Generalized Regularity and Stochastic Processes

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Workshop “Regularity of Stochastic Processes”

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UNIVERSITÉ DU LUXEMBOURG

(Pointwise) Hölder spaces

Let $x_0 \in \mathbb{R}^d$; a function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to the Hölder space $C^\alpha(x_0)$ $(0 < \alpha < 1)$ if there exists $C > 0$ s.t., for j large enough,

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If $f \in L^\infty([a,b])$ and the constant $C > 0$ is uniform, then f belongs to the uniform Hölder space $C^\alpha([a,b])$

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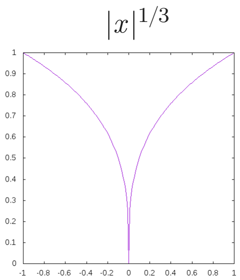

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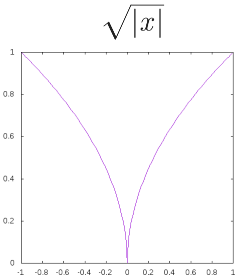

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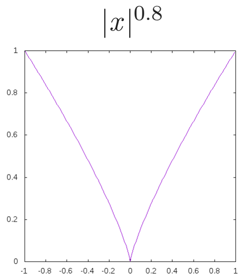

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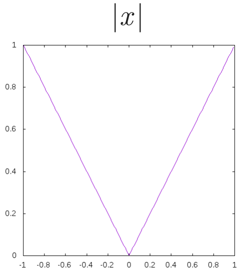

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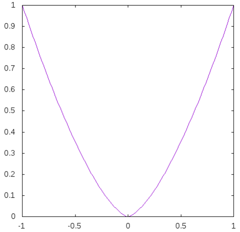
$|x|^{3/2}$



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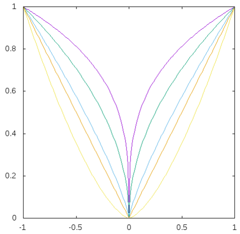
$$|x|^{1/3}, \sqrt{|x|}, |x|^{0.8}, |x|, |x|^{3/2}.$$



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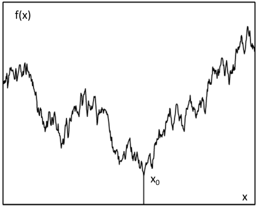

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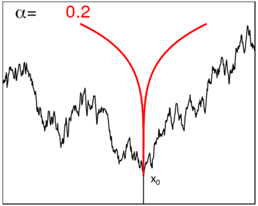

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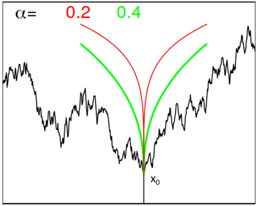

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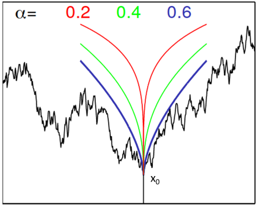

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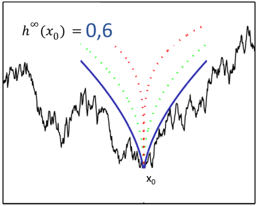

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$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

Khintchine law of iterated logarithm

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Let $\{B(t)\}_{t\in\mathbb{R}}$ be a (two-sided) Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, almost surely, for almost every $t \in \mathbb{R}$

$$\limsup_{r\to 0} \frac{|B(t+r) - B(t)|}{\sqrt{|r| \log\log |r|^{-1}}} = \sqrt{2} \tag{1}$$

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This means that, almost surely, for almost every t, $h_B(t) = \frac{1}{2}$ but $B \notin C^{\frac{1}{2}}(t)$.

Usual Hölder spaces are **unadapted** to detect precise pointwise behaviours that we wish to put into evidence.

Admissible sequences

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A sequence $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ of real positive numbers is called admissible if there exists a positive constant C such that

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One sets

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

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so that for any $\varepsilon > 0$, there exists $C > 0$ s.t. for all j, k

$$C^{-1}2^{j(\underline{s}(\sigma)-\varepsilon)} \leqslant \frac{\sigma_{j+k}}{\sigma_k} \leqslant C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

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$$\lim_{t \to 0} \frac{\psi(rt)}{\psi(t)} = 1,$$

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Generalized Hölder spaces

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Definition

A modulus of continuity is an increasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\theta(0) = 0$ and for which there is $C > 0$ such that $\theta(2x) \leqslant C\theta(x)$ for all $x \in \mathbb{R}^+$.

