Multifractional Hermite processes: definition and first properties

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then the d-multiple Wiener-Itô integral of f with respect to the Brownian motion $\{B(t)\}_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as the $L^2(\Omega)$ random variable.

$$I_d(f) := \sum_{j_1,\ldots,j_d=1}^{n} a_{j_1,\ldots,j_d} (B(t_{j_1}) - B(s_{j_1})) \times \ldots (B(t_{j_d}) - B(s_{j_d})). \tag{2}$$

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Functions of the form (1) are dense among symmetric $L^2(\mathbb{R}^d)$ function and the corresponding sequence of random variables (2) converge in $L^2(\Omega)$.

Hermite processes

Given $h \in (\frac{1}{2}, 1)$ and $d \in \mathbb{N}^*$, we define, for all $s \geq 0$, the function

$$f_h(s, \bullet) \; : \; \mathbb{R}^d \to \mathbb{R}_+ \; : \; \mathbf{x} \mapsto \prod_{\ell=1}^{d} (s - x_\ell)_+^{\frac{h-1}{d} - \frac{1}{2}}. \tag{3}$$

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is symmetric and belongs to $L^2(\mathbb{R}^d)$.Then, the Hermite process of order d and Hurst parameter h is defined as

$$\left\{ I_d \left(\int_0^t f_h(s, \bullet)\, ds \right) \right\}_{t \in \mathbb{R}_+}. \tag{4}$$

Hermite processes – properties

$$\{X_d(t,h)\}_{t\in\mathbb{R}_+} = \left\{ I_d\left(\int_0^t f_h(s,\bullet)\, ds \right) \right\}_{t\in\mathbb{R}_+}.$$

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2. **Stationarity of increments:** for any $r > 0$, the processes $\{X_d(t+r,h) - X_d(t,h)\}_{t\in\mathbb{R}_+}$ and $\{X_d(t,h)\}_{t\in\mathbb{R}_+}$ are equal in law.

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3. **Covariance function:** For all $s,t \in \mathbb{R}_+$, $\mathbb{E}[X_d(t,h)X_d(s,h)] = c_h(t^{2h} + s^{2h} - |t-s|^{2h})$.

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4. **Hölder regularity:** $\{X_d(t,h)\}_{t\in\mathbb{R}_+}$ has a version with almost sure (uniform) Hölder exponent h.

A function f defined on I belongs to the Hölder space $C^\alpha(I)$ if there exists $c > 0$ such that, for all $x, y \in I$

$$|f(x) - f(y)| \le c|x-y|^\alpha.$$

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$\{X_1(t,h)\}_{t\in\mathbb{R}_+}$ is the Brownian motion of Hurst parameter h.

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If $d > 1$, $\{X_d(t,h)\}_{t\in\mathbb{R}_+}$ is non-Gaussian.

Generators of multifractional Hermite processes

Definition

Given $d \in \mathbb{N}^*$, the *generator of the multifractional Hermite process of order* d is the real-valued centred field $\{X_d(t,h)\}_{(t,h)\in\mathbb{R}_+\times(\frac{1}{2},1)}$ defined, for all $(t,h) \in \mathbb{R}_+ \times (\frac{1}{2},1)$, by the multiple Wiener-Itô integral

$$X_d(t,h) := I_d\left(\int_0^t f_h(s,\bullet)\, ds\right). \tag{5}$$

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Proposition

Let $d \in \mathbb{N}^*$, K be a compact set of $(\frac{1}{2}, 1)$ and I be a compact interval of $\mathbb{R}_+$. There exist a positive deterministic constant c_1 only depending on d an K and a positive deterministic constant c_2, only depending on d, K and I, such that, for all $t, u \in I$ and $h_1, h_2 \in K$,

$$\|X_d(t, h_1) - X_d(u, h_2)\|_{L^2(\Omega)}$$

is bounded from above by $c_1|t - u|^{\min\{h_1,h_2\}} + c_2|h_1 - h_2|$ and from below by $c_1|t - u|^{\min\{h_1,h_2\}} - c_2|h_1 - h_2|$.

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Idea: assume $h_1 < h_2$ and write

$$\|X_d(t, h_1) - X_d(u, h_1)\|_{L^2(\Omega)} - \|X_d(u, h_1) - X_d(u, h_2)\|_{L^2(\Omega)} \leq \|X_d(t, h_1) - X_d(u, h_2)\|_{L^2(\Omega)}$$
$$\leq \|X_d(t, h_1) - X_d(u, h_1)\|_{L^2(\Omega)} + \|X_d(u, h_1) - X_d(u, h_2)\|_{L^2(\Omega)}.$$

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$$\|X_d(t, h_1) - X_d(u, h_2)\|_{L^p(\Omega)} \leq c_p \left(|t - u|^{\min\{h_1, h_2\}} + |h_1 - h_2| \right). \qquad (6)$$

It is a consequence of the hypercontractivity property: for every $p > 0$ and $d \geq 1$, there exists a constant $0 < k(p, d) < \infty$ such that, for every random variable F with the form of a d-multiple Wiener-Itô integral

$$\|F\|_{L^p(\Omega)} \leq k(p, d)\|F\|_{L^2(\Omega)}.$$

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Consequence of Kolmogorov Theorem

Given $d \in \mathbb{N}^*$, there exist a modification of the field $\{X_d(t,h)\}_{(t,h)\in\mathbb{R}_+\times(1/2,1)}$, also denoted by $\{X_d(t,h)\}_{(t,h)\in\mathbb{R}_+\times(1/2,1)}$, and Ω^*, an event of probability 1, such that, on Ω^*, given I, a compact interval of $\mathbb{R}_+$, and K, a compact set of $(\frac{1}{2}, 1)$, for all $0 < a < \inf K$, there exists a finite positive random variable C such that, for all $t, u \in I$ and $h_1, h_2 \in K$,

$$|X_d(t,h_1) - X_d(u,h_2)| \leq C(|t-u| + |h_1 - h_2|)^a. \tag{7}$$

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Multifractional Hermite processes

On the event Ω^* of probability 1,

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Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a function $H : \mathbb{R}_+ \to K$, the *multifractional Hermite process of order* d *and Hurst function* H is the process $\{X_d^{H(\cdot)}(t)\}_{t \in \mathbb{R}_+}$ defined, for all $t \in \mathbb{R}_+$, by

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First observations:

1. On the event Ω^*, $\{X_d^{H(\cdot)}(t)\}_{t\in\mathbb{R}_+}$ is always continuous at 0.

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3. If H is discontinuous at a point $t_0 \neq 0$, almost surely, $\{X_d^{H(\cdot)}(t)\}_{t\in\mathbb{R}_+}$ is discontinuous at t_0.

Hölder regularity (lower bound)

On the event Ω^* of probability 1,

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A first Condition for H

Given $d \in \mathbb{N}^*$ and a compact set K of $(\frac{1}{2}, 1)$, we say that the Hurst function $H : \mathbb{R}_+ \to K$ satisfies the uniform $\min$-Hölder regularity condition if, for all compact interval I of $\mathbb{R}_+$, there exists $\gamma \in (\underline{H}(I), 1)$ such that $H \in C^\gamma(I)$, where we set $\underline{H}(I) := \min\{H(I)\}$

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Under this condition, it is clear that, on Ω^*, for all interval I, the Hölder exponent of $\{X_d^{H(\cdot)}(t)\}_{t\in\mathbb{R}_+}$ on I is at least $\underline{H}(I)$.

Modulus of continuity

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Theorem (L.L.)

Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the uniform $\min$-Hölder regularity condition, there exists Ω_1^*, an event of probability 1, such that, on Ω_1^*, for all compact interval I of $\mathbb{R}_+$

$$\limsup_{r \to 0^+} \frac{\sup_{t_0 \in I} \ \mathrm{Osc}(X_d^{H(\cdot)}, [t_0 - r, t_0 + r] \cap I)}{r^{\underline{H}(I)} (\log r^{-1})^{\frac{d}{2}}} < +\infty.$$

$\mathrm{Osc}(f, I) := \sup_{t,s \in I} |f(t) - f(s)|.$

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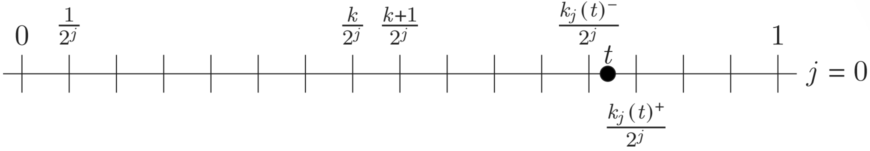
Important fact

For all $d \geq 1$, there exists an universal deterministic constant $c_d > 0$ such that, for any random variable X in the Wiener chaos of order d, and $y \geq 2$,

$$\mathbb{P}(|X| \geq y \|X\|_{L^2(\Omega)}) \leq \exp(-c_d y^{\frac{2}{d}}).$$

Modulus of continuity – Idea of the proof



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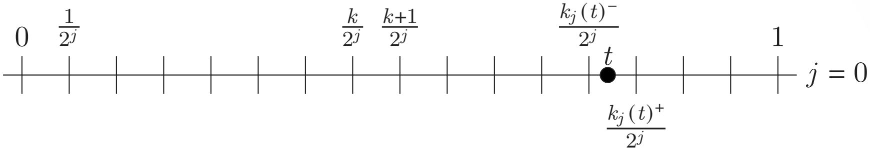
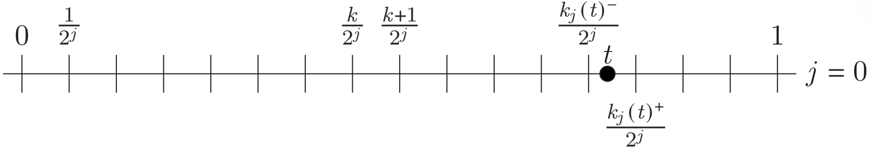


If I is a compact interval in $[0, n]$ and $t \in I$, there exists $J_2 \in \mathbb{N}$ such that, for all $j \geq J_2$, $k_j^-(t) \in I$ or $k_j(t)^+ \in I$. We choose $k_j(t) \in \{k_j^-(t), k_j^+(t)\}$ such that $k_j(t) \in I$.

Modulus of continuity – Idea of the proof



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On Ω^*, for all t and $j_0 \geq J_2$, we write

$$X_d^{H(\cdot)}(t) = X_{j_0, k_{j_0}(t)} + \sum_{j \geq j_0} \left(X_{j+1, k_{j+1}(t)} - X_{j, k_j(t)} \right)$$

with $X_{j,k} := X_d^{H(\cdot)}(k 2^{-j})$

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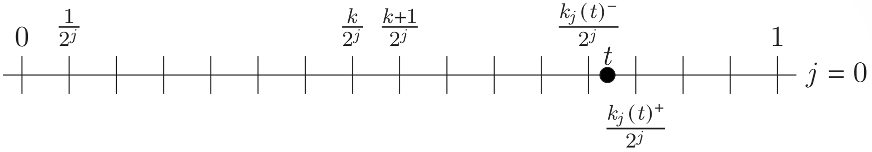

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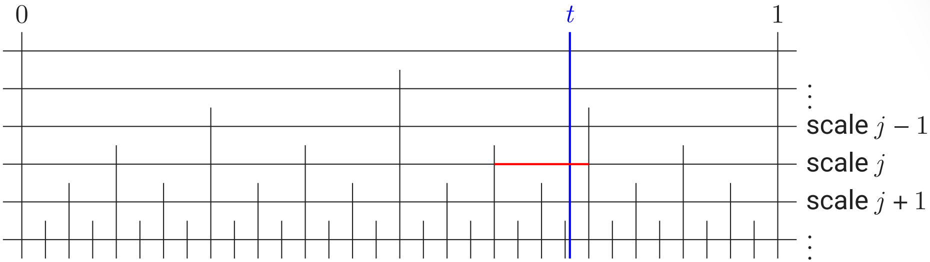
$$\sum_j \mathbb{P}\left(\exists 0 \leq k \leq n2^j, k' \in \{2k, 2k \pm 1, 2k \pm 2\} : \frac{|X_{j+1,k'} - X_{j,k}|}{\|X_{j+1,k'} - X_{j,k}\|_{L^2(\Omega)}} \geq cj^{\frac{d}{2}}\right) < \infty.$$

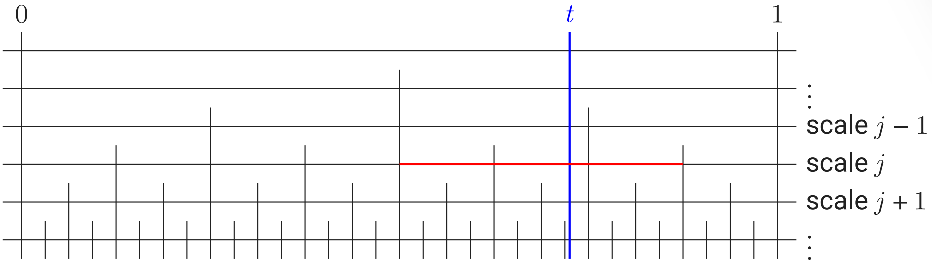
Lower bound for oscillations

Lower bound for oscillations



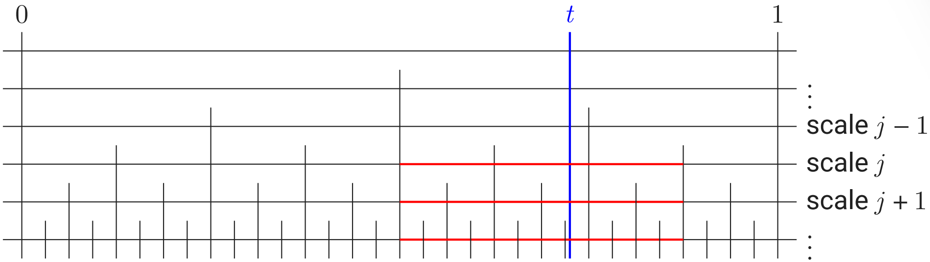
$\lambda_j(t) = [k_j^-(t)2^{-j}, k_j^+(t)2^{-j})$ is the unique dyadic interval at scale j containing t

Lower bound for oscillations



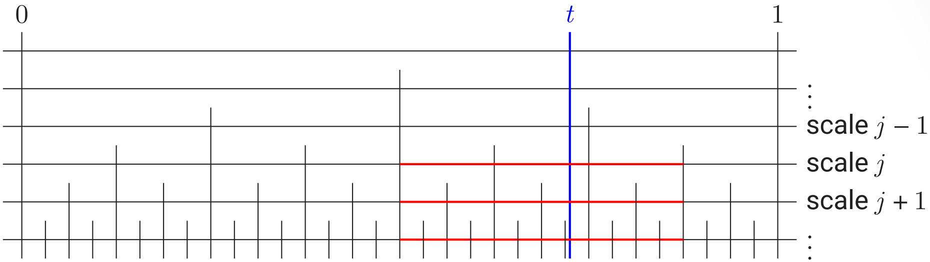
$3\lambda_j(t)$ is $\lambda_j(t)$ and its neighbours.

Lower bound for oscillations



$\lambda \subset 3\lambda_j(t)$.

