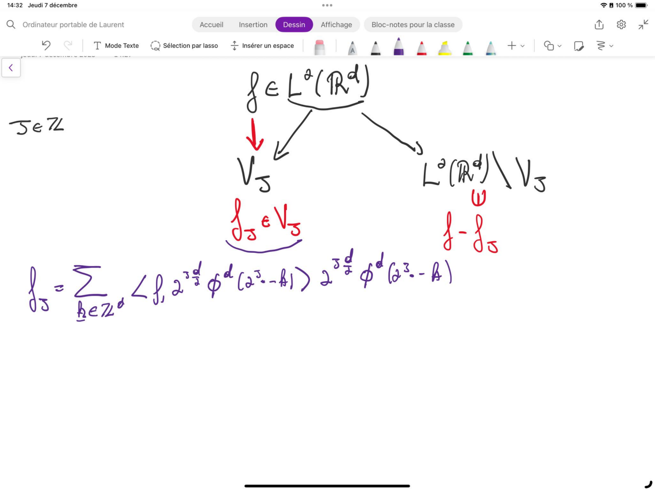

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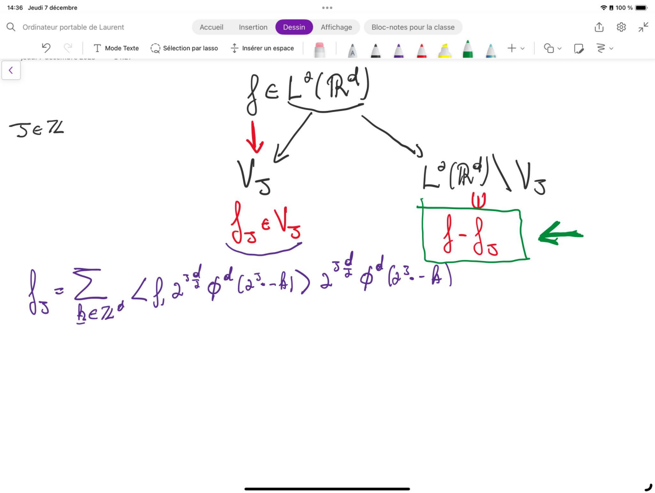

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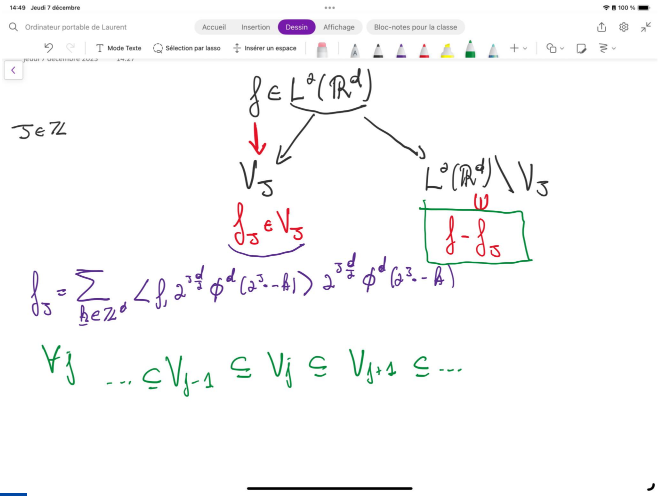

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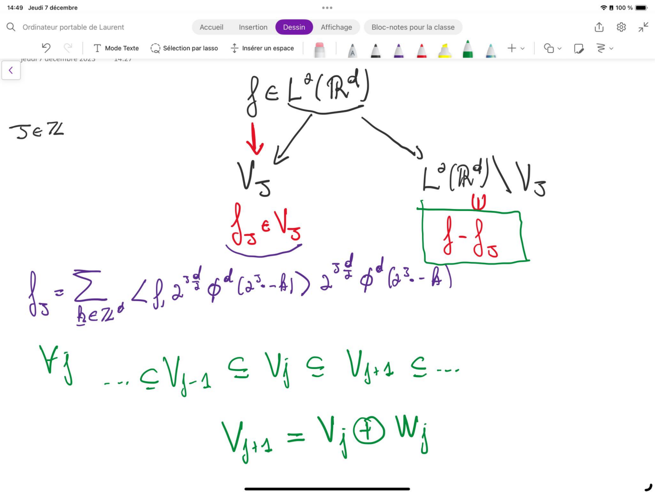