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Proposition (D. Kreit and S. Nicolay, 2017)

Let $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence. There exists a modulus of continuity θ such that $\sigma_j = \theta(2^{-j})$ if and only if $\boldsymbol{\sigma}$ is a non-increasing sequence. Moreover, we can choose θ continuous at 0 if and only if $\boldsymbol{\sigma}$ converges to 0.

Wavelet analysis

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the admissibility condition

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|}{|\xi|} \, d\xi < \infty, \tag{2}$$

Wavelet analysis

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Any function $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot - k), \tag{3}$$

where

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The expansion (3) holds true in many function spaces.

Wavelet analysis on Hölder spaces

Characterization of the uniform spaces (D. Kreit and S. Nicolay)

A function $f \in L^\infty(\mathbb{R}^d)$ belongs to the generalized Hölder space $C^{\boldsymbol{\sigma}}(\mathbb{R}^d)$ if and only if there exists $C > 0$ such that, for all $j \in \mathbb{N}$,

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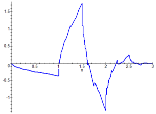
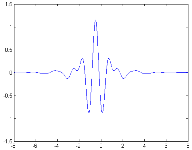
If ψ is a wavelet,





$\psi_{j,k} = 2^j \psi(2^j \cdot - k)$ is "localized" around the dyadic interval

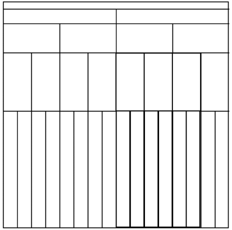
$$\lambda_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$$

Wavelet analysis on Hölder spaces

Definition

The **wavelet leader** of scale j at the point x_0 is the quantity

$$d_j(x_0) = \max_{\lambda \in 3\lambda_j(x_0)} \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|.$$



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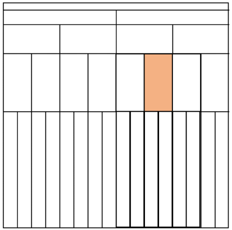

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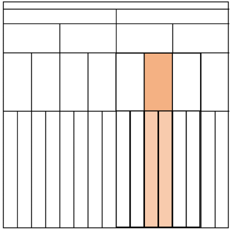

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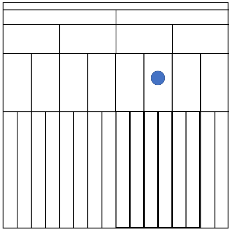

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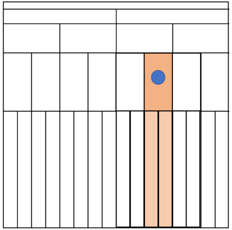

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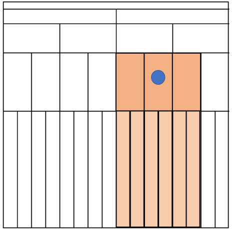

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Nearly-characterization of the pointwise spaces (D. Kreit, L.L. and S. Nicolay)

If f belongs to the space $C^{\boldsymbol{\sigma}}(x_0)$, then there exists $C > 0$ such that, for all $j \in \mathbb{N}$,

$$d_j(x_0) \leqslant C\sigma_j \tag{4}$$

Conversely, if σ_j tends to 0 as j tends to ∞, if f belongs to $C^{\varepsilon}(\mathbb{R}^d)$ for some $\varepsilon > 0$, then (4) implies $f \in C^{\boldsymbol{\sigma}}_{\log}(x_0)$:

$$\|f - f(x_0)\|_{L^\infty(B(x_0,2^{-j}))} \leqslant C\sigma_j |\log(\sigma_j)|.$$

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$\Rightarrow$ One cannot define a proper counterpart to the Lebesgue measure on such spaces.

In the Euclidean space, it is well known that one can associate a probability measure μ to a Borel set B such that $\mu(B + x)$ vanishes for very $x \in \mathbb{R}^n$ if and only if the Lebesgue measure $\mathcal{L}(B)$ of B also vanishes

Prevalence

Prevalence

Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B + x) = 0$ for every $x \in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set.

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Prevalence

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Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B + x) = 0$ for every $x \in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set.

- if E is finite-dimensional, B is Haar-null if and only if $\mathcal{L}(B) = 0$,
- if E is infinite-dimensional, the compact sets of E are Haar-null,
- the translated of a Haar-null set is Haar-null,
- a prevalent set is dense in E,
- the intersection of a countable collection of prevalent sets is still prevalent.