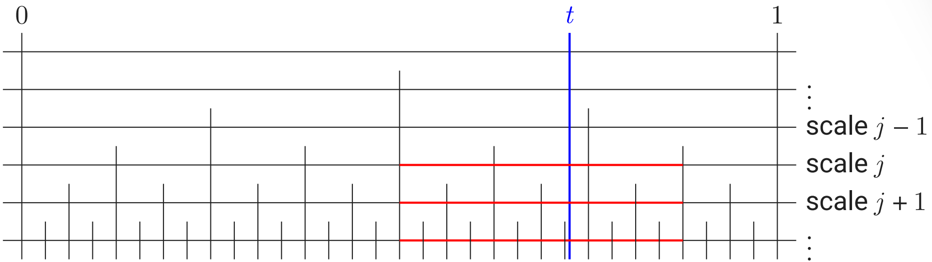
Lower bound for oscillations



If $\lambda = [k2^{-j}, (k+1)2^{-j})$ is a dyadic interval, we set

$$\Delta_{j,k} := X_d^{H(\cdot)}\left(\frac{k+1}{2^j}\right) - X_d^{H(\cdot)}\left(\frac{k}{2^j}\right)$$

Lower bound for oscillations



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$$\Delta_{j,k} := X_d^{H(\cdot)}\left(\frac{k+1}{2^j}\right) - X_d^{H(\cdot)}\left(\frac{k}{2^j}\right)$$

$$\sup_{\lambda \subseteq 3\lambda_j(t)} |\Delta_\lambda| \leq \operatorname{Osc}(X_d^{H(\cdot)}, [t - 22^{-j}, t + 22^{-j}]).$$

A brilliant idea from Antoine

$$\Delta_{j,k} = X_d\left(\frac{k+1}{2^j}, H\left(\frac{k+1}{2^j}\right)\right) - X_d\left(\frac{k}{2^j}, H\left(\frac{k+1}{2^j}\right)\right) + X_d\left(\frac{k}{2^j}, H\left(\frac{k+1}{2^j}\right)\right) - X_d\left(\frac{k}{2^j}, H\left(\frac{k}{2^j}\right)\right)$$

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Given an integer $M \geq 0$, for all $(j, k) \in \mathbb{N} \times \{0, \dots, 2^j - 1\}$, we consider the enlarged dyadic interval

$$\lambda_{j,k}^{M} := \left(\frac{k - M}{2^j}, \frac{k + 1}{2^j} \right]^d$$

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$$\widetilde{\Delta_{j,k}}^M := I_d \left(\mathbb{1}_{\lambda_{j,k}^M} \left(\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f_{H\left(\frac{k+1}{2^j}\right)}(s, \bullet) \, ds \right) \right)$$

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$$\Delta_{j,k} = \widetilde{\Delta_{j,k}}^M + \underbrace{\Delta_{j,k}}^M + \widehat{\Delta_{j,k}}.$$

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$$\Delta_{j,k} = \overset{\frown}{\Delta_{j,k}}^{M} + \overset{\smile}{\Delta_{j,k}}^{M} + \widehat{\Delta_{j,k}}.$$

If $M_1,\ldots,M_n$ are fixed positive real numbers, the random variables $\overset{\frown}{\Delta_{j_1,k_1}}^{M_1},\ldots,\overset{\frown}{\Delta_{j_n,k_n}}^{M_n}$ are independent as soon as the condition

$$\lambda_{j_\ell,k_\ell}^{M_\ell}\cap\lambda_{j_{\ell'},k_{\ell'}}^{M_{\ell'}} = \emptyset \text{ for all } 1\le \ell,\ell'\le n \qquad (9)$$

is satisfied.

Dominent random variables

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Proposition (L.L.)

Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$, there exists a positive deterministic constant c, only depending on d and K, such that, for all $(j,k) \in \mathbb{N} \times \{0, \ldots, 2^j - 1\}$ and $M > 0$, one has

1. $c^{-1} 2^{-H\left(\frac{k+1}{2^j}\right)j} \leq \|\widetilde{\Delta_{j,k}}^M\|_{L^2(\Omega)} \leq c 2^{-H\left(\frac{k+1}{2^j}\right)j}$;
2. $\|\overset{\smile}{\Delta_{j,k}}^M\|_{L^2(\Omega)} \leq c M^{\frac{H\left(\frac{k+1}{2^j}\right)-1}{d}} 2^{-H\left(\frac{k+1}{2^j}\right)j}$;
3. $\|\widehat{\Delta_{j,k}}\|_{L^2(\Omega)} \leq c \operatorname{Osc}(H, \lambda_{j,k})$.

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3. $\|\widehat{\Delta_{j,k}}\|_{L^2(\Omega)} \leq c \operatorname{Osc}(H, \lambda_{j,k})$.

A pointwise Condition for H

Given $d \in \mathbb{N}^*$ and a compact set K of $(\frac{1}{2}, 1)$, we say that the Hurst function $H : \mathbb{R}_+ \to K$ satisfies the pointwise Hölder regularity condition if, for all $t \in \mathbb{R}_+$, there exists $\gamma \in (H(t), 1)$ such that $H \in C^\gamma(t)$.

Lower bound for oscillations

Theorem (L.L.)
Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the pointwise regularity condition, there exists $\underline{\Omega}$, an event of probability 1, such that, on $\underline{\Omega}$, for all $t_0 \in \mathbb{R}_+$,

$$\limsup_{r \to 0^+} \frac{\mathrm{Osc}(X_d^{H(\cdot)}, [t_0 - r, t_0 + r] \cap \mathbb{R}_+)}{r^{H(t_0)} (\log r^{-1})^{\frac{-d^2 H(t_0)}{2(1-H(t_0))}}} > 0 \tag{10}$$

Lower bound for oscillations

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Important fact

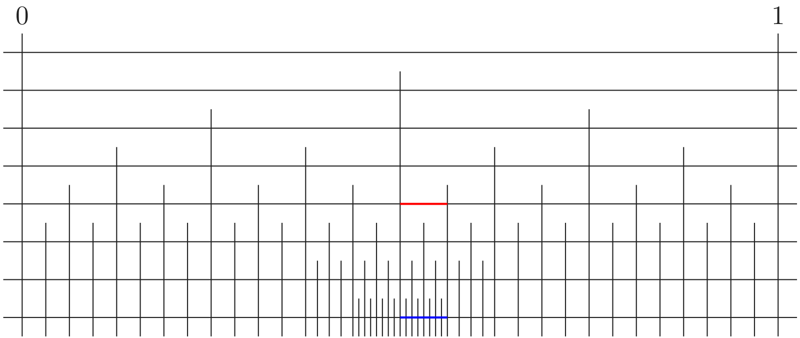
Given $d \in \mathbb{N}^*$, there exists an universal deterministic constant $\gamma_d \in [0, 1)$ such that, for any random variable X in the Wiener chaos of order d, one has

$$\mathbb{P}\left(|X| \leq \frac{1}{2}\|X\|_{L^2(\Omega)}\right) \leq \gamma_d.$$

Roadmap through the dyadic intervals

If $\lambda = \lambda_{j,k}$ is a dyadic interval and $m \in \mathbb{N}$,

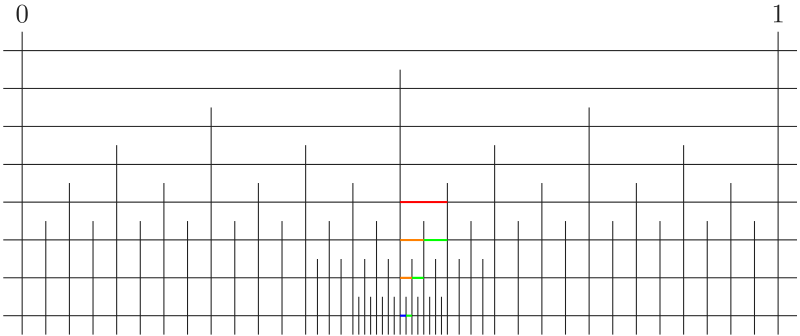
$$\mathcal{S}_{\lambda,m} = \mathcal{S}_{j,k,m} := \{\lambda \in \Lambda_{j+m} \ : \ \lambda \subset \lambda_{j,k}\}.$$



