Rosenblatt process and wavelets

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Lille – Scale Invariance and Randomness

07-06-2022

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Motivation – a bit of history

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Let $\{B(t,\cdot)\}_{t\in\mathbb{R}}$ denote the standard two-sided Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The Khinchin law of the iterated logarithm allows to control the behavior of B at a given point, in the sense that for every $t\in\mathbb{R}$, it holds

$$\limsup_{r\to 0}\frac{|B(t+r)-B(t)|}{\sqrt{|r|\log\log|r|^{-1}}}=\sqrt{2} \tag{1}$$

on an event of probability one.

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Almost surely, for almost every $t \in \mathbb{R}$,

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This contrasts with the uniform Hölder condition obtained by Paul Lévy: almost surely, one has

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There exists exceptional points, called rapid points, where the law of the iterated logarithm fails.

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while

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Kahane used the expansion of the Brownian motion in the well-chosen Faber-Schauder system, to propose an easy way to study its regularity and irregularity properties.

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Kahane used the expansion of the Brownian motion in the well-chosen Faber-Schauder system, to propose an easy way to study its regularity and irregularity properties. It allows to recover the law of the iterated logarithm and the estimation of the modulus of continuity of the Brownian motion. Furthermore, Kahane obtained the existence of a third category of points, presenting a slower oscillation:

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and there exist points, called slow points, satisfy the condition

$$\limsup_{r\to 0} \frac{|B(t+r)-B(t)|}{\sqrt{|r|}} < +\infty.$$

Fractional Brownian Motion

C. Esser & L.L. (2022)

For all $H \in (0,1)$, there exists an event Ω_H of probability 1 satisfying the following assertions for all $\omega \in \Omega_H$ and every non-empty interval I of $\mathbb{R}$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log\log|t-s|^{-1}}} < +\infty.$$

- There exists a dense set of *rapid* points $t \in I$ such that

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log|t-s|^{-1}}} < +\infty.$$

- There exists a dense set of *slow* points $t \in I$ such that

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H} < +\infty.$$

Does this feature depend from the Gaussianity of the process?

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The Rosenblatt process

- belongs to the class of Hermite processes

$$X_H^{(d)}(t,\cdot) := \frac{1}{\Gamma(H-1/2)^d} \int'_{\mathbb{R}^d} \left(\int_0^t \prod_{p=1}^{d} (s - x_p)_+^{H-3/2} \, ds \right) dB(x_1) \dots dB(x_d) \tag{2}$$

with $1 - \frac{1}{2d} < H < 1$.

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with $1 - \frac{1}{2d} < H < 1$.

Wavelet-type expansion

We consider the *generalized* Rosenblatt process $R_{H_1,H_2}(t,\cdot)$ given by

$$\frac{1}{\Gamma\left(H_1-\frac{1}{2}\right)\Gamma\left(H_2-\frac{1}{2}\right)}\int_{\mathbb{R}^2}^{\prime}\int_0^t (s-x_1)_+^{H_1-\frac{3}{2}}(s-x_2)_+^{H_2-\frac{3}{2}}\,ds\,dB(x_1)dB(x_2)$$

where $H_1, H_2 \in (\frac{1}{2}, 1)$ are such that $H_1 + H_2 > \frac{3}{2}$.

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Observations:

- if ψ is the Meyer wavelet,

$$\{2^{\frac{j_1+j_2}{2}}\psi(2^{j_1}x_1-k_1)\psi(2^{j_2}x_2-k_2)\ :\ (j_1,j_2,k_1,k_2)\in\mathbb{Z}^4\}$$

is an orthonormal basis in $L^2(\mathbb{R}^2)$.

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is an orthonormal basis in $L^2(\mathbb{R}^2)$.

- for all $H(1/2,1)$

$$\psi_H(s) := \frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbb{R}}(s-x)_+^{H-3/2}\psi(x)\,dx.$$

is well-defined as the antiderivative of order $H-1/2$ of ψ.

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Let $\varepsilon_{j_1,j_2}^{k_1,k_2}$ be the second order Wiener chaos random variable defined by

$$2^{\frac{j_1+j_2}{2}}\int_{\mathbb{R}^2}'\psi(2^{j_1}x_1-k_1)\psi(2^{j_2}x_2-k_2)\,dB(x_1)dB(x_2).$$

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Ayache & Esmili (2020)

Let ψ be the Meyer wavelet and I be any compact interval of $\mathbb{R}_+$. Almost surely, the random series

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_0^t \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx \qquad (3)$$

converges uniformly to R_{H_1,H_2} on the interval I.