Prevalence of the logarithmic correction

$$E^{\varepsilon}_{\infty}(x_0) = \{f \in C^{\varepsilon}(\mathbb{R}^d) \,:\, (\sigma_j^{-1} d_j(x_0))_j \in \ell^{\infty}\},$$

equipped with the norm

$$\|\cdot\|_{E^{\varepsilon}_{\infty}(x_0)} \,:\, E^{\varepsilon}_{\infty}(x_0) \to [0, +\infty) \,:\, f \mapsto \|f\|_{C^{\varepsilon}(\mathbb{R}^d)} + \|(\sigma_j^{-1} d_j(x_0))_j\|_{\ell^{\infty}}.$$

Prevalence of the logarithmic correction

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L.L. & S. Nicolay (2022)

If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \frac{\underline{s}(\boldsymbol{\sigma})}{4}$, from the the prevalence point of view, almost every function of $E_\infty^\varepsilon(x_0)$ belongs to $C^{\boldsymbol{\sigma}}_{\log}(x_0) \backslash C^{\boldsymbol{\sigma}}_{/_s \log}(x_0)$.

Nevertheless...

C. Esser & L.L. (2022)

1. If ψ is compactly supported then, for all j,

$$|d_j(x_0)| \leqslant C\|f - f(x_0)\|_{L^\infty(B(x_0,R2^{-j}))}$$

   where R is computed from the support of the wavelet and C is a positive constant only depending on the wavelet.

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2. If $\psi \in \mathcal{S}(\mathbb{R})$ and $\boldsymbol{\sigma}$ is an admissible sequence such that

$$\limsup_{j\to+\infty} \frac{\|f - f(x_0)\|_{L^\infty(B(x_0,R2^{-j}))}}{\sigma_j} < \infty$$

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for all γ admissible such that $\gamma_j = o(\sigma_j)$ if $j \to +\infty$.

Regularity of Stochastic Processes

Let $\{X_t\}_{t\in I}$ (I interval in $\mathbb{R}$) be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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Kolmogorov-Censtov

If there exist $\alpha > 0$, $\beta > 0$ and a constant $C > 0$ such that, for all $s, t \in I$,

$$\mathbb{E}[(X_s - X_t)^\alpha] \leqslant C|s-t|^\beta$$

then there is a version of $\{X_t\}_{t\in I}$ whose paths belong to $C^\gamma(x_0)$, for all $x_0 \in I$ and $0 < \gamma < \frac{\beta}{\alpha}$.

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then there is a version of $\{X_t\}_{t\in I}$ whose paths belong to $C^\gamma(x_0)$, for all $x_0 \in I$ and $0 < \gamma < \frac{\beta}{\alpha}$.It means that, almost surely, for all $x_0 \in I$, $h_X(x_0) \geqslant \frac{\beta}{\alpha}$.

Stochastic Processes and Wavelets : How?

How to obtain wavelet coefficients and wavelet (type) expansion for stochastic processes?

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$$\int f(t, \mathbf{s})\, \mu(d\mathbf{s})$$

   where μ is a random measure and f is a deterministic function such that, for all t, $\mathbf{s} \mapsto f(t, \mathbf{s})$ is square-integrable

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   - For all t, one can apply the wavelet expansion to $\mathbf{s} \mapsto f(t,\mathbf{s})$
   - **After some works** we can get random series converging to X involving the wavelet coefficient $c_\lambda^{(t)}$ of $\mathbf{s} \mapsto f(t,\mathbf{s})$ and the random variables $\int \psi_\lambda(s)\,\mu(d\mathbf{s})$.

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3. One can use the information provided by wavelet leaders

Some results concerning FBM...

C. Esser & L.L. (2022)

For all $H \in (0,1)$, there exists an event Ω_H of probability 1 satisfying the following assertions for all $\omega \in \Omega_H$ and every non-empty interval I of $\mathbb{R}$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s \to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log\log|t-s|^{-1}}} < +\infty.$$

- There exists a dense set of *rapid* points $t \in I$ such that

$$0 < \limsup_{s \to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log|t-s|^{-1}}} < +\infty.$$

- There exists a dense set of *slow* points $t \in I$ such that

$$0 < \limsup_{s \to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H} < +\infty.$$

... and Rosenblatt processes

L. Daw & L.L. (2022)

For all $H_1, H_2 \in (\frac{1}{2}, 1)$ such that $H_1 + H_2 > \frac{3}{2}$, there exists an event Ω_{H_1,H_2} of probability 1 satisfying the following assertions for all $\omega \in \Omega_{H_1,H_2}$ and every $I \neq \varnothing$.