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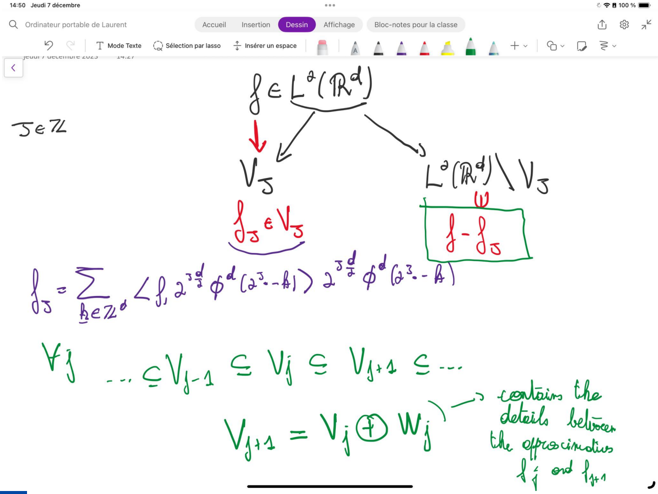

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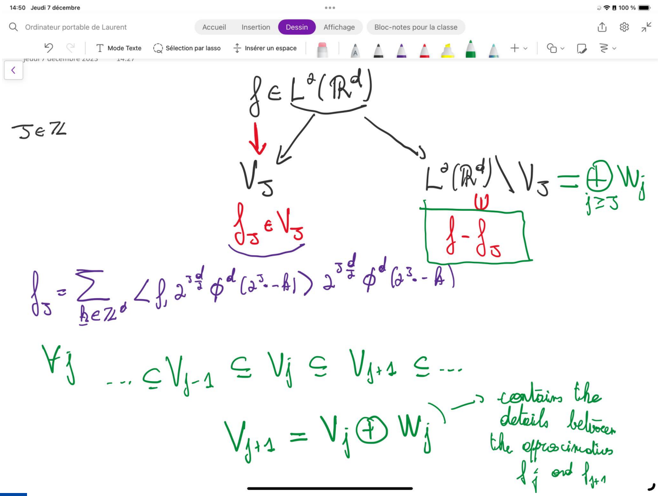

A base for the details spaces

In dimension 1 (Meyer)

There exists a function ψ, called mother wavelet, belonging to W_0^1 and such that, for all $j \in \mathbb{Z}$, the sequence $(2^{j/2}\psi(2^j \cdot -k))_{k\in\mathbb{Z}}$ is an orthonormal basis in W_j^1. Moreover, for all $J \in \mathbb{Z}$, the family

$$\{2^{J/2}\phi^1(2^J x - k) : k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z},\, j \geq J\}$$

is a base in $L^2(\mathbb{R})$, called wavelet base.

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In higher dimension: tensor products

Usually, we use $\{2^{J\frac{d}{2}}\Phi^d(2^J x - k) : k \in \mathbb{Z}^d\} \cup \{2^{j\frac{d}{2}}\Psi(2^j x - k) : k \in \mathbb{Z}^d,\, j \geq J\}$ with Φ^d, d-tensor products of ϕ with itself and Ψ, d-tensor products of ϕ and ψ where at least one term is ψ.

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Better in our context

$$\{2^{J\frac{d}{2}}\Phi^d(2^J x - k) : k \in \mathbb{Z}^d\} \cup \{\prod_{\ell=1}^{d} 2^{\frac{j_\ell}{2}}\psi(2^{j_\ell}x_\ell - k_\ell) : k \in \mathbb{Z}^d,\, \max_{1\leq \ell \leq d} j_\ell \geq J\}$$

Processes in the dth Wiener chaos

General definition

$$\{I_d(K(t,\bullet))\}_{t\geq 0}$$

where, for all $t \geq 0$, the function $K(t,\bullet) \in L^2(\mathbb{R}^d)$.

