Wavelet-type approximation of Hermite processes

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Multifractal analysis and self-similarity

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$$\frac{1}{\Gamma\left(\frac{1}{2} + \frac{h-1}{d}\right)^d} \int_{\mathbb{R}^d}' \left(\int_0^t \prod_{\ell=1}^{d} (s - x_\ell)_+^{\frac{h-1}{d} - \frac{1}{2}} \, ds \right) dB(x_1) \cdots dB(x_d)$$

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5. Enjoyable properties: self-similarity, stationnarity of increments, Hölder regularity,...

Generalized Hermite processes

Fix $d \in \mathbb{N}^*$, and $\mathbf{h} = (h_1, \cdots, h_d)$ satisfying

$$h_1, \cdots, h_d \in (1/2, 1) \text{ and } \sum_{\ell=1}^{d} h_\ell > d - \frac{1}{2}. \tag{1}$$

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1. Numerical simulation;
2. Study precisely the pointwise regularity.

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Alternative wavelet-type expansion of (generalized) Rosenblatt process.

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2. We would like to judge the rate of convergence for the approximation. For the Rosenblatt process, it was unknown before Ayache-Esmili (2020).

A reminder about wavelet analysis - I

Given a multiresolution analysis

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of $L^2(\mathbb{R}^d)$

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and we can apply the following approximation procedure.

$$\begin{array}{ccc} & f & \\ \swarrow & & \searrow \\ f_J & + & f_J^{\perp} \\ \cap & & \cap \\ V_J^{(d)} & & \oplus_{j \geq J} W_j^{(d)} \end{array}$$

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If $(V_j^{(1)})_{j\in\mathbb{Z}}$ is a multiresolution of $L^2(\mathbb{R})$,

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is a multiresolution analysis of $L^2(\mathbb{R}^d)$. Thus, if ϕ is the scaling function associated with $(V_j^{(1)})_{j\in\mathbb{Z}}$, for all $J \in \mathbb{Z}$

$$\left\{(x_1,\ldots,x_d) \mapsto \prod_{\ell=1}^{d} 2^{J\frac{d}{2}}\phi(2^J x_\ell - k_\ell) \;:\; \mathbf{k} = (k_1,\ldots,k_d) \in \mathbb{Z}^d\right\}$$

is an orthonormal basis in $V_J^{(d)}$.

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(Meyer)
There exists a function ψ, called mother wavelet, belonging to W_0^1 and such that, for all $j \in \mathbb{Z}$, the sequence $(2^{j/2}\psi(2^j \cdot -k))_{k\in\mathbb{Z}}$ is an orthonormal basis in W_j^1. Moreover, for all $J \in \mathbb{Z}$, the family

$$\{2^{J/2}\phi(2^J x - k) \; : \; k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) \; : \; k \in \mathbb{Z}, \, j \geq J\}$$

is a base in $L^2(\mathbb{R})$, called wavelet base.

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Combining these facts, one can show that, for all J,

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Remark: this basis is “unusual” in the literature.

Strategy

$$X_{\mathbf{h}}^{(d)}(t) := \int_{\mathbb{R}^d}' K_{\mathbf{h}}^{(d)}(t, x_1, \ldots, x_d) dB(x_1) \cdots dB(x_d)$$

with, in this case,

$$K_{\mathbf{h}}^{(d)}(t, x_1, \ldots, x_d) := \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} \int_0^t \prod_{j=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2} \, ds.$$

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approximation — details

with $X_{\mathbf{h},J}^{(d)}(t) = \int'_{\mathbb{R}^d} K_{\mathbf{h},J}^{(d)}(t, x_1, \ldots, x_d) dB(x_1) \cdots dB(x_d)$ and
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L^2 convergences

Thanks to the Wiener isometry, we know that, if $I_d(f)$ stands for the d-uple stochastic integral of the $L^2(\mathbb{R}^d)$ function,

$$\mathbb{E}[I_d(f)^2] = d!\|\widetilde{f}\|^2_{L^2(\mathbb{R}^d)} \leq d!\|f\|^2_{L^2(\mathbb{R}^d)},$$