- Almost every $t \in I$ is *ordinary*:

$$0 < \limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log\log|t-s|^{-1}} < +\infty.$$

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Comparison of the methods and results

	FBM	Rosenblatt
Representation		
Finiteness of the limits		
Positiveness of the limits		
Results		

Comparison of the methods and results

	FBM	Rosenblatt
Representation	Wavelet series	
Finiteness of the limits		
Positiveness of the limits		
Results		

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-Hj}\xi_{j,k}\psi_{H+1/2}(2^j\cdot-k)+R$$

Comparison of the methods and results

	FBM	Rosenblatt
Representation	Wavelet series	Wavelet-type series
Finiteness of the limits		
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$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_0^t \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

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Representation	Wavelet series	Wavelet-type series
Finiteness of the limits	Find nice upper bounds on the oscillation $\lvert B_H(t,\omega) - B_H(s,\omega)\rvert$	Find nice upper bounds on the oscillation $\lvert R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)\rvert$
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Results		

$$c_{j,k} = \widetilde{c_{j,k}}^{M} + \widetilde{c_{j,k}}^{M}$$

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Positiveness of the limits	Using wavelet leaders directly from the wavelet series	Using wavelet leaders estimated from the stochastic integral
Results		

$$\widetilde{c_{j,k}}^M = c_{H_1,H_2} \int'_{\lambda_{j,k}^M} \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2), \quad \lambda_{j,k}^M := \left] \frac{k-NM}{2^j}, \frac{k+N}{2^j} \right[^2$$

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Positiveness of the limits	Using wavelet leaders directly from the wavelet series	Using wavelet leaders estimated from the stochastic integral
Results	Three types of points	Three types of points

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Positiveness of the limits	Using wavelet leaders directly from the wavelet series	Using wavelet leaders estimated from the stochastic integral
Results	Three types of points with $\sqrt{\log(\cdot)}$ and $\sqrt{\log(\log(\cdot))}$ corrections	Three types of points

Comparison of the methods and results

	FBM	Rosenblatt
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Finiteness of the limits	Find nice upper bounds on the oscillation $\lvert B_H(t,\omega) - B_H(s,\omega)\rvert$	Find nice upper bounds on the oscillation $\lvert R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)\rvert$
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Stochastic processes serving Generalized Hölder spaces

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To show that a property $\mathcal{P}$ holds only on an Haar-null set of a complete metric vector space E, one can exhibit a process X whose sample paths lies in a compact subset of E and such that, for all $f \in E$, almost surely the property $\mathcal{P}$ does not hold for $X + f$.

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C. Esser & L.L. (2022)

Let $(\alpha_j)_{j \in \mathbb{N}}$ be a non-decreasing sequence of $(0, h)$ with tends to h. The subset

$$K = \left\{ f \in C^{\nearrow h} : \max_{k \in \{0,\dots,2^j-1\}} |c_{j,k}| \leqslant 2^{-\alpha_j j} \ \forall j \in \mathbb{N} \right\}$$

is compact in $C^{\nearrow h}$

Stochastic processes serving Generalized Hölder spaces

If $(j_n)_{n\in\mathbb{N}}$ is a sequence satisfying $j_{n+1} > j_n + \lfloor \log_2 j_n^2 \rfloor + 1$ and if we set, set $\alpha_n = h - \frac{1}{\sqrt{j_n}}$ for every $n \in \mathbb{N}$. The random wavelets series

$$X = \sum_{n\in\mathbb{N}} \sum_{j=j_n}^{j_{n+1}-1} \sum_{k=0}^{2^j-1} 2^{-\alpha_n j} \varepsilon_{j,k} \psi_{j,k},$$

where $(\varepsilon_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ is a sequence of independent $\mathcal{U}([-1,1])$ random variables, takes its values in a compact subset of $C^{\nearrow h}$.

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where $(\varepsilon_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ is a sequence of independent $\mathcal{U}([-1,1])$ random variables, takes its values in a compact subset of $C^{\nearrow h}$. For all $f \in C^{\nearrow h}$, the wavelet coefficient of $X + f$ is given by

$$2^{-\alpha_n j} \varepsilon_{j,k} + c_{j,k} = 2^{-\alpha_n j} (\varepsilon_{j,k} + 2^{\alpha_n j} c_{j,k}),$$

where $(c_{j,k})$ are the wavelet coefficients of f. Using Borel-Cantelli Lemma, we show that, for all M,

$$d_{j_n,k} > M 2^{-h j_n} \text{ if } n \text{ is large enough.}$$

Genericity results

C. Esser & L.L. (2022)
Let $h > 0$. The set of functions f such that $f \notin C^h(t)$ for every $t \in [0,1]$ is prevalent in $C^{\nearrow h}$.

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Let $h > 0$. The set of functions f such that $f \notin C^h(t)$ for every $t \in [0,1]$ is Baire-residual in $C^{\nearrow h}$.

Baire-residual means that it contains a countable intersection of dense open sets.

Generalized Regularity and Stochastic Processes

Laurent Loosveldt

Workshop “Regularity of Stochastic Processes”

May 19th 2022

UNIVERSITÉ DU LUXEMBOURG