Roadmap through the dyadic intervals

If the dyadic interval $\lambda_{j,k}$ and $m \in \mathbb{N}$ are fixed and $S \in \mathcal{S}_{j,k,m}$, we define the sequences of dyadic intervals $(I_n)_{0 \leq n \leq m}$ and $(T_n)_{1 \leq n \leq m}$ in the following way:

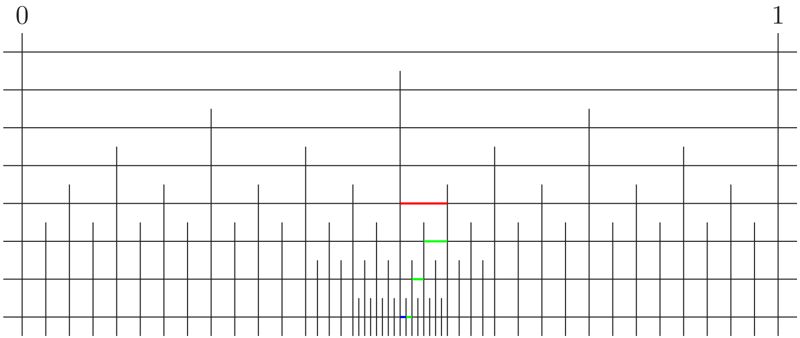
- $I_0 = \lambda_{j,k}$;
- $I_m = S$;
- for all $1 \leq n \leq m$, $I_{n-1} = I_n \cup T_n$.



Roadmap through the dyadic intervals

For all $\lambda_{j,k} \in \Lambda$, we define

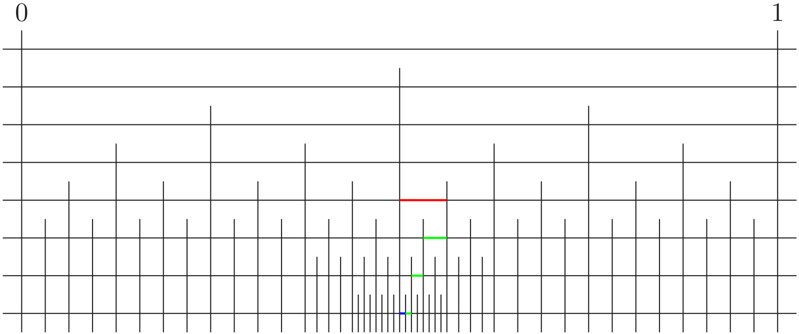
$$M_\lambda := (8c^2 c' j^{\frac{d}{2}})^{\frac{d}{1-H\left(\frac{k+1}{2^j}\right)}}. \qquad (11)$$



Roadmap through the dyadic intervals

For any $1 \leq n \leq m$, there are ℓ_d dyadic intervals $(T_n^\ell = \lambda_{j_n^{(\ell)},k_n^{(\ell)}})_{1\leq \ell \leq \ell_d}$ in $S_{T_n,\lfloor \log_2(\ell_d M_{T_n})\rfloor+1}$ such that, for all $1 \leq \ell \leq \ell_d$

$$\left(\frac{k_n^{(\ell)} - M_{T_n}}{2^{j_n^{(\ell)}}}, \frac{k_n^{(\ell)}+1}{2^{j_n^{(\ell)}}}\right) \subseteq T_n.$$



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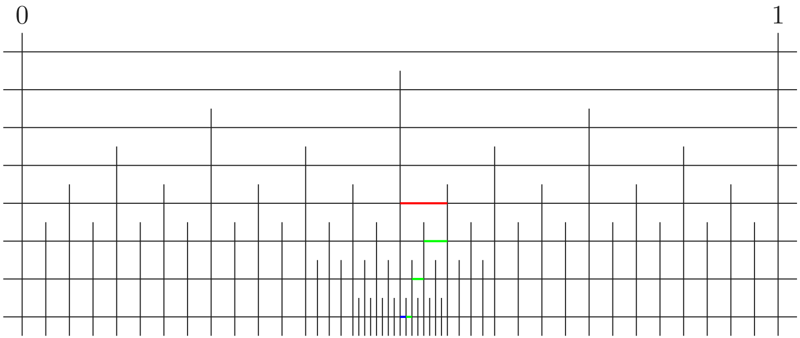
$$\left(\frac{k_n^{(\ell)} - M_{T_n}}{2^{j_n^{(\ell)}}}, \frac{k_n^{(\ell)} + 1}{2^{j_n^{(\ell)}}} \right) \cap \left(\frac{k_n^{(\ell')} - M_{T_n}}{2^{j_n^{(\ell')}}}, \frac{k_n^{(\ell')} + 1}{2^{j_n^{(\ell')}}} \right) = \emptyset.$$



Consequences of this construction

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Let us set consider the random variable

$$\mathcal{G}_{j,k,m} = \sum_{S\in\mathcal{S}_{j,k,m}} \mathcal{B}_{j,k,m}(S),$$

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As a consequence, $\Omega_1 = \bigcap_{j\in\mathbb{N}, 0\le k<2^j} \{\omega \,:\, \liminf_{m\to+\infty} \mathcal{G}_{j,k,m}(\omega) = 0\}$ is an event of probability 1.

Consequences of this construction

For all $\omega \in \Omega_1$ and $t_0 \in [0,1)$, there exist infinitely many $j \in \mathbb{N}$ such that there is $\lambda \in 3\lambda_j(t_0)$ and $\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(\ell_d M_\lambda) \rfloor + 1}$ for which

$$|\widetilde{\Delta_{\lambda'}^{M_\lambda}}(\omega)| \geq \frac{1}{2} \|\widetilde{\Delta_{\lambda'}^{M_\lambda}}\|_{L^2(\Omega)}. \tag{11}$$

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We deduce from Borel-Cantelli Lemma the existence of Ω_2, an event of probability 1 such that, for all $\omega \in \Omega_2$, there exists $J_2 \in \mathbb{N}$ such that, for all $j \geq J_2$, $\lambda \in \Lambda_j$ and $\lambda' \in S_{\lambda,\lfloor \log_2(\ell_d M_\lambda)\rfloor+1}$,

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and Ω_3, an event of probability 1 such that, for all $\omega \in \Omega_3$, there exists $J_3 \in \mathbb{N}$ such that, for all $j \geq J_3$, $\lambda \in \Lambda_j$ and $\lambda' \in S_{\lambda, \lfloor \log_2(\ell_d M_\lambda) \rfloor + 1}$,

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Consequences of this construction

There exist infinitely many $j \in \mathbb{N}$ such that there is $\lambda \in 3\lambda_j(t_0)$ and $\lambda' \in S_{\lambda, \lfloor \log_2(\ell_d M_\lambda) \rfloor + 1}$ for which

$$|\Delta_{\lambda'}(\omega)| \geq \frac{c^{-1}}{8}(8c^2 c' j^{\frac{d}{2}})^{-\frac{dH\left(\frac{k'+1}{j'}\right)}{1-H\left(\frac{k+1}{2^j}\right)}} 2^{-jH\left(\frac{k'+1}{j'}\right)}. \tag{11}$$

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As a consequence

$$\limsup_{j \to +\infty} \frac{\operatorname{Osc}(X_d^{H(\cdot)}, [t_0 - 22^{-j}, t_0 + 22^{-j}] \cap \mathbb{R}_+)}{2^{-jH(t_0)} j^{-\frac{d^2 H(t_0)}{2(1-H(t_0))}}} > 0.$$

Law of iterated logarithm – upper bound

A local Condition for H

Given $d \in \mathbb{N}^*$ and a compact set K of $(\frac{1}{2}, 1)$, we say that the Hurst function $H : \mathbb{R}_+ \to K$ satisfies the local Hölder regularity condition if, for all $t \in \mathbb{R}_+$, there exist $t \in I_t \subseteq \mathbb{R}_+$ and $\gamma \in (H(t), 1)$ such that $H \in C^\gamma(I_t)$.