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Hermite process

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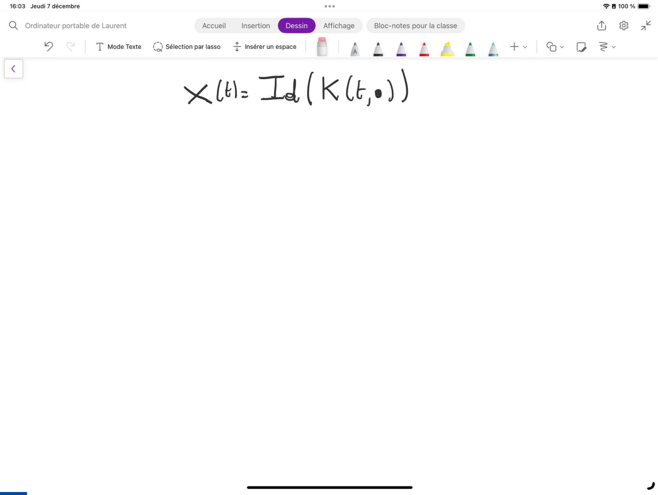
Generalized Hermite processes (Bai, Taqqu)

$$K(t,x_1,\ldots,x_d) = \frac{1}{\prod_{\ell=1}^{d}\Gamma(h_\ell - 1/2)} \int_0^t \prod_{j=1}^{d} (s-x_\ell)_+^{h_\ell - 3/2}\, ds$$

Expansion strategy



$$X(t) = I_d(K(t,\cdot))$$

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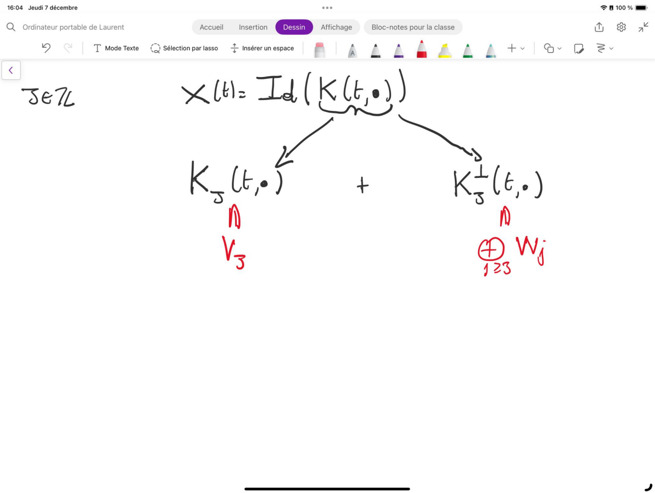

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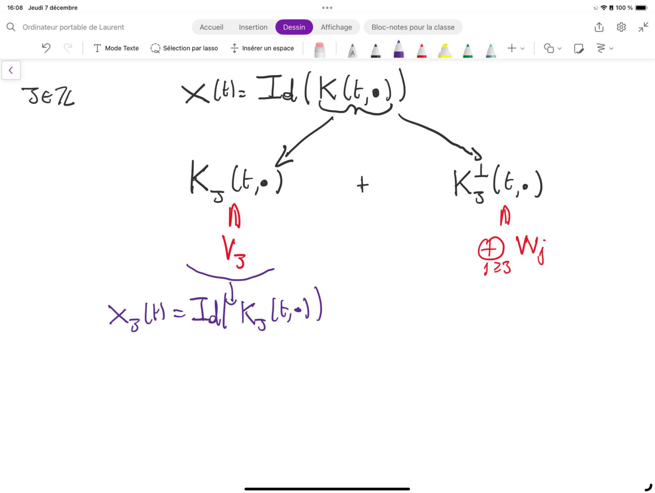

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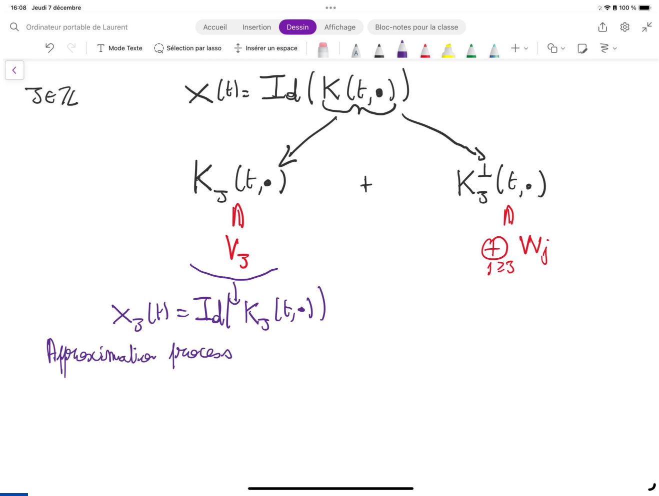

