Points lents, ordinaires et rapides pour les séries gaussiennes d'ondelettes et applications à l'étude de processus stochastiques

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Results of Kahane about Brownian motion

Let B denote the standard Brownian motion on $\mathbb{R}$. The Khinchin law of the iterated logarithm allows to control the behavior of B at a given point, in the sense that for every $t \in \mathbb{R}$, it holds

$$\limsup_{r \to 0} \frac{|B(t+r) - B(t)|}{\sqrt{|r| \log \log |r|^{-1}}} = \sqrt{2} \tag{1}$$

on an event of probability one.

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This contrasts with the uniform Hölder condition obtained by Paul Lévy: almost surely, one has

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There exists exceptional points, called fast points, where the law of the iterated logarithm fails.

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Kahane used the expansion of the Brownian motion in the well-chosen Faber-Schauder system, to propose an easy way to study its regularity and irregularity properties. It allows to recover the law of the iterated logarithm and the estimation of the modulus of continuity of the Brownian motion. Furthermore, Kahane obtained the existence of a third category of points, presenting a slower oscillation:

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there exist points, called slow points, satisfy the condition

$$\limsup_{r \to 0} \frac{|B(t+r) - B(t)|}{\sqrt{|r|}} < +\infty.$$

The Faber-Schauder System

$$\Lambda(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

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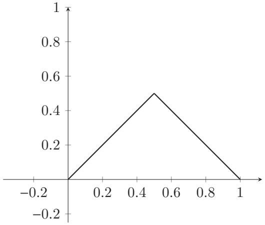

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Let ε and $(\varepsilon_{j,k})_{j\in\mathbb{N}, 0\le k\le 2^j-1}$ be iid $\mathcal{N}(0,1)$ random variables, for all $t \in [0,1]$, set

$$B(t) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \varepsilon_{j,k} 2^{-j/2} \Lambda(2^j t - k) + \varepsilon t.$$

$B = \{B(t) \, : \, t \in [0,1]\}$ is a Brownian motion on $[0,1]$.

The Faber-Schauder System

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is the Haar wavelet.
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Wavelets

Under some general assumptions, there exist two functions ϕ and ψ, called wavelets, which generate two orthonormal bases of $L^2(\mathbb{R})$.Namely

$$\{\phi(\cdot - k)\}_{k\in\mathbb{Z}} \cup \{\psi(2^j \cdot -k) : j \in \mathbb{N}, k \in \mathbb{Z}\}$$

and

$$\{\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}.$$

Wavelets

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Any function $f \in L^2(\mathbb{R})$ can be decomposed as follows,

$$f = \sum_{k\in\mathbb{Z}} C_k \phi(\cdot - k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot - k) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot - k)$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbb{R}^n} f(x)\phi(x-k)\, dx.$$

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- ϕ and ψ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$.
- ϕ and ψ are compactly supported.

Wavelets

One can also work with biorthogonal wavelets bases : a couple of two two Riesz wavelet bases of $L^2(\mathbb{R})$ generated respectively by ψ and $\widetilde{\psi}$ and such that

$$2^{j/2}2^{j'/2}\int_{\mathbb{R}}\psi(2^j x-k)\widetilde{\psi}(2^{j'}x-k')\,dx=\delta_{j,j'}\delta_{k,k'}.$$

In that case, any function $f\in L^2(\mathbb{R})$ can be decomposed as

$$f=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}c_{j,k}\psi(2^j\cdot-k)$$

where

$$c_{j,k}=2^j\int_{\mathbb{R}}f(x)\widetilde{\psi}(2^j x-k)\,dx.$$

Wavelets expansion of Fractional Brownian Motion

Y. Meyer – F. Sellan – M. S. Taqqu (1999)

If ψ is a Lemarié-Meyer wavelet (in the Schwartz class), any fractional Brownian motion B_h of Hurst index $h \in (0,1)$ can be written as

$$B_h(t) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} 2^{-hj} \xi_{j,k} \psi_{h+1/2}(2^j t - k) + R(t) \tag{2}$$

where R is a smooth process, $(\xi_{j,k})_{j \in \mathbb{N}, k \in \mathbb{Z}}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables, and ψ_α is a fractional primitive of ψ. Note that such a function leads to a biorthogonal wavelet basis.

Systematic study of gaussian wavelet series

We consider any function of the form

$$f_h = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \xi_{j,k} 2^{-hj} \psi(2^j \cdot -k) \tag{3}$$

where $(\xi_{j,k})_{(j,k) \in \mathbb{N} \times \mathbb{Z}}$ denote a sequence of independent $\mathcal{N}(0,1)$ random variables and where ψ is any compactly supported or smooth wavelet.

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$$f_h = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \xi_{j,k} 2^{-hj} \psi(2^j \cdot -k) \tag{3}$$

We will study the precise pointwise regularity of f_h with the help of different moduli of continuity: an increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\omega(0) = 0$ and for which there is $C > 0$ such that $\omega(2x) \leq C\omega(x)$ for all $x \in \mathbb{R}^+$.