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Proposition (L.L.)

Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the local Hölder condition, there exists $\overline{\Omega}_1$, an event of probability 1, such that on $\overline{\Omega}_1$, for (Lebesgue) almost every $t_0 \in \mathbb{R}_+$, we have

$$\limsup_{r \to 0^+} \frac{\mathrm{Osc}(X_d^{H(\cdot)}, [t_0 - r, t_0 + r] \cap \mathbb{R}_+)}{r^{H(t_0)} (\log(\log r^{-1}))^{\frac{d}{2}}} < \infty. \tag{12}$$

Ideas of the proof

If $s, t \in [t_0 - r, t_0 + r]$ with $2^{-(j_0+1)} \leq r \leq 2^{-j_0}$, for any $j \geq j_0$ and $x \in \{s, t\}$, $\lambda_j(x) \subseteq 3\lambda_{j_0}(t_0)$ and we write

$$X_d^{H(\cdot)}(t) - X_d^{H(\cdot)}(s) = X_{j_0,k_{j_0}^-}(t) - X_{j_0,k_{j_0}^-}(s) + \sum_{j \geq j_0} \left(X_{j+1,k_{j+1}^-}(t) - X_{j+1,k_{j+1}^-}(s) - X_{j,k_j^-}(t) + X_{j,k_j^-}(s) \right).$$

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$$\mathbb{P}\left(\exists j \geq j_0, \lambda_{k,j}, \lambda_{k',j} \subseteq 3\lambda_{j_0}(t_0) : \frac{|X_{j,k'} - X_{j,k}|}{\|X_{j,k'} - X_{j,k}\|_{L^2(\Omega)}} \geq c \log(j_0)^{\frac{d}{2}} (j - j_0 + 1)^{\frac{d}{2}} \right)$$
$$\leq \sum_{j \geq j_0} 32^{j-j_0} \exp(-c_d c^{\frac{2}{d}} \log(j_0)(j - j_0 + 1))$$
$$\leq c' \exp(-c_d c^{\frac{2}{d}} \log(j_0))$$

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If $s, t \in [t_0 - r, t_0 + r]$ with $2^{-(j_0+1)} \leq r \leq 2^{-j_0}$, for any $j \geq j_0$ and $x \in \{s, t\}$, $\lambda_j(x) \subseteq 3\lambda_{j_0}(t_0)$ and we write

$$X_d^{H(\cdot)}(t) - X_d^{H(\cdot)}(s) = X_{j_0,k_{j_0}^-(t)} - X_{j_0,k_{j_0}^-(s)} + \sum_{j \geq j_0} \left(X_{j+1,k_{j+1}^-(t)} - X_{j+1,k_{j+1}^-(s)} - X_{j,k_j^-(t)} + X_{j,k_j^-(s)} \right).$$

$$\mathbb{P}\left(\exists j \geq j_0, \lambda_{k,j}, \lambda_{k',j} \subseteq 3\lambda_{j_0}(t_0) : \frac{|X_{j,k'} - X_{j,k}|}{\|X_{j,k'} - X_{j,k}\|_{L^2(\Omega)}} \geq c \log(j_0)^{\frac{d}{2}} (j - j_0 + 1)^{\frac{d}{2}} \right)$$
$$\leq \sum_{j \geq j_0} 32^{j-j_0} \exp(-c_d c^{\frac{2}{d}} \log(j_0)(j - j_0 + 1))$$
$$\leq c' \exp(-c_d c^{\frac{2}{d}} \log(j_0))$$

and we conclude using Borel-Cantelli Lemma and Fubini Theorem.

Law of iterated logarithm – lower bound (for probabilities)

We want to bound from below the probabilities

$$\mathbb{P}(|\widetilde{\Delta}_{j,k}^{M}| \geq y2^{-jH\left(\frac{k+1}{2^j}\right)}) \tag{13}$$

for $(j,k) \in \mathbb{N} \times \{0,\ldots,2^j - 1\}$ and $M > 0$.

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for $(j,k) \in \mathbb{N} \times \{0, \ldots, 2^j - 1\}$ and $M > 0$. We know that for any random variable X in the Wiener chaos of order d, there exist two deterministic constants $y_0 \geq 0$ and $c > 0$ such that, for all $y \geq y_0$,

$$\mathbb{P}(|X| \geq y) \geq \exp(-cy^{\frac{2}{d}}).$$

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$$\mathbb{P}(|X| \geq y) \geq \exp(-cy^{\frac{2}{d}}).$$

But, unfortunately, these constants depend on the law of X and are not universal, which is undesirable in our context.

Law of iterated logarithm – lower bound (for probabilities)

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Lemma

Let $d \in \mathbb{N}^*$, K be a compact set of $(\frac{1}{2}, 1)$ and $H : \mathbb{R}_+ \to K$ be a continuous Hurst function. For all $t_0 \in \mathbb{R}_+$, there exist four deterministic constants $c_{t_0} > 0$, $y_{t_0} > 0$, $j_0 \in \mathbb{N}$ and $M_0 > 0$ such that, for all $\lambda_{j,k} \subseteq 3\lambda_{j_0}(t_0)$, $M \geq M_0$ and $y > y_{t_0}$, we have

$$\mathbb{P}(|\widetilde{\Delta}_{j,k}^M| \geq y 2^{-jH\left(\frac{k+1}{2^j}\right)}) \geq \exp(-c_{t_0} y^{\frac{2}{d}}). \tag{13}$$

Law of iterated logarithm – lower bound

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Proposition (L.L.)

Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the pointwise Hölder regularity Condition, there exists $\overline{\Omega}_2$, an event of probability 1, such that on $\overline{\Omega}_2$, for (Lebesgue) almost every $t_0 \in \mathbb{R}_+$, we have

$$0 < \limsup_{r \to 0^+} \frac{\mathrm{Osc}(X_d^{H(\cdot)}, [t_0 - r, t_0 + r] \cap \mathbb{R}_+)}{r^{H(t_0)} (\log(\log r^{-1}))^{\frac{d}{2}}}.$$

Law of iterated logarithm

Theorem (L.L.)
Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$ and a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the local Hölder regularity Condition, there exists $\overline{\Omega}$, an event of probability 1, such that on $\overline{\Omega}$, for (Lebesgue) almost every $t_0 \in \mathbb{R}_+$, we have

$$0 < \limsup_{r \to 0^+} \frac{\mathrm{Osc}(X_d^{H(\cdot)}, [t_0 - r, t_0 + r] \cap \mathbb{R}_+)}{r^{H(t)} (\log(\log r^{-1}))^{\frac{d}{2}}} < \infty. \tag{14}$$

Local asymptotic self-similiraty

Definition

A real-valued stochastic process $\{X(t)\}_{t\in\mathbb{R}_+}$ is *weakly locally asymptotically self-similar* of order $h > 0$ at the point t_0 with tangent process $\{Y(t)\}_{t\geq 0}$ if the sequence of process $\{\varepsilon^{-h}(X(t_0+\varepsilon t) - X(t_0))\}_{t\in\mathbb{R}_+}$ converges to the process $\{Y(t)\}_{t\in\mathbb{R}_+}$ in finite dimensional distributions, as $\varepsilon \to 0^+$.

