Wavelet-type approximation of Hermite processes

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Who?

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Fix $d \in \mathbb{N}^*$, and $\mathbf{h} = (h_1, \cdots, h_d)$ satisfying

$$h_1, \cdots, h_d \in (1/2, 1) \text{ and } \sum_{\ell=1}^{d} h_\ell > d - \frac{1}{2}. \tag{1}$$

$$X_{\mathbf{h}}^{(d)}(t) := \int_{\mathbb{R}^d}' K_{\mathbf{h}}^{(d)}(t, x_1, \ldots, x_d)\, dB(x_1) \cdots dB(x_d) \tag{2}$$

where the kernel function $K_{\mathbf{h}}$ is given, for every $(t, x_1, \ldots, x_d) \in \mathbb{R}_+ \times \mathbb{R}^d$, by

$$K_{\mathbf{h}}^{(d)}(t, x_1, \cdots, x_d) := \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} \int_0^t \prod_{j=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2}\, ds \tag{3}$$

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- $h_\ell = 1 + \frac{H-1}{d} \; \forall 1 \leq \ell \leq d \Rightarrow$ (Standard) Hermite process with Hurst parameter H

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2. Simulations
3. Methods to study the pointwise regularity.

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Wavelet-type expansion of a general Hermite process was still an open problem.

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1. For practical reasons, we would like to approximate the process using so-called FARIMA sequences and fractional scaling functions. Until now it was unclear how such quantities could appear in the approximation procedure.
2. We would like to judge the rate of convergence for the approximation. For the Rosenblatt process, it was unknown before Ayache-Esmili (2020).

The idea behind wavelet analysis

Multiresolution analysis (MRA)

A multiresolution analysis of the Hilbert space $L^2(\mathbb{R}^d)$ is given by a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that

(a) for all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$;

(b) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$;

(c) for all $j \in \mathbb{Z}$, $V_j = \{f(2^j\cdot) : f \in V_0\}$;

(d) there exists a scaling function $\phi^d \in V_0$ such that the sequence $(\phi^d(\cdot - k))_{k\in\mathbb{Z}^d}$ is an orthogonal basis of V_0.

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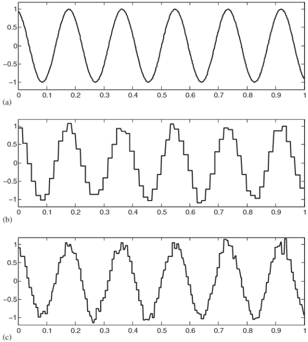
3. For all J, $V_J \oplus \bigoplus_{j\geq J} W_j$ is dense in $L^2(\mathbb{R}^d)$.

An example

Haar MRA (d=1)

$$V_j^1 = \{f \in L^2(\mathbb{R}) \ : \ \forall k \in \mathbb{Z}, f \text{ is constant on } [k\, 2^{-j}, (k+1)\, 2^{-j})\ \}$$



The scaling function is $\mathbb{1}_{[0,1)}$.

Here come the wavelets

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(Meyer)
There exists a function ψ, called mother wavelet, belonging to W_0^1 and such that, for all $j \in \mathbb{Z}$, the sequence $(2^{j/2}\psi(2^j \cdot -k))_{k\in\mathbb{Z}}$ is an orthonormal basis in W_j^1. Moreover, for all $J \in \mathbb{Z}$, the family

$$\{2^{J/2}\phi^1(2^J x - k) \ : \ k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) \ : \ k \in \mathbb{Z}, \, j \geq J\}$$

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- Meyer MRA leads to wavelets in the Schwartz class: for all $m \in \mathbb{N}_0$ and $L > 0$ we have

$$\sup_{x\in\mathbb{R}} \left\{(3 + |x|)^L |D^m\psi(x)|\right\} < +\infty. \tag{4}$$

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(d>1)

Usually, we use

$$\{2^{J\frac{d}{2}}\Phi^d(2^J x - k) \,:\, k \in \mathbb{Z}^d\} \cup \{2^{J\frac{d}{2}}\Psi(2^j x - k) \,:\, k \in \mathbb{Z}^d,\, j \geq J\}$$

- Φ^d, d-tensor products of ϕ with itself
- Ψ, d-tensor products of ϕ and ψ where at least one term is ψ.

Here come the wavelets

(Meyer)

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is a base in $L^2(\mathbb{R})$, called wavelet base.