Objective

We want to find functions (moduli of continuity) θ_o, θ_r and θ_{sl} and an event Ω' on probability 1 such that, for all $\omega \in \Omega'$ and for every non-empty interval I of $\mathbb{R}$

- for almost every $t \in I$

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta_o(|t-s|)} < \infty$$

- for a dense set of $t \in I$

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta_r(|t-s|)} < \infty$$

- for a dense set of $t \in I$

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta_{sl}(|t-s|)} < \infty$$

Methods

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta(|t-s|)} < \infty$$

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It suffices to show that there exists $C > 0$ such that

$$|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)| \leqslant C\theta(|t-s|).$$

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One can write $R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)$ as the series

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) \int_s^t \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx$$

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Methods

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta(|t-s|)} < \infty$$

We use wavelet leaders!

Methods

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{\theta(|t-s|)} < \infty$$

If

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \underbrace{\left(2^j \int_{\mathbb{R}} f(x) \Psi(2^j x - k)\, dx\right)}_{c_{j,k}} \Psi(2^j \cdot - k),$$

is the expansion of the function f using a compactly supported wavelet Ψ.

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is the expansion of the function f using a compactly supported wavelet Ψ. ,
$\Psi(2^j \cdot -k)$ has its support in a multiple of the dyadic interval

$$\lambda = [k2^{-j}, (k+1)2^{-j})$$

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Definition

The **wavelet leader** of scale j at the point x_0 is the quantity

$$d_j(x_0) = \max_{\lambda \in 3\lambda_j(x_0)} \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|.$$

Methods

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$$f = \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda_j} c_\lambda \Psi_\lambda,$$

For all j,

$$|d_j(t)| \leqslant C \sup_{s \in B(t, R2^{-j})} |f(t) - f(s)|$$

where R is computed from the support of the wavelet and C is a positive constant only depending on the wavelet.

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$$f = \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda_j} c_\lambda \Psi_\lambda,$$

$$\left(0 < \limsup_{j\to+\infty} \frac{|d_j(t)|}{\theta(2^{-j})}\right) \Rightarrow \left(0 < \limsup_{s\to t} \frac{|f(t)-f(s)|}{\theta(|t-s|)}\right)$$

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!! We can not use the wavelet TYPE series

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} \int_0^t \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx$$

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Estimation of the wavelet coefficients

Let Ψ be a wavelet with compact support included in $[-N, N]$, we have

$$c_{j,k} = C_{H_1,H_2} \int_A' \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2)$$

where $A := \left]-\infty, \frac{k+N}{2^j}\right]^2$, $C_{H_1,H_2} := \frac{1}{\Gamma\left(H_1 - \frac{1}{2}\right)\Gamma\left(H_2 - \frac{1}{2}\right)}$ and for $s, x_1, x_2 \in \mathbb{R}$

$$f_{H_1,H_2}(s, x_1, x_2) = (s - x_1)_+^{H_1 - 3/2} (s - x_2)_+^{H_2 - 3/2}$$

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Estimation of the wavelet coefficients

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Given an integer $M \geqslant 0$, $c_{j,k}$ can be written as following

$$c_{j,k} = \widetilde{c_{j,k}}^M + \undertilde{c_{j,k}}^M$$

where

$$\widetilde{c_{j,k}}^M = c_{H_1,H_2} \int_{\lambda_{j,k}^M}' \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s,x_1,x_2)\, ds\, dx\, dB(x_1)\, dB(x_2) \qquad (4)$$

with

$$\lambda_{j,k}^M := \left]\frac{k-NM}{2^j}, \frac{k+N}{2^j}\right[^2$$

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$c_{j,k} = \widetilde{c_{j,k}}^M + \overset{\smile}{c_{j,k}}^M$ where

$$\widetilde{c_{j,k}}^M = c_{H_1,H_2} \int_{\lambda_{j,k}^M}' \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2) \qquad (4)$$

$$(\lambda_{j,k}^M \cap \lambda_{j',k'}^M = \emptyset) \Rightarrow \left(\widetilde{c_{j,k}}^M \perp\!\!\!\perp \widetilde{c_{j',k'}}^M\right)$$