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$\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\omega(0) = 0$ and for which there is $C > 0$ such that $\omega(2x) \leq C\omega(x)$ for all $x \in \mathbb{R}^+$. Wavelet characterizations of regularity require the following additional regularity property for moduli of continuity:

$$\begin{cases} \sum_{j=J}^{\infty} 2^{Nj}\omega(2^{-j}) \leq C 2^{NJ}\omega(2^{-J}) \\ \sum_{j=-\infty}^{J} 2^{(N+1)j}\omega(2^{-j}) \leq C 2^{(N+1)J}\omega(2^{-J}) \end{cases} \tag{4}$$

Systematic study of gaussian wavelet series

We consider any function of the form

$$f_h = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \xi_{j,k} 2^{-hj} \psi(2^j \cdot -k) \tag{3}$$

- the modulus of continuity ω_r of the rapid points is defined by

$$\omega_r^{(h)}(x) = |x|^h \sqrt{\log |x|^{-1}}$$

- the modulus of continuity ω_o of the ordinary points is defined by

$$\omega_o^{(h)}(x) = |x|^h \sqrt{\log \log |x|^{-1}}$$

- the modulus of continuity ω_s of the slow points is defined by

$$\omega_s^{(h)}(x) = |x|^h.$$

Regularity properties

We want to bound $|f(s) - f(t)|$ for $(s, t) \in (0, 1)$, by the mean of one of the three moduli of continuity.

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$$f_{h,j} := \sum_{k \in \mathbb{Z}} \xi_{j,k} 2^{-hj} \psi(2^j \cdot -k)$$

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3. For all $j \leq \nu$, use mean value theorem to get $x \in (s,t)$ such that
$$|f_{h,j}(t) - f_{h,j}(s)| \leq |t-s||Df_{h,j}(x)| \leq 2^{(1-h)j}|t-s| \sum_{k\in\mathbb{Z}} |\xi_{j,k}||\psi(2^j x - k)|$$

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4. For all $j > \nu$, we bound separately $|f_{h,j}(t)|$ and $|f_{h,j}(s)|$.

Rapid points

A. Ayache – M.S. Taqqu (2003)

There are an event Ω^* of probability 1 and a positive random variable C_2 of finite moment of every order such that, for all $\omega \in \Omega^*$ and $(j,k) \in \mathbb{Z}^2$, the inequality

$$|\xi_{j,k}(\omega)| \leq C_2(\omega)\sqrt{\log(3+|j|+|k|)}$$

holds.

Rapid points

C. Esser – L.L. (2021)

Almost surely, there exists a constant $C_1 > 0$ such that, for all $t, s \in (0,1)$ we have

$$|f_h(s) - f_h(t)| \leq C_1 |t-s|^h \sqrt{\log |t-s|^{-1}}.$$

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Idea of the proof : we use the fast decay of the Wavelet to reduce to bound terms of the form

$$\sum_{|k| \leq 2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(1+|2^j x - k|)^4} + \sum_{|k| > 2^{j+1}} \frac{\sqrt{\log(3+j+|k|)}}{(1+|2^j x - k|)^5}$$

Ordinary points

If $j \in \mathbb{N}_0$, and $t \in (0,1)$, we denote by $k_j(t)$ the unique positive integer in $\{0, \ldots, 2^j - 1\}$ such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$.
Using a reindexation of $\mathbb{N} \times Z$, almost surely, there are an event Ω_t^* of probability 1 and a positive random variable C_t of finite moment of every order such that, for all $\omega \in \Omega^*$ and $(j,k) \in \mathbb{N} \times \mathbb{Z}$,

$$|\xi_{j,k}| \leq C_t \sqrt{\log(3 + j + |k - k_j(t)|)}.$$

Ordinary points

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Almost surely, for almost every $t \in (0,1)$,

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{|t-s|^h \sqrt{\log\log|t-s|^{-1}}} < +\infty.$$

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Idea of the proof: fix t, on Ω_t^*, we reduce to bound terms of the form

$$\sum_{k \in \kappa_j^t(n)} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(1 + |2^j x - k|)^4} + \sum_{k \notin \kappa_j^t(n)} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(1 + |2^j x - k|)^4}$$

where

$$\kappa_j^t(n) = \{k \in \mathbb{Z} \, : \, |k - k_j(t)| \leq n\}$$

for some n well chosen. Conclusion by Fubini Theorem.

Slow points

C. Esser – L.L. (2021)

Almost surely, there exists $t \in (0,1)$ such that

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{|x|^h} < +\infty.$$

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

At distance 0, is the realisation of the associated g.r.v. between μ and 2μ?

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

At distance 2^m is the realisation of the associated g.r.v. between 2μ and 4μ?

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

At distance 2^{lm} is the realisation of the associated g.r.v. between $2^l\mu$ and $2^{l+1}\mu$?