(d>1) (A. Ayache, L.L, J. Hamonier)

But one can show that

$$\{2^{J\frac{d}{2}}\Phi^d(2^J x - k) \;:\; k \in \mathbb{Z}^d\} \cup \{\prod_{\ell=1}^{d} 2^{\frac{j_\ell}{2}}\psi(2^{j_\ell} x_\ell - k_\ell) \;:\; k \in \mathbb{Z}^d,\, \max_{1\leq\ell\leq d} j_\ell \geq J\}$$

is also an orthonormal base in $L^2(\mathbb{R}^d)$.

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3. we get a sequence of approximating processes

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4. the quality of this approximation can be judged through the detail processes

$$X_{\mathbf{h},J}^{(d)\perp}(t) := \int_{\mathbb{R}^d}' K_{\mathbf{h},J}^{(d)\perp}(t, x_1, \ldots, x_d) dB(x_1) \cdots dB(x_d)$$

   where $K_{\mathbf{h},J}^{(d)\perp}(t, x_1, \ldots, x_d)$ is the projection of the kernel onto $V_J^\perp$.

Approximation process

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$$X^{(d)}_{\mathbf{h},J}(t) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mu_{J,\mathbf{k}}\mathcal{K}_{J,\mathbf{k}}(t)$$

with

$$\mu_{J,\mathbf{k}} = 2^{J\frac{d}{2}}\int'_{\mathbb{R}^d}\phi(2^Jx_1-k_1)\cdots\phi(2^Jx_d-k_d)\,dB(x_1)\ldots dB(x_d)$$

and

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Approximation process - random part

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By Wiener isometry,

$$\left(g_{J,k} = 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k)\, dx \right)_{k \in \mathbb{Z}}$$

is a family of i.i.d. $\mathcal{N}(0,1)$ random variables, since the function $(2^{J/2}\phi(2^J \cdot - k))_k$ are orthogonal.

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$$\mu_{J,\mathbf{k}} = \prod_{\ell=1}^{p} H_{n_\ell}(g_{J,\widetilde{k}_\ell})$$

where n_ℓ is the multiplicity of $\widetilde{k}_\ell$ in $\mathbf{k}$ and H_n is the nth Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2/2} D^n e^{-x^2/2}.$$

Approximation process - deterministic part

$$\mathcal{K}_{J,\mathbf{k}}(t) = \frac{1}{\prod_{\ell=1}^{d} \Gamma(h_\ell - 1/2)} 2^{J\frac{d}{2}} \int_0^t \int_{\mathbb{R}^d} \prod_{\ell=1}^{d} (s - x_\ell)_+^{h_\ell - 3/2} \phi(2^J x_\ell - k_d)\, dx_1 \ldots dx_d.$$

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is the fractional antiderivative of ϕ.

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$$\mathcal{K}_{J,\mathbf{k}}(t) = 2^{-J(h_1 + \ldots + h_d - d)} \int_0^t \prod_{\ell=1}^{d} \phi_{h_\ell}(2^J s - k_\ell) \, ds$$

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$$\mathcal{K}_{J,\mathbf{k}}(t) = 2^{-J(h_1 + \ldots + h_d - d)} \int_0^t \prod_{\ell=1}^{d} \phi_{h_\ell}(2^J s - k_\ell)\, ds$$

BAD NEWS

The function ϕ_h is not well-adapted for approximation purposes (badly localized, not in the Schwartz class,...).

Work in the frequency domain

GOOD NEWS

The *fractional scaling function* of order $\delta \in (0, 1/2)$ of the Meyer scaling function defined through its Fourier transform by

$$\widehat{\Phi}_{\Delta}^{(\delta)}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^{\delta} \widehat{\phi}(\xi) \; \forall \, \xi \neq 0 \text{ and } \widehat{\Phi}_{\Delta}^{(\delta)}(0) = 1$$

is much better!

Work in the frequency domain

UNIVERSITÉ DU LUXEMBOURG

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is much better!

With a bit of Fourier analysis tools and tricks, we get to

$$K_{\mathbf{h},J}^{(d)}(t, \mathbf{x}) = 2^{-J(h_1 + \cdots + h_d - d)} \sum_{\mathbf{k} \in \mathbb{Z}^d} \beta_{\mathbf{k}}^{(\mathbf{h})} 2^{J\frac{d}{2}} \prod_{\ell=1}^{d} \Phi^{-(h_\ell - 1/2)}(2^J x_\ell - k_\ell)$$

where

$$\beta_{\mathbf{k}}^{(\mathbf{h})} := \int_0^t \prod_{\ell=1}^{d} \Phi_{\Delta}^{(h_\ell - 1/2)}(2^J s - k_\ell)\, ds \text{ and } \widehat{\Phi}^{-(\delta)}(\xi) = (1 - e^{i\xi})^{-\delta} \widehat{\phi}(\xi)$$