Estimation of the wavelet coefficients

Let Ψ be a wavelet with compact support included in $[-N, N]$, we have

$$c_{j,k} = C_{H_1,H_2} \int_A' \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2)$$

L. Daw & L.L. (2022)

There exist three strictly positive deterministic constants C_{Ψ,H_1,H_2}, C'_{Ψ,H_1,H_2} and C^*_{Ψ,H_1,H_2} such that for all $(j,k) \in \mathbb{N} \times \mathbb{Z}$ and $M \geqslant 2$ one has

$$C_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} \leqslant \left\| \widetilde{c_{j,k}}^M \right\|_{L^2(\Omega)} \leqslant C'_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)}$$

and

$$\left\| \widetilde{c_{j,k}}^M \right\|_{L^2(\Omega)} \leqslant C^*_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} M^{\max\{H_1,H_2\}-1}$$

Results

Janson (1997)

Let us fix $n \in \mathbb{N}$.

There exists a strictly positive universal deterministic constant $\overset{\star}{C}$ such that, for every random variable X belonging to the Wiener chaos of order n and for each real number $y \geqslant 2$, one has

$$\mathbb{P}(|X| \geqslant y\|X\|_{L^2(\Omega)}) \leqslant \exp(-\overset{\star}{C} y^{2/n}).$$

If X is a random variable belonging to the Wiener chaos of order n, there exist $a, b, y_0 > 0$ such that, for all $y \geqslant y_0$,

$$\exp(-ay^{2/n}) \leqslant \mathbb{P}(|X| \geqslant y) \leqslant \exp(-by^{2/n}).$$

Results

L. Daw & L.L. (2022)

For all $H_1, H_2 \in (\frac{1}{2}, 1)$ such that $H_1 + H_2 > \frac{3}{2}$, there exists an event Ω_{H_1,H_2} of probability 1 satisfying the following assertions for all $\omega \in \Omega_{H_1,H_2}$ and every $I \neq \emptyset$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log\log |t-s|^{-1}} < +\infty.$$

- There exists a dense set of *rapid* points $t \in I$ such that

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log |t-s|^{-1}} < +\infty.$$

- There exists a dense set of *slow* points $t \in I$ such that

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty.$$

Rapid points

First we write $R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)$ as the series

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) \int_s^t \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx$$

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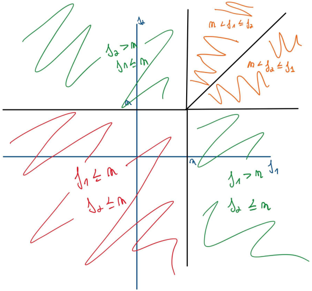
If n is such that $2^{-n-1} < |t-s| \leqslant 2^{-n}$, we split the sums over j_1 and j_2 according to the regions



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$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)\int_s^t \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

Ayache & Esmili (2020)

There exist an event Ω^* of probability 1 and a positive random variable C_1 with finite moment of any order, such that, for all $\omega \in \Omega^*$ and for each $(j_1,j_2,k_1,k_2) \in \mathbb{Z}^4$,

$$|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)| \leqslant C_1(\omega)\sqrt{\log(3+|j_1|+|k_1|)}\sqrt{\log(3+|j_2|+|k_2|)}. \qquad (4)$$

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Rapid points

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L. Daw & L.L. (2022)

There exists an event Ω_{rap} of probability 1 such that for all $\omega \in \Omega_{\text{rap}}$ there exists $C_R(\omega) > 0$ such that, for all $t, s \in (0,1)$, we have

$$|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)| \leqslant C_R(\omega)|t-s|^{H_1+H_2-1} \log|t-s|^{-1}. \qquad (4)$$

Rapid points

L. Daw & L.L. (2022)

There exists a deterministic constant $C > 0$ such that for all M there is $\Omega_2 \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_2$ there exist $t \in (0,1)$ such that

$$\limsup_{j \to +\infty} \frac{\left| \widetilde{c_{\lambda_j(t)}}^{M}(\omega) \right|}{j 2^{-j(H_1+H_2-1)}} \geqslant C.$$

Rapid points

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Proof combined Baire's theorem, Borel-Cantelli Lemma and Janson estimates