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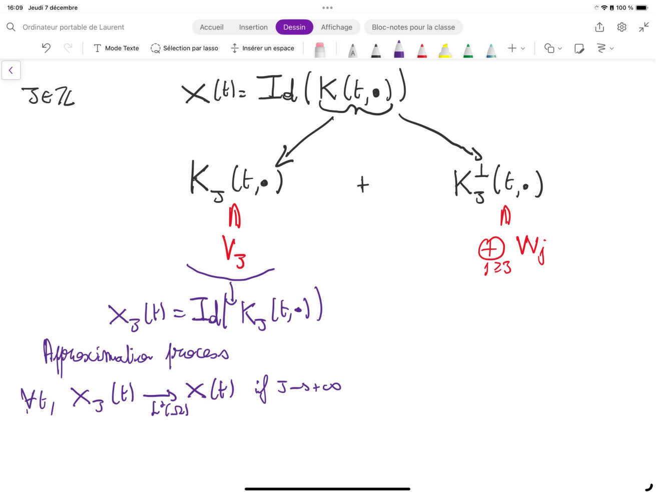

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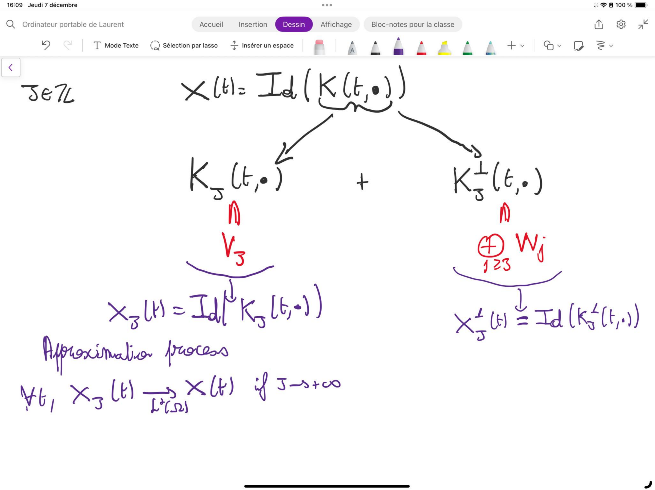

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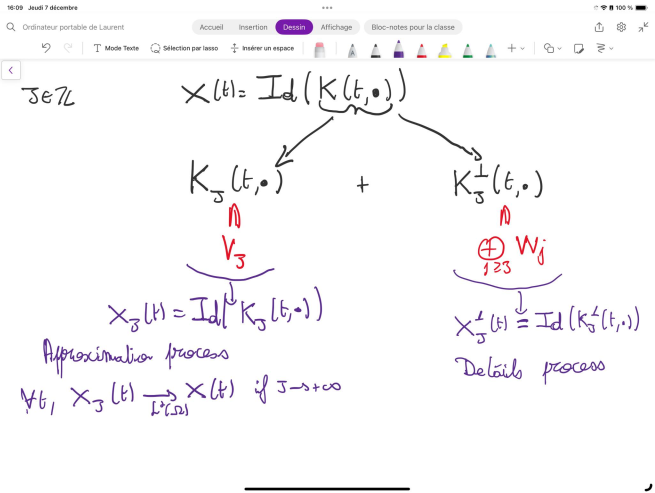

