An automaton generating parageometric automorphisms?

Pierre Stas

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Preliminary notions

- Fundamental group of a topological space: $\pi_1(X, x)$.
- Free group of rank n: F_n and its boundary: ∂F_n.
- Automorphism of a group: $\mathrm{Aut}(G)$.
- Outer automorphisms of a group: $\mathrm{Out}(G)$.
  - $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G) = \{\varphi \in \mathrm{Aut}(G) \mid \exists x \in G \text{ s.t. } \varphi(y) = x^{-1}yx\}$.
  - In graphs, they can be seen as automorphisms of the fundamental group without base-point.
- Fully irreducible outer automorphisms are the generic case.

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A brief journey in Outer Space

A brief journey in Outer Space

“Definition”

Outer space is the set of of homothetic conjugacy classes of proper, simplicial F_n-trees.

For any $\Phi \in \mathrm{Out}(F_n)$, there exist two F_n-trees T_- and T_+ (the repelling and attracting tree respectively) such that for any x in outer space,

$$\lim_{n \to -\infty} \Phi^n(x) = T_- \qquad \text{and} \qquad \lim_{n \to +\infty} \Phi^n(x) = T_+.$$

These trees are said to be *geometric* if there exists a simplicial 2-complex K with a metric foliation in which their dual can be seen.

A brief journey in Outer Space

Geometric and parageometric automorphisms

Let Φ be an outer automorphism of F_n. It is said to be *geometric* if there exists a self-homeomorphism of a compact surface that induces Φ on its fundamental group $(\star)$. Equivalently, T_+ and T_- are geometric.

Definition

The outer automorphism Φ is *parageometric* if its attracting tree in the boundary of outer space is geometric but the repelling tree is not.

A brief journey in Outer Space

Indices of outer automorphisms

For any $\varphi \in \mathrm{Aut}(F_n)$, we define $\partial\varphi : \partial F_n \to \partial F_n$ by applying φ element by element.

Definition

For $\Phi \in \mathrm{Out}(F_n)$ and $\varphi \in \Phi$, we define

- $\mathrm{ind}(\varphi) = |\,\text{Attracting fixed points of } \partial\varphi| - 2.$
- $\mathrm{ind}(\Phi) = \sum_{[\psi]_\sim \in \Phi^k} \mathrm{ind}(\psi)$, where $\psi \sim \psi'$ if $\exists u \in F_n$ s.t. $\psi' = \iota_u \circ \psi \circ \iota_{u^{-1}}$ $(\star)$.

A brief journey in Outer Space

Indices of outer automorphisms

Theorem

For any $\Phi \in Out(F_n)$*, we have* $ind(\Phi) \leq 2n - 2$*.*

These indices give us the following equivalences:

Theorem

For any $\Phi \in Out(F_n)$*,*

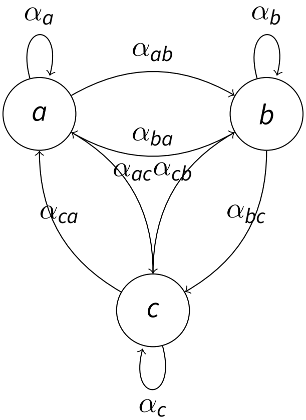
- Φ *geometric* $\Leftrightarrow$ $ind(\Phi)$ *and* $ind(\Phi^{-1})$ *are maximal.*
- Φ *parageometric* $\Leftrightarrow$ $ind(\Phi)$ *is maximal and* $ind(\Phi^{-1})$ *is not.*

Leroy’s automaton

Leroy's automaton

Definition

We are interested in studying the automorphisms generated by paths in $\mathcal{A}$ ($\star$):



Leroy's automaton

Definition

Any substitution (morphism of a free monoïd onto itself) generates the set of bi-infinite words $\{(a_n)_{n\in\mathbb{Z}} | a_{[k,l]}$ appears in an iterated image of a letter by φ. $\}$.

Example

The *Tribonacci shift*

$$\ldots 0102010010201010201001020 1020\ldots$$

is generated by the substitution $\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$.

Leroy’s automaton

Key feature

This automaton has some features that apply to its infinite paths. As such, these properties translate well when considering the automorphisms φ that correspond to a loop γ[1] in $\mathcal{A}$.

Theorem

The set of bi-infinite words generated by a φ accepted by $\mathcal{A}$ has exactly one infinite right special factor with three infinite extensions and two infinite left special factors with two right extensions each.

[1]We should take γ “complicated enough” so that it is primitive and later even fully irreducible.

Leroy's automaton

Illustrating the previous theorem

We know that there are exactly two sequences of letters $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ denoted $\dots a_{-2}a_{-1}.a_0a_1a_2\dots$ and $\dots b_{-2}b_{-1}.b_0b_1b_2\dots$ such that there is an n_a and an n_b for which

$$a_{n_a+k} = b_{n_b+k}, \forall k \in \mathbb{N}.$$

Generating parageometric automorphisms

Generating parageometric automorphisms

Intuition

The special factors mentioned in the previous theorem seem to be linked to the attracting fixed points that define the index of an outer automorphism.

Indeed, it is direct for the right special factor $(\star)$. It is not so simple for the left special factors. The problem comes from the composition by an inner automorphism.

Generating parageometric automorphisms

Intuition

Left special case

Let us look at the attracting points of the automorphism Φ such that $\alpha_{ab}\alpha_{bc}\alpha_{ca} = \varphi \in \Phi$.

We have $\varphi : \begin{cases} a \mapsto ac \\ b \mapsto bac \\ c \mapsto cbac \end{cases}$. We see that a is left special and extended by b or c. Let us explore this more closely $(\star)$.

So we need an automaton that keeps more information.

Generating parageometric automorphisms

A richer automaton