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It means that the convergence in $L^2(\mathbb{R}^d)$ of a sequence of functions implies the convergence in $L^2(\Omega)$ for the corresponding sequence of stochastic integrals.
Also, if $f \in L^2(\mathbb{R}^d)$, we have

$$f_J = \sum_{\Bbbk \in \mathbb{Z}^d} \langle f, 2^{J\frac{d}{2}} \Phi^d(2^J \cdot - k)\rangle 2^{J\frac{d}{2}} \Phi^d(2^J \cdot - \Bbbk),$$

where $\Phi_d = \phi^1 \otimes \cdots \otimes \phi^1$, and thus

$$I_d(f_J) = \sum_{\Bbbk \in \mathbb{Z}^d} \langle f, 2^{J\frac{d}{2}} \Phi^d(2^J \cdot - k)\rangle 2^{J\frac{d}{2}} I_d\left(\Phi^d(2^J \cdot - \Bbbk)\right),$$

with convergence in $L^2(\Omega)$.

Approximation process

We apply this fact to

$$X^{(d)}_{\mathbf{h},J}(t) := \int'_{\mathbb{R}^d} K^{(d)}_{\mathbf{h},J}(t, x_1, \dots, x_d)\, dB(x_1) \cdots dB(x_d)$$

and, using Fubini Theorem, for all $t \in \mathbb{R}_+$, in $L^2(\Omega)$, we have

$$X^{(d)}_{\mathbf{h},J}(t) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mu_{J,\mathbf{k}} \mathcal{K}_{J,\mathbf{k}}(t)$$

with

$$\mu_{J,\mathbf{k}} = 2^{J\frac{d}{2}} \int'_{\mathbb{R}^d} \phi(2^J x_1 - k_1) \cdots \phi(2^J x_d - k_d)\, dB(x_1) \dots dB(x_d)$$

and

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By Wiener isometry,

$$\left(g_{J,k} = 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k)\, dB(x) \right)_{k \in \mathbb{Z}}$$

is a family of i.i.d. $\mathcal{N}(0,1)$ random variables, since the function $(2^{J/2}\phi(2^J \cdot - k))_k$ are orthogonal.

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$$\mu_{J,\mathbf{k}} = \prod_{\ell=1}^{p} H_{n_\ell}(g_{J,\widetilde{k}_\ell})$$

where n_ℓ is the multiplicity of $\widetilde{k}_\ell$ in $\mathbf{k}$ and H_n is the nth Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2/2} D^n e^{-x^2/2}.$$

Approximation process - deterministic part

$$\mathcal{K}_{J,\mathbf{k}}(t) = \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} 2^{J\frac{d}{2}} \int_0^t \int_{\mathbb{R}^d} \prod_{\ell=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2} \phi(2^J x_\ell - k_d)\, dx_1 \dots dx_d.$$

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$$\mathcal{K}_{J,\mathbf{k}}(t) = 2^{-J(h_1 + \dots + h_d - d)} \int_0^t \prod_{\ell=1}^{d} \phi_{h_\ell}(2^J s - k_\ell)\, ds$$

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$$\mathcal{K}_{J,\mathbf{k}}(t) = 2^{-J(h_1 + \ldots + h_d - d)} \int_0^t \prod_{\ell=1}^{d} \phi_{h_\ell}(2^J s - k_\ell)\, ds$$

BAD NEWS

The function ϕ_h is not well-adapted for approximation purposes (badly localized, not in the Schwartz class,...).

Work in the frequency domain

GOOD NEWS

The *fractional scaling function* of order $\delta \in (0, 1/2)$ of the Meyer scaling function defined through its Fourier transform by

$$\widehat{\Phi}_{\Delta}^{(\delta)}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi) \; \forall \xi \neq 0 \text{ and } \widehat{\Phi}_{\Delta}^{(\delta)}(0) = 1$$

is much better!

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is much better!