New random part

$$2^{J\frac{d}{2}} \int_{\mathbb{R}^d}' \prod_{\ell=1}^{d} \Phi^{-(h_\ell - 1/2)}(2^J x_\ell - k_\ell)\, dB(x_1) \ldots dB(x_d)$$

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Lemma (A. Ayache, L.L., J. Hamonier)
For all $\delta \in (0, \frac{1}{2})$, we have

$$\Phi^{-(\delta)}(x) = \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} \phi(x+p) \tag{4}$$

with convergence in $L^2(\mathbb{R})$, where

$$\gamma_p^{(\delta)} := \frac{\delta\, \Gamma(p+\delta)}{\Gamma(p+1)\Gamma(\delta+1)}.$$

New random part

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(2^{J\frac{d}{2}}\int_{\mathbb{R}^d}'\prod_{\ell=1}^{d}\phi(2^J x_\ell+p_\ell-k_\ell)\,dB(x_1)\dots dB(x_d)\right)$$

New random part

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\mu_{J,\mathbf{k}-\mathbf{p}}$$

where we recall that, for all $\mathbf{k}\in\mathbb{Z}^d$,

$$\mu_{J,\mathbf{k}}=\prod_{\ell=1}^{p}H_{n_\ell}(g_{J,\widetilde{k}_\ell})$$

New random part

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FARIMA (autogressive fractionally integrated moving average)

Let $\{Z_j\}_{j\in\mathbb{Z}}$ be a sequence of i.i.d. centred Gaussian random variables and $\delta\in(-\frac{1}{2},\frac{1}{2})$. The Gaussian FARIMA $(0,\delta,0)$, denoted by $\{Z_j^{(\delta)}\}_{j\in\mathbb{Z}}$, is defined, for all $j\in\mathbb{Z}$ as

$$Z_j^{(\delta)} := \sum_{p=0}^{+\infty}\gamma_p^{(\delta)} Z_{j-p} \tag{4}$$

Random part with FARIMA

Hermite polynomials and partitions

Tthe dth Hermite polynomials can be written as

$$H_d(x) = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m a_m^{(d)} x^{d-2m},$$

where $a_m^{(d)}$ is the number of partitions of $\{1, \ldots, d\}$ with m (non ordered) pairs and $d - 2m$ singletons.

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$\mathcal{P}_m^{(d)}$ is the set of partitions of $\{1, \ldots, d\}$ with m (non ordered) pairs and $d - 2m$ singletons.

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$\mathcal{P}_m^{(d)}$ is the set of partitions of $\{1,\ldots,d\}$ with m (non ordered) pairs and $d-2m$ singletons.

(A. Ayache, L.L., J. Hamonier)

For all $J \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, we have

$$\mu_{J,\mathbf{k}} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[g_{J,k_{l_r}} g_{J,k_{l'_r}}] \prod_{s=m+1}^{d-m} g_{J,k_{l''_s}} \text{ with } g_{J,k} = 2^{J/2} \int_{\mathbb{R}} \phi(2^J x - k)\, dx$$

Expansion for the approximation process

$$\sum_{\mathbf{p}\in\mathbb{N}_0^d}\left(\prod_{\ell=1}^{d}\gamma_{p_\ell}^{(h_\ell-1/2)}\right)\left(\sum_{m=0}^{\lfloor d/2\rfloor}(-1)^m\sum_{P\in\mathcal{P}_m^{(d)}}\prod_{r=1}^{m}\mathbb{E}[g_{J,k_{l_r}-p_{l_r}}\,g_{J,k_{l'_r}-p_{l'_r}}]\prod_{s=m+1}^{d-m}g_{J,k_{l''_s}-p_{l''_s}}\right)$$

Expansion for the approximation process

For all $(J, \mathbf{k}) \in \mathbb{Z} \times \mathbb{Z}^d$, we define the random variable

$$\sigma_{J,\mathbf{k}}^{(\mathbf{h})} = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \sum_{P \in \mathcal{P}_m^{(d)}} \prod_{r=1}^{m} \mathbb{E}[\varepsilon_{J,k_{l_r}}^{(h_{l_r}-1/2)} \varepsilon_{J,k_{l'_r}}^{(h_{l'_r}-1/2)}] \prod_{s=m+1}^{d-m} \varepsilon_{J,k_{l''_r}}^{(h_{l''_r}-1/2)}.$$

where $\varepsilon_{J,k}^{(\delta)} := \sum_{p=0}^{+\infty} \gamma_p^{(\delta)} g_{J,k-p}$ is the FARIMA sequence associated to $(g_{J,k})_k$.