Local asymptotic self-similiraty

Proposition (L.L.)
Let $d \in \mathbb{N}^*$, K be a compact set of $(\frac{1}{2}, 1)$ and $H : \mathbb{R}_+ \to K$ be a Hurst function. If H satisfies the pointwise Hölder regularity Condition then, for all $t_0 \geq 0$, the multifractional Hermite process $\{X_d^{H(\cdot)}(t) : t \geq 0\}$ is weakly locally asymptotically self-similar of order $H(t_0)$ at t_0 with tangent process $\{X_d(t, H(t_0)) : t \geq 0\}$, the Hermite process of order d and Hurst parameter $H(t_0)$.

We write

$$\varepsilon^{-H(t_0)}\left(X_d^{H(\cdot)}(t_0+\varepsilon t) - X_d^{H(\cdot)}(t_0)\right) = \varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0+\varepsilon t)) - X_d(t_0+\varepsilon t, H(t_0))\right) + \varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0)) - X_d(t_0, H(t_0))\right).$$

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$$\varepsilon^{-H(t_0)}\left(X_d^{H(\cdot)}(t_0+\varepsilon t) - X_d^{H(\cdot)}(t_0)\right) = \varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0+\varepsilon t)) - X_d(t_0+\varepsilon t, H(t_0))\right)$$
$$+ \varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0)) - X_d(t_0, H(t_0))\right).$$

We know that

$$\{\varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0)) - X_d(t_0, H(t_0))\right)\}_{t\geq 0}$$

is equal in finite-dimensional distribution to

$$\{X_d(t, H(t_0))\}_{t\geq 0}.$$

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Let $d \in \mathbb{N}^*$, K be a compact set of $(\frac{1}{2}, 1)$ and $H : \mathbb{R}_+ \to K$ be a Hurst function. If H satisfies the pointwise Hölder regularity Condition then, for all $t_0 \geq 0$, the multifractional Hermite process $\{X_d^{H(\cdot)}(t) : t \geq 0\}$ is weakly locally asymptotically self-similar of order $H(t_0)$ at t_0 with tangent process $\{X_d(t, H(t_0)) : t \geq 0\}$, the Hermite process of order d and Hurst parameter $H(t_0)$.

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$$+ \varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0)) - X_d(t_0, H(t_0))\right).$$

On the other hand,

$$\|\varepsilon^{-H(t_0)}\left(X_d(t_0+\varepsilon t, H(t_0+\varepsilon t)) - X_d(t_0+\varepsilon t, H(t_0))\right)\|_{L^2(\Omega)}$$
$$\leq c_2 \varepsilon^{-H(t_0)} |H(t_0+\varepsilon t) - H(t_0)|.$$

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Let $d \in \mathbb{N}^*$, K be a compact set of $(\frac{1}{2}, 1)$ and $H : \mathbb{R}_+ \to K$ be a Hurst function. If H satisfies the pointwise Hölder regularity Condition then, for all $t_0 \geq 0$, the multifractional Hermite process $\{X_d^{H(\cdot)}(t) : t \geq 0\}$ is weakly locally asymptotically self-similar of order $H(t_0)$ at t_0 with tangent process $\{X_d(t, H(t_0)) : t \geq 0\}$, the Hermite process of order d and Hurst parameter $H(t_0)$.

We write

$$\varepsilon^{-H(t_0)} \left(X_d^{H(\cdot)}(t_0 + \varepsilon t) - X_d^{H(\cdot)}(t_0) \right) = \varepsilon^{-H(t_0)} \left(X_d(t_0 + \varepsilon t, H(t_0 + \varepsilon t)) - X_d(t_0 + \varepsilon t, H(t_0)) \right) + \varepsilon^{-H(t_0)} \left(X_d(t_0 + \varepsilon t, H(t_0)) - X_d(t_0, H(t_0)) \right).$$

In particular, for all fixed $t \geq 0$, the sequence of random variables

$$\left(\varepsilon^{-H(t_0)} \left(X_d(t_0 + \varepsilon t, H(t_0 + \varepsilon t)) - X_d(t_0 + \varepsilon t, H(t_0)) \right) \right)_{\varepsilon > 0}$$

converges to 0 in $L^2(\Omega)$, and thus in probability, when $\varepsilon \to 0^+$.

Strong local asymptotic self-similiraty

Definition

When $\{X(t)\}_{t\in\mathbb{R}_+}$ and $\{Y(t)\}_{t\in\mathbb{R}_+}$ have, almost surely, continuous path and if the previous convergence also holds in the sense of continuous function over an arbitrary compact set of $\mathbb{R}_+$, we say that $\{X(t)\}_{t\in\mathbb{R}_+}$ is *strongly locally asymptotically self-similar* of order $h > 0$ at the point t_0, with tangent process $\{Y(t)\}_{t\in\mathbb{R}_+}$.

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If $\{X(t)\}_{t\in\mathbb{R}_+}$ is weakly locally asymptotically self-similar of order $h > 0$ at the point t_0 with tangent process $\{Y(t)\}_{t\in\mathbb{R}_+}$, it suffices to show that, for all $a > 0$ and $\delta > 0$

$$\lim_{\eta\to 0^+} \limsup_{\varepsilon\to 0^+} \mathbb{P}\left(\sup_{s,t\in[0,a],|t-s|\leq\eta} \left| \frac{X(t_0+\varepsilon t) - X(t_0+\varepsilon s)}{\varepsilon^h} \right| \geq \delta \right) = 0. \tag{15}$$

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It is the so-called Prohorov's criterion.

Markov and Garsia-Rodemich-Rumsey inequalities

The Markov inequality entails, for any $p \geq 1$,

$$\mathbb{P}(\varepsilon, \eta, \delta) \leq \delta^{-p} \varepsilon^{-pH(t_0)} \mathbb{E}\left[\sup_{s,t \in [0,a], |t-s| \leq \eta} \left| X^{H(\cdot)}(t_0 + \varepsilon t) - X^{H(\cdot)}(t_0 + \varepsilon s) \right|^p \right].$$

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Now, we use the so called Garsia-Rodemich-Rumsey inequality to write, for $\alpha \geq \frac{1}{p}$,

$$\mathbb{E}\left[\sup_{s,t \in [0,a], |t-s| \leq \eta} \left| X^{H(\cdot)}(t_0 + \varepsilon t) - X^{H(\cdot)}(t_0 + \varepsilon s) \right|^p \right]$$
$$\leq c_{a,p,\alpha} \eta^{\alpha p - 1} \iint_{[0,a]^2} \mathbb{E}\left[\left| X^{H(\cdot)}(t_0 + \varepsilon t) - X^{H(\cdot)}(t_0 + \varepsilon s) \right|^p \right] |t-s|^{-\alpha p - 1} \, ds dt.$$

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$$\begin{aligned}
&\mathbb{E}\left[\sup_{s,t \in [0,a], |t-s| \leq \eta} \left| X^{H(\cdot)}(t_0 + \varepsilon t) - X^{H(\cdot)}(t_0 + \varepsilon s) \right|^p \right] \\
&\quad \leq c_{a,p,\alpha} \eta^{\alpha p - 1} \iint_{[0,a]^2} \mathbb{E}\left[\left| X^{H(\cdot)}(t_0 + \varepsilon t) - X^{H(\cdot)}(t_0 + \varepsilon s) \right|^p \right] |t-s|^{-\alpha p - 1} \, ds dt.
\end{aligned}$$

If H satisfies the local Hölder regularity Condition,

$$\mathbb{P}(\varepsilon, \eta, \delta) \leq 2 c_{a,p,\alpha} \delta^{-p} \eta^{\alpha p - 1} \iint_{[0,a]^2} |t-s|^{p(\inf K - \alpha) - 1} \, ds dt.$$

Fractal dimensions

Definition

Given $d \in \mathbb{N}^*$, a set $A \subseteq \mathbb{R}^d$ and $\varepsilon, h > 0$, the quantity

$$\mathcal{H}^h_\varepsilon(A) := \inf\{\sum_j \operatorname{diam}^h(A_j) : A \subseteq \bigcup_j A_j \text{ and, } \forall j, \operatorname{diam}(A_j) < \varepsilon\}$$

where, as usual, diam stands for the diameter, is called the (h, ε)-*Hausdorff outer measure* of A. Moreover, for all $h > 0$, the application $\varepsilon \mapsto \mathcal{H}^h_\varepsilon(A)$ is decreasing and it follows that the h-*dimensional Hausdorff outer measure*

$$\mathcal{H}^h(A) := \lim_{\varepsilon \to 0^+} \mathcal{H}^h_\varepsilon(A)$$

is well-defined.