With a bit of Fourier analysis tools and tricks, we get to

$$K_{\mathbf{h},J}^{(d)}(t,\mathbf{x}) = 2^{-J(h_1+\cdots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \beta_{\mathbf{k}}^{(\mathbf{h})} 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \Phi^{-(h_\ell-1/2)}(2^J x_\ell - k_\ell)$$

where

$$\beta_{\mathbf{k}}^{(\mathbf{h})} := \int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds \text{ and } \widehat{\Phi}^{-(\delta)}(\xi) = (1-e^{i\xi})^{-\delta}\widehat{\phi}(\xi)$$

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New random part

$$2^{J\frac{d}{2}} \int_{\mathbb{R}^d}' \prod_{\ell=1}^{d} \Phi^{-(h_\ell - 1/2)}(2^J x_\ell - k_\ell)\, dB(x_1) \ldots dB(x_d)$$

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Lemma (A. Ayache, L.L., J. Hamonier)
For all $\delta \in (0, \frac{1}{2})$, we have

$$\Phi^{-(\delta)}(x) = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} \phi(x+p) \tag{2}$$

with convergence in $L^2(\mathbb{R})$, where

$$\gamma_p^{(\delta)} := \frac{\delta\, \Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}.$$

New random part

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(2^{J\frac{d}{2}}\int_{\mathbb{R}^d}'\prod_{\ell=1}^{d}\phi(2^J x_\ell+p_\ell-k_\ell)\,dB(x_1)\ldots dB(x_d)\right)$$

New random part

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d} \left(\prod_{\ell=1}^{d} \gamma_{p_\ell}^{(h_\ell - 1/2)} \right) \mu_{J,\mathbf{k}-\mathbf{p}}$$

where we recall that, for all $\mathbf{k} \in \mathbb{Z}^d$,

$$\mu_{J,\mathbf{k}} = \prod_{\ell=1}^{p} H_{n_\ell}(g_{J,\widetilde{k}_\ell})$$

New random part

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\mu_{J,\mathbf{k}-\mathbf{p}}$$

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FARIMA (autogressive fractionally integrated moving average)

Let $\{Z_j\}_{j\in\mathbb{Z}}$ be a sequence of i.i.d. centred Gaussian random variables and $\delta\in(-\frac{1}{2},\frac{1}{2})$.
The Gaussian FARIMA $(0,\delta,0)$, denoted by $\{Z_j^{(\delta)}\}_{j\in\mathbb{Z}}$, is defined, for all $j\in\mathbb{Z}$ as

$$Z_j^{(\delta)}:=\sum_{p=0}^{+\infty}\gamma_p^{(\delta)}Z_{j-p} \qquad (2)$$

Random part with FARIMA

Hermite polynomials and partitions

Tthe dth Hermite polynomials can be written as

$$H_d(x) = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m a_m^{(d)} x^{d-2m},$$

where $a_m^{(d)}$ is the number of partitions of $\{1, \ldots, d\}$ with m (non ordered) pairs and $d - 2m$ singletons.

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(A. Ayache, L.L., J. Hamonier)

For all $J \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, we have

$$\mu_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[g_{J,k_{l_r}} g_{J,k_{l'_r}}] \prod_{s=m+1}^{d-m} g_{J,k_{l''_s}}$$

Expansion for the approximation process

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(\sum_{m=0}^{\lfloor d/2\rfloor}(-1)^m\sum_{P\in\mathcal{P}_m^{(d)}}\prod_{r=1}^{m}\mathbb{E}[g_{J,k_{l_r}-p_{l_r}}\,g_{J,k_{l'_r}-p_{l'_r}}]\prod_{s=m+1}^{d-m}g_{J,k_{l''_s}-p_{l''_s}}\right)$$

Expansion for the approximation process

For all $(J, \mathbf{k}) \in \mathbb{Z} \times \mathbb{Z}^d$, we define the random variable

$$\sigma^{(\mathbf{h})}_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}^{(d)}_m} \prod_{r=1}^{m} \mathbb{E}[\varepsilon^{(h_{l_r}-1/2)}_{J,k_{l_r}} \varepsilon^{(h_{l'_r}-1/2)}_{J,k_{l'_r}}] \prod_{s=m+1}^{d-m} \varepsilon^{(h_{l''_r}-1/2)}_{J,k_{l''_r}}.$$

where $\varepsilon^{(\delta)}_{J,k} := \sum_{p=0}^{+\infty} \gamma^{(\delta)}_p g_{J,k-p}$ is the FARIMA sequence associated to $(g_{J,k})_k$.