Rapid points

L. Daw & L.L. (2022)
There exists a deterministic constant $C' > 0$ such that for all M there is $\Omega_2' \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_2'$ there exist $J \in \mathbb{N}$ such that, for all $j \geqslant J$, for all $\lambda \in \Lambda_j$, $\lambda \subseteq [0,1]$,

$$\left| \check{c}_\lambda{}^M(\omega) \right| \leqslant C' M^{\max\{H_1,H_2\}-1} j 2^{j(H_1+H_2-1)}$$

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Proof combined Borel-Cantelli Lemma and Janson estimates

Rapid points

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L. Daw & L.L. (2022)
There exists $\Omega_2^* \subset \Omega$ with probability 1 such that, for all $\omega \in \Omega_2^*$, there exist $t \in (0,1)$ such that

$$\limsup_{j \to +\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)} j} > 0.$$

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There exists $\Omega_2^* \subset \Omega$ with probability 1 such that, for all $\omega \in \Omega_2^*$, there exist $t \in (0,1)$ such that

$$\limsup_{j\to+\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}j} > 0.$$

Almost surely, there is $t \in (0,1)$ such that

$$0 < \limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log|t-s|^{-1}} < +\infty.$$

Ordinary points

Fix $t \in (0,1)$, there exists Ω_t^*, an event of probability 1, and $C_{t,1}$, a positive random variable with finite moment of any order, such that, for all $\omega \in \Omega_t^*$ and for each $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, one has

$$|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)| \leqslant C_{t,1}(\omega)\sqrt{\log(3+|j_1|+|k_1-k_{j_1}(t)|)}\sqrt{\log(3+|j_2|+|k_2-k_{j_2}(t)|)}. \tag{5}$$

where $k_j(t)$ is the unique integer such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$.

Ordinary points

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where $k_j(t)$ is the unique integer such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$. For all $s \in (0,1)$ and $\omega \in \Omega_t^*$ we have

$$|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)| \leqslant C_t(\omega)|t-s|^{H_1+H_2-1}\log|\log|t-s|^{-1}|.$$

Ordinary points

L. Daw & L.L. (2022)

There exists an event Ω_{ord} of probability 1 such that for all $\omega \in \Omega_{\text{ord}}$, for almost every $t \in (0,1)$,

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log|\log|t-s|^{-1}|} < +\infty.$$

Ordinary points

L. Daw & L.L. (2022)

There exists an event Ω_{ord} of probability 1 such that for all $\omega \in \Omega_{\text{ord}}$, for almost every $t \in (0,1)$,

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log|\log|t-s|^{-1}|} < +\infty.$$

L. Daw & L.L. (2022)

There exists $\Omega_1^* \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_1^*$ and Lebesgue almost every $t \in (0,1)$ one has

$$\limsup_{j\to+\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}\log j} > 0.$$

Ordinary points

Almost surely, for almost every $t \in (0,1)$ such that

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log|\log|t-s|^{-1}|} < +\infty.$$

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Slow points

The finiteness of the limit

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}}$$

is more tricky.

Slow points

The finiteness of the limit

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}}$$

is more tricky. It needs a selection within the dyadic subintervals of $(0,1)$. The idea is that a dyadic interval is “killed” if some associate random variables are big, which happens with a low probability.

Slow points

The finiteness of the limit

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}}$$

is more tricky. It needs a selection within the dyadic subintervals of $(0,1)$. The idea is that a dyadic interval is "killed" if some associate random variables are big, which happens with a low probability. The proof is a combination of

- Cantor Theorem,
- Borel-Cantelli Lemma
- Tchebycheff's inequality
- $\ldots$

Slow points

L. Daw & L.L. (2022)

There exists an event Ω_{slo} of probability 1 such that for all $\omega \in \Omega_{\text{slo}}$ there exist $t \in (0,1)$ such that

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty.$$

Slow points

L. Daw & L.L. (2022)

There exists an event Ω_{slo} of probability 1 such that for all $\omega \in \Omega_{\text{slo}}$ there exist $t \in (0,1)$ such that

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty.$$

The positiveness of the limit is still open, the difficulty being within the fact that it seems difficult to find optimal bounds both for $\tilde{c}_\lambda^{\,M}$ and $\check{c}_\lambda^{\,M}$.

Rosenblatt process and wavelets

Laurent Loosveldt

Lille – Scale Invariance and Randomness

07-06-2022

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