Fractal dimensions

Definition

Given $d \in \mathbb{N}^*$ and a non-empty set $A \subseteq \mathbb{R}^d$, *the Hausdorff dimension of* A is

$$\dim_{\mathcal{H}}(A) = \sup\{h > 0 : \mathcal{H}^h(A) = \infty\} = \inf\{h > 0 : \mathcal{H}^h(A) = 0\},$$

while, by convention, $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

Fractal dimensions

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Definition

Given $d \in \mathbb{N}^*$, a non-empty bounded set $A \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, let $N_\varepsilon(A)$ be the smallest number of sets of diameter at most ε which can cover A. The quantities

$$\underline{\dim}_{\mathcal{B}}(A) := \liminf_{\varepsilon \to 0^+} \frac{\log(N_\varepsilon(A))}{-\log(\varepsilon)} \text{ and } \overline{\dim}_{\mathcal{B}}(A) := \limsup_{\varepsilon \to 0^+} \frac{\log(N_\varepsilon(A))}{-\log(\varepsilon)}$$

are, respectively, the *lower and upper box-counting dimensions* of A. If they are equal, the common value is refereed as the *box-counting dimension* of A and we denote it $\dim_{\mathcal{B}}(A)$.

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Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$, a Hurst function $H : \mathbb{R}_+ \to K$ and a compact interval $I \subset \mathbb{R}_+$, we are interested in the dimensions of the graph

$$\mathcal{G}_d(I) := \{(t, X_d^{H(\cdot)}(t)) : t \in I\}.$$

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$$\dim_{\mathcal{H}}(A) \leq \underline{\dim}_{\mathcal{B}}(A) \leq \overline{\dim}_{\mathcal{B}}(A). \tag{16}$$

Fractal dimensions

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$$(A \subseteq B) \Rightarrow \dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{H}}(B). \tag{17}$$

Upper bound for box-counting dimension

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Lemma (Falconer)

Let $I \subset \mathbb{R}_+$ be a compact interval and $f : I \to \mathbb{R}$ be a continuous function for which there exist $c \geq 0$ and $1 \leq \alpha \leq 2$ such that, for all $s, t \in I$,

$$|f(s) - f(t)| \leq c|t - s|^{2-\alpha},$$

then

$$\overline{\dim}_{\mathcal{B}} \left(\{(t, X_d^{H(\cdot)}(t)) : t \in I\} \right) \leq \alpha.$$

Upper bound for box-counting dimension

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$$\overline{\dim}_{\mathcal{B}}\left(\{(t, X_d^{H(\cdot)}(t)) : t \in I\}\right) \leq \alpha.$$

Proposition (L.L.)

Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$, a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the uniform $\min$-regularity Condition and a compact interval $I \subset \mathbb{R}_+$, there exists $\widetilde{\Omega}_1$, an event of probability 1, such that, on $\widetilde{\Omega}_1$, we have

$$\overline{\dim}_{\mathcal{B}}\left(\mathcal{G}_d(I)\right) \leq 2 - \underline{H}(I).$$

Lower bound for Hausdorff dimension

Let $t_0 \in I$ be such that $H(t_0) = \underline{H}(I)$, for $j \in \mathbb{N}$, we have, for all $t, r \geq 0$ such that $t, t+r \in [t_0 - j^{-1}, t_0 + j^{-1}] \cap I$ and $s > 0$

$$\begin{aligned}
&\mathbb{E}\left[\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)|^2 + r^2\right)^{-\frac{s}{2}}\right] \\
&= \int_0^{r^{-s}} \mathbb{P}\left(\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)|^2 + r^2\right)^{-\frac{s}{2}} \geq x\right) \, dx \\
&= s \int_0^{+\infty} y(y^2 + r^2)^{-\frac{s}{2}-1} \mathbb{P}\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)| \leq y\right) \, dy
\end{aligned}$$

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$$\mathbb{E}\left[\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)|^2 + r^2\right)^{-\frac{s}{2}}\right]$$
$$= s\int_0^{+\infty} y(y^2+r^2)^{-\frac{s}{2}-1}\mathbb{P}\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)| \leq y\right)\, dy$$

Lemma (Carbery- Wright)

There is an absolute deterministic constant $c > 0$ such that, for any $n, d \geq 1$, $1 < p < \infty$ any polynomial $Q : \mathbb{R}^d \to \mathbb{R}$ of degree at most n, any Gaussian random vector $(X_1, \ldots, X_d)$ and any $x > 0$,

$$\mathbb{E}[|Q(X_1,\ldots,X_d)|^{\frac{p}{n}}]^{\frac{1}{p}}\mathbb{P}(|Q(X_1,\ldots,X_d)| \leq x) \leq cpx^{\frac{1}{n}}.$$

Lower bound for Hausdorff dimension

For all $t, u \in I$, we write

$$f_{t,u}^{H(\cdot)} : \mathbb{R}^d \to \mathbb{R} : \mathbf{w} \mapsto \int_0^t f_{H(t)}(s, \mathbf{w})\, ds - \int_0^u f_{H(u)}(s, \mathbf{w})\, ds$$

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Given $\{e_j\}_{j \in \mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R})$, the sequence of functions

$$\left(f_{t,u}^{H(\cdot),J} := \sum_{j_1,\ldots,j_d=1}^{J} \langle f_{t,u}^{H(\cdot)}, e_{j_1} \odot \cdots \odot e_{j_d} \rangle e_{j_1} \odot \cdots \odot e_{j_d} \right)_J$$

converges to $f_{t,u}^{H(\cdot)}$ in $L^2(\mathbb{R}^d)$.

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$$I_d\left(e_{j_1} \odot \cdots \odot e_{j_d}\right) = \prod_{\ell=1}^{p} H_{n_\ell}\left(\int_{\mathbb{R}} e_{\tilde{j}_\ell}(x)\, dB(x) \right),$$

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Wiener isometry, Fatou's Lemma and Carbery- Wright inequality give

$$\mathbb{P}\left(|X_d^{H(\cdot)}(t) - X_d^{H(\cdot)}(u)| \le x \right) \le c2dx^{\frac{1}{d}} \left\| X_d^{H(\cdot)}(t) - X_d^{H(\cdot)}(u) \right\|_{L^2(\Omega)}^{-\frac{1}{d}}$$

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As $H(t_0) = \underline{H}(I)$, one can find $\xi > 0$ such that, for all $0 < \varepsilon < \xi$ and $t, u \in I \cap [t_0 - \varepsilon, t_0 + \varepsilon]$

$$\left\| X_d^{H(\cdot)}(t) - X_d^{H(\cdot)}(u) \right\|_{L^2(\Omega)} \geq c_1 |t-u|^{\min\{H(t),H(u)\}} - c_2 |H(t) - H(u)|$$
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Therefore,

$$\mathbb{E}\left[\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)|^2 + r^2\right)^{-\frac{s}{2}}\right] \leq c'' r^{\frac{1}{d} - s - \frac{\overline{H}(t_0, j^{-1})}{d}}.$$

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$$\mathbb{E}\left[\left(|X_d^{H(\cdot)}(t+r) - X_d^{H(\cdot)}(t)|^2 + r^2 \right)^{-\frac{s}{2}} \right] \leq c'' r^{\frac{1}{d} - s - \frac{\overline{H}(t_0, j^{-1})}{d}}.$$

Thus, if we consider the random measure $\mu_{X,j}$ defined for all Borel sets $A \subseteq \mathbb{R}^2$ by

$$\mu_{X,j}(A) := \mathcal{L}\{t \in [t_0 - j^{-1}, t_0 + j^{-1}] \cap I \ : \ (t, X_d^{H(\cdot)}(t)) \in A\},$$

$$\mathbb{E}\left(\iint \frac{d\mu_{X,j}(x) d\mu_{X,j}(y)}{|x-y|^s} \right) < \infty$$

for all $s < 1 + \frac{1 - \overline{H}(t_0, j^{-1})}{d}$.