Expansion for the approximation process

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$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[\varepsilon_{J,k_{l_r}}^{(h_{l_r}-1/2)} \varepsilon_{J,k_{l'_r}}^{(h_{l'_r}-1/2)}] \prod_{s=m+1}^{d-m} \varepsilon_{J,k_{l''_r}}^{(h_{l''_r}-1/2)}.$$

where $\varepsilon_{J,k}^{(\delta)} := \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} g_{J,k-p}$ is the FARIMA sequence associated to $(g_{J,k})_k$.

Theorem (A. Ayache, L.L., J. Hamonier)

The approximation process can be expressed, for all $t \in \mathbb{R}_+$, as:

$$X_{\mathbf{h},J}^{(d)}(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds. \tag{3}$$

Moreover, the series (3) is almost surely uniformly convergent in t on compact intervals.

Details process

We recall that $\{\prod_{\ell=1}^{d} 2^{\frac{j_\ell}{2}} \psi(2^{j_\ell} x_\ell - k_\ell) \ : \ k \in \mathbb{Z}^d, \max_{1 \leq \ell \leq d} j_\ell \geq J\}$ is a base of $V_J^{\perp}$.

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This time, the fractional antiderivative

$$\psi_h(s) := \frac{1}{\Gamma(h-1/2)} \int_{\mathbb{R}} (s-x)_+^{h-3/2} \psi(x) \, dx$$

belongs to the Schwartz class!

Details process

For each fixed $t \in \mathbb{R}_+$, we write

$$X_{\mathbf{h},J}^{(d),\perp}(t) = \sum_{\substack{(\mathbf{j},\mathbf{k})\in(\mathbb{Z}^d)^2 \\ (\max_{1\le \ell\le d} j_\ell)\ge J}} \varepsilon_{\mathbf{j},\mathbf{k}}\, 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell}s - k_\ell)\, ds, \qquad (4)$$

with convergence in $L^2(\Omega)$ and where

$$\varepsilon_{\mathbf{j},\mathbf{k}} = \prod_{\ell=1}^p H_{n_\ell}\left(2^{j_\ell/2}\int_{\mathbb{R}} \psi(2^{j_\ell}x - k_\ell)\, dB(x)\right)$$

and n_ℓ is the multiplicity of (j_ℓ, k_ℓ) in $(\mathbf{j}, \mathbf{k})$.

Rate of convergence

Theorem (A. Ayache, L.L., J. Hamonier)
For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X_{\mathbf{h}}^{(d)} - X_{\mathbf{h},J}^{(d)}\|_{I,\infty} = \|X_{\mathbf{h},J}^{(d),\perp}\|_{I,\infty} \leq CJ^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)} \tag{5}$$

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Basic ideas:

1. There exist an event Ω^* of probability 1 and C_d^* a positive random variable of finite moment of any order, such that on Ω^* one has, for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^2)^d$,

$$|\varepsilon_{\mathbf{j},\mathbf{k}}| \leq C_d^* \prod_{m=1}^{d} \sqrt{\log(3 + |j_m| + |k_m|)}. \quad (6)$$

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2. The random variables $\varepsilon_{\mathbf{j},\mathbf{k}}$ and $\varepsilon_{\mathbf{r},\mathbf{s}}$ are correlated if and only if $(\mathbf{j},\mathbf{k})$ is a permutation of $(\mathbf{r},\mathbf{s})$.

Wavelet-type approximation of Hermite processes

Laurent Loosveldt

joint work with A. Ayache and J. Hamonier (Université de Lille)

Multifractal analysis and self-similarity

Marseille – 28th June 2023

UNIVERSITÉ DU LUXEMBOURG