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Theorem (A. Ayache, L.L., J. Hamonier)
For each fixed $t \in \mathbb{R}_+$, one has almost surely

$$X_{\mathbf{h},J}^{(d)}(t) = 2^{-J(h_1+\ldots+h_d-d)} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sigma_{J,\mathbf{k}}^{(\mathbf{h})} \int_0^t \prod_{\ell=1}^{d} \Phi_\Delta^{(h_\ell-1/2)}(2^J s - k_\ell)\, ds. \qquad (5)$$

Moreover, the series (5) is almost surely uniformly convergent on compact intervals.

Details process

We recall that $\{\prod_{\ell=1}^{d} 2^{\frac{j_\ell}{2}} \psi(2^{j_\ell} x_\ell - k_\ell) \,:\, k \in \mathbb{Z}^d,\ \max_{1\le \ell \le d} j_\ell \ge J\}$ is a base of $V_J^{\perp}$.

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This time, the fractional antiderivative

$$\psi_h(s) := \frac{1}{\Gamma(h - 1/2)} \int_{\mathbb{R}} (s - x)_+^{h-3/2} \psi(x)\, dx$$

belongs to the Schwartz class!

Details process

Theorem (A. Ayache, L.L., J. Hamonier)

For each fixed $t \in \mathbb{R}_+$, one has almost surely

$$X_{\mathbf{h},J}^{(d),\perp}(t) = \sum_{\substack{(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^d)^2 \\ (\max_{1 \le \ell \le d} j_\ell) \ge J}} \varepsilon_{\mathbf{j},\mathbf{k}} \, 2^{j_1(1-h_1)+\cdots+j_d(1-h_d)} \int_0^t \prod_{\ell=1}^d \psi_{h_\ell}(2^{j_\ell} s - k_\ell)\, ds, \qquad (6)$$

where

$$\varepsilon_{\mathbf{j},\mathbf{k}} = \prod_{\ell=1}^p H_{n_\ell}\left(2^{j_\ell/2} \int_{\mathbb{R}} \psi(2^{j_\ell} x - k_\ell)\, dB(x)\right)$$

and n_ℓ is the multiplicity of (j_ℓ, k_ℓ) in $(\mathbf{j}, \mathbf{k})$. Moreover, the series (6) is almost surely uniformly convergent on compact intervals.

Rate of convergence

Theorem (A. Ayache, L.L., J. Hamonier)
For any compact interval $I \subset \mathbb{R}_+$, there exists an almost surely finite random variable C (depending on I) for which one has, almost surely, for each $J \in \mathbb{N}$,

$$\|X_{\mathbf{h}}^{(d)} - X_{\mathbf{h},J}^{(d)}\|_{I,\infty} = \|X_{\mathbf{h},J}^{(d),\perp}(t)\|_{I,\infty} \leq CJ^{\frac{d}{2}} 2^{-J(h_1+\cdots+h_d-d+1/2)} \tag{7}$$

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Basic ideas:

1. There exist an event Ω^* of probability 1 and C_d^* a positive random variable of finite moment of any order, such that on Ω^* one has, for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^2)^d$,

$$|\varepsilon_{\mathbf{j},\mathbf{k}}| \leq C_d^* \prod_{m=1}^{d} \sqrt{\log(3 + |j_m| + |k_m|)}. \tag{8}$$

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1. There exist an event Ω^* of probability 1 and C_d^* a positive random variable of finite moment of any order, such that on Ω^* one has, for all $(\mathbf{j},\mathbf{k}) \in (\mathbb{Z}^2)^d$,

$$|\varepsilon_{\mathbf{j},\mathbf{k}}| \leq C_d^* \prod_{m=1}^{d} \sqrt{\log(3+|j_m|+|k_m|)}. \quad (8)$$

2. The random variables $\varepsilon_{\mathbf{j},\mathbf{k}}$ and $\varepsilon_{\mathbf{r},\mathbf{s}}$ are correlated if and only if $(\mathbf{j},\mathbf{k})$ is a permutation of $(\mathbf{r},\mathbf{s})$.

Wavelet-type approximation of Hermite processes

Laurent Loosveldt
joint work with A. Ayache and J. Hamonier (Université de Lille)

Luxembourg Workshop in Stochastic Analysis

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UNIVERSITÉ DU LUXEMBOURG