Fractal dimensions of the graph

Theorem (L.L.)
Given $d \in \mathbb{N}^*$, a compact set K of $(\frac{1}{2}, 1)$, a Hurst function $H : \mathbb{R}_+ \to K$ satisfying the uniform $\min$-Hölder regularity Condition and a compact interval $I \subset \mathbb{R}_+$, there exists $\widetilde{\Omega}$, an event of probability 1, such that on $\widetilde{\Omega}$, we have

$$1 + \frac{1 - \underline{H}(I)}{d} \leq \dim_{\mathcal{H}}(\mathcal{G}_d(I)) \leq \overline{\dim}_{\mathcal{B}}(\mathcal{G}_d(I)) \leq 2 - \underline{H}(I).$$

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When $d = 2$, any symmetric function $f \in L^2(\mathbb{R}^2)$ can be written as

$$f = \sum_{j \in \mathbb{N}} \lambda_{f,j} \, e_{f,j} \otimes e_{f,j},$$

with convergence in $L^2(\mathbb{R}^2)$, where $\{e_{f,j}\}_{j \in \mathbb{N}}$ are the eigenvectors with corresponding eigenvalues $\{\lambda_{f,j}\}_{j \in \mathbb{N}}$ of the Hilbert-Schmidt operator

$$\mathcal{A}_f : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : g \mapsto \int_{\mathbb{R}} f(\cdot, y) g(y) \, dy.$$

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The language of Malliavin calculus

If F is a cylindrical random variables of the form

$$F = g(I_1(f_1), \dots, I_1(f_n)) \tag{18}$$

with $n \geq 1$, $f_j \in L^2(\mathbb{R})$ and g infinitely differentiable such that all its partial derivatives have polynomial growth, the mth Malliavin derivative of F is the element of $L^2(\Omega, L^2(\mathbb{R}^m))$ defined by

$$D^m F = \sum_{j_1,\dots,j_m=1}^{n} \frac{\partial^m g}{\partial x_{j_1} \dots \partial x_{j_m}} (I_1(f_1), \dots, I_1(f_n)) f_{j_1} \otimes \cdots \otimes f_{j_m}.$$

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For all $m \geq 1$ and $p \geq 1$, $\mathbb{D}^{m,p}$ denote the closure of $\mathcal{S}$ (the set of cylindrical random variables) with respect to the norm

$$\| \cdot \|_{m,p} \ : \ F \mapsto \left(\mathbb{E}[|F|^p] + \sum_{j=1}^{m} \mathbb{E}[\|D^j F\|^p_{L^2(\mathbb{R}^j)}] \right)^{\frac{1}{p}}. \tag{19}$$

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Boundedness and continuity of density

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Lemma (Hu-Lu-Nualart)

If $F \in \mathbb{D}^{2,s}$ is such that $\mathbb{E}[|F|^{2p}] < \infty$ and $\mathbb{E}[\|DF\|_{L^2(\mathbb{R})}^{-2r}] < \infty$ for $p, r, s > 1$ satisfying $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$, then F has continuous and bounded density f_F with

$$\sup_{x \in \mathbb{R}} |f_F(x)| \leq c_p \left\| \|DF\|_{L^2(\mathbb{R})}^{-2} \right\|_{L^r(\Omega)} \|F\|_{2,s},$$

where $c_p > 0$ is a deterministic constant only depending on p.

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In our context,

$$I_2(f) = \sum_{j \in \mathbb{N}} \lambda_{f,j} \left(I_1(e_{f,j})^2 - 1 \right)$$

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$$DI_2(f) = 2 \sum_{j \in \mathbb{N}} \lambda_{f,j} I_1(e_{f,j}) e_{f,j}$$

Boundedness and continuity of density

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In our context,

$$\|DI_2(f)\|_{L^2(\mathbb{R})} = 2 \left(\sum_{j \in \mathbb{N}} \lambda_{f,j}^2 I_1(e_{f,j})^2 \right)^{\frac{1}{2}}. \quad (20)$$

Boundedness and continuity of density

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where $c_p > 0$ is a deterministic constant only depending on p.

Lemma (Hu-Lu-Nualart)

Let $G := \left(\sum_{j \in \mathbb{N}} \lambda_j X_j^2\right)^{\frac{1}{2}}$ where $\{\lambda_j\}_{j \in \mathbb{N}}$ satisfies $|\lambda_j| \geq |\lambda_{j+1}|$ for all $j \geq 1$ and $\{X_j\}_{j \in \mathbb{N}}$ are i.i.d. standard normal. For all $r > 1$, $\mathbb{E}[G^{-2r}] < \infty$ if and only if there exists $N > 2r$ such that $|\lambda_N| > 0$ and, in this case,

$$\mathbb{E}[G^{-2r}] \leq c_p N^{-r} |\lambda|^{-2r}, \quad (20)$$

with $c_r > 0$ a deterministic constant only depending on r.

The case of the multifractional Rosenblatt process

We keep the notation

$$f_{t,u}^{H(\cdot)} : \mathbb{R}^2 \to \mathbb{R} : \mathbf{w} \mapsto \int_0^t f_{H(t)}(s, \mathbf{w})\, ds - \int_0^u f_{H(u)}(s, \mathbf{w})\, ds$$

and also write

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If $\{\lambda_j\}_{j\in\mathbb{N}}$ are the eigenvalues of the Hilbert-Schmidt operator $\mathcal{A}_{f_{1,0}^{H(t)}}$, $\{|t-u|^{H(t)}\lambda_j\}_{j\in\mathbb{N}}$ are the eigenvalues of $\mathcal{A}_{f_{t,u}^{H(t)}}$

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If $\{\lambda_j\}_{j\in\mathbb{N}}$ are the eigenvalues of the Hilbert-Schmidt operator $\mathcal{A}_{f_{1,0}^{H(t)}}$, $\{|t-u|^{H(t)}\lambda_j\}_{j\in\mathbb{N}}$ are the eigenvalues of $\mathcal{A}_{f_{t,u}^{H(t)}}$. We also know that $\lambda_3 \neq 0$. Thus, if $\{\xi_j^{t,u}\}_{j\in\mathbb{N}}$ are the eigenvalues of the Hilbert-Schmidt operator $\mathcal{A}_{f_{t,u}^{H(u)}}$ ordered with $|\xi_j^{t,u}| \geq |\xi_{j+1}^{t,u}|$

$$|\xi_3^{t,u}| > |t-u|^{H(t)}|\lambda_3| - \|f_{t,u}^{H(\cdot)} - f_{t,u}^{H(t)}\|_{L^2(\mathbb{R}^2)}.$$

As $H(t_0) = \underline{H}(I)$, one can find $\xi > 0$ such that, for all $0 < \varepsilon < \xi$ and $t, u \in I \cap [t_0 - \varepsilon, t_0 + \varepsilon]$

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Multifractional Hermite processes: definition and first properties

Laurent Loosveldt

Lille – Séminaire de probabilités et statistique

21 mars 2023

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