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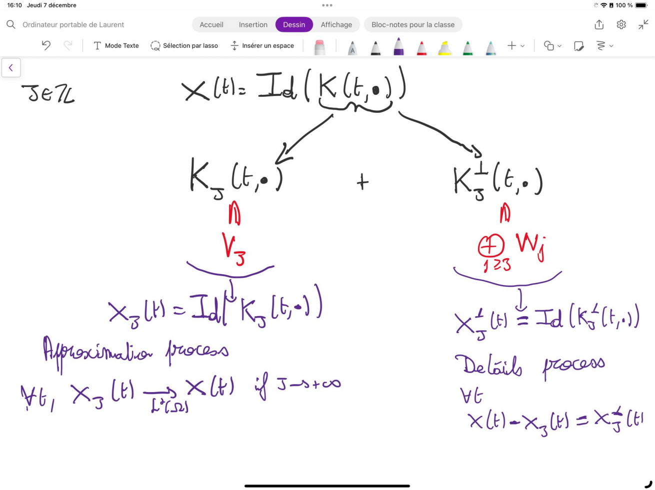

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$$\text{for all } t \geq 0 \quad X_J^{\perp}(t) = \sum_{\substack{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2 \\ \max\limits_{\ell \in [\![1,d]\!]} j_\ell \geq J}} \langle K(t,\bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \rangle I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \right) \text{ in } L^2(\Omega).$$

Explicit expression of the details process

$$\langle K(t,\bullet), \bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell} \rangle = 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \int_0^t \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds$$

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A.Ayache, J.Hamonier, L.L.

$$I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\right) = \prod_{\ell=1}^{p} H_{n_\ell}\left(I_1\left(2^{j_\ell/2}\psi(2^{j_\ell}\cdot - k_\ell)\right)\right)$$

where n_ℓ is the multiplicity of (j_ℓ, k_ℓ) in $(\mathbf{j},\mathbf{k})$.

Application to generalized Hermite processes

Tools for the approximation process

- fractional scaling function :

$$\widehat{\Phi}_{\Delta}^{(\delta)}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi) \ \forall \xi \neq 0 \text{ and } \widehat{\Phi}_{\Delta}^{(\delta)}(0) = 1$$

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$$\mu_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathscr{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[g_{J,k_{l_r}} g_{J,k_{l'_r}}] \prod_{s=m+1}^{d-m} g_{J,k_{l''_s}} \text{ with } g_{J,k} = I_1(2^{J/2}\phi(2^J \cdots - k))$$

Tools for the approximation process

A.Ayache, J.Hamonier, L.L.

The approximation process can be expressed, for all $t \in \mathbb{R}_+$, as:

$$X_J(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\int_0^t \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \right) \sigma_{J,\mathbf{k}}^{(\mathbf{h})}, \tag{6}$$

where the series is convergent in $L^2(\Omega)$. Moreover this series is also almost surely uniformly convergent in t on each compact interval of $\mathbb{R}_+$.

Rate of convergence

A.Ayache, J.Hamonier, L.L.

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X - X_J\|_{I,\infty} = \|X^{(d,\perp)}_{\mathbf{h},J}\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{7}$$

Towards numerical simulation

In the proof of the last Theorem, we notice that the rate of convergence is mainly governed by the terms

$$2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} I_d\left(\bigotimes_{\ell=1}^{d} \psi_{j_\ell,k_\ell}\right)\int_0^t \prod_{\ell=1}^{d} \psi_{h_\ell}(2^{j_\ell}s-k_\ell)\,ds$$

for which there is $\ell \in [[1,d]]$ and

$$k_\ell \in D_j^1(t) := \{k \in \mathbb{Z} : [k2^{-j}-2^{-ja}, k2^{-j}+2^{-ja}] \subseteq [0,t]\},$$

with $\frac{1}{2} < a < 1$.

Towards numerical simulation

Definition

$$\mathscr{J}_J^1(t) := \{\mathbf{k} \in (D_J^1(t))^d : \max_{\ell,\ell' \in [\![1,d]\!]} |k_\ell - k_{\ell'}| \leq 2^{\varepsilon J}\},$$

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The simulation process at scale J

$$S_J(t) = 2^{-J(h_1+\ldots+h_d-d+1)} \sum_{\mathbf{k}\in\mathscr{J}_J^1(t)} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_{\mathbb{R}} \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(s-k_\ell)\, ds.$$

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A.Ayache, J.Hamonier, L.L.

For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X - S_J\|_{I,\infty} \leq C J^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)}. \tag{8}$$