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

If we answer NO at each step, we keep the dyadic interval

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

If we answer YES at one step

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

If we answer YES at one step, we kill the dyadic interval

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

We denote by F_j^μ the union of remaining intervals at step j, we want to show that, almost surely, there exists $\mu \in \mathbb{N}$ such that

$$S_{\text{low}}^\mu(\omega) = \bigcap_j F_j^\mu \neq \emptyset$$

which is equivalent to the fact that, for all $J \in \mathbb{N}_0$,

$$S_{\text{low},J}^\mu(\omega) = \bigcap_{j \leq J} F_j^\mu$$

is non-empty.

Slow points – Kahane procedure

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We denote by $N_J^\mu(\omega)$ the number of subintervals of $S^\mu_{\text{low},J}$, we want to show that

$$\mathbb{P}(\bigcup_\mu \bigcap_J (N_J^\mu \geq 1)) = 1.$$

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ fixed (for now)

If $N_J^{\mu} = N$, deciding wether erasing a given interval because of an interval at distance between 2^{ml} and $2^{m(l+1)}$ is a Bernoulli trial of parameter

$$p_l(\mu) = \mathbb{P}(2^l \mu < |\xi| \leq 2^{l+1}\mu).$$

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If $N_J^\mu = N$, deciding wether erasing a given interval because of an interval at distance between $2^l\mu$ and $2^{l+1}\mu$ is a Bernoulli trial of parameter

$$p_l(\mu) = \mathbb{P}(2^l\mu < |\xi| \leq 2^{l+1}\mu).$$

By Tchebycheff's inequality applied on $\mathcal{B}(2N, p_l(\mu))$ laws, this will remove at most

$$2N(2^{ml+1}+1)(p_l(\mu) + l\sqrt{p_l(\mu)(1-p_l(\mu))})$$

intervals with probability greater than $1 - N^{-1}$.

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intervals with probability greater than $1 - N^{-1}$. In total, we remove at most

$$2N\sum_{l=0}^{+\infty}(2^{ml+1} + 1)\left(p_l(\mu) + l\sqrt{p_l(\mu)(1 - p_l(\mu))}\right)$$

intervals with probability greater than $1 - N^{-1}$.

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ great enough.

$$\mathbb{P}(N^{\mu}_{J+1} \geq \frac{3}{2} N_J | N^{\mu}_J = N) \geq 1 - N^{-1}$$

Slow points – Kahane procedure

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$$
\begin{aligned}
\mathbb{P}(N^\mu_{J+1} \geq (\frac{3}{2})^{J+1}) &\geq \mathbb{P}((N^\mu_{J+1} \geq \frac{3}{2} N^\mu_J) \cap (N^\mu_J \geq (\frac{3}{2})^J)) \\
&= \sum_{N \geq (\frac{3}{2})^J} \mathbb{P}(N^\mu_{J+1} \geq \frac{3}{2} N_J | N^\mu_J = N)\mathbb{P}(N^\mu_J = N) \\
&\geq \sum_{N \geq (\frac{3}{2})^J} (1 - N^{-1})\mathbb{P}(N^\mu_J = N) \\
&\geq (1 - (\frac{2}{3})^J) \sum_{N \geq (\frac{3}{2})^J} \mathbb{P}(N^\mu_J = N) \\
&= (1 - (\frac{2}{3})^J)\mathbb{P}(N^\mu_J \geq (\frac{3}{2})^J).
\end{aligned}
$$

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ great enough.

For all $J \geq J_1$

$$\mathbb{P}(N_J^\mu \geq 1) \geq \mathbb{P}(N_J^\mu \geq (\frac{3}{2})^J)$$
$$\geq \mathbb{P}(N_{J_1}^\mu \geq (\frac{3}{2})^{J_1}) \left(\prod_{j=J_1}^{J} (1 - (\frac{2}{3})^j) \right).$$

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m such that $\frac{1}{m} < h$, μ great enough.

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$$\mathbb{P}(N_J^\mu \geq 1) \geq (\frac{3}{2})^{J_1}) \left(\prod_{j=J_1}^{J} (1 - (\frac{2}{3})^j) \right).$$

For all $\varepsilon > 0$, as

$$\lim_{J_1 \to +\infty} \prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) = 1,$$

one can choose J_1 such that

$$\prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) > 1 - \varepsilon.$$

Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ great enough.

For all $\varepsilon > 0$, J_1 great enough and $J \geq J_1$

$$\mathbb{P}(N_J^\mu \geq 1) \geq (\frac{3}{2})^{J_1}) \left(\prod_{j=J_1}^{J} (1 - (\frac{2}{3})^j) \right) \quad \prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) > 1 - \varepsilon.$$

By increasing μ if necessary, we can choose to remove the intervals $[0, 2^{-J_1}]$ and $[1 - 2^{-J_1}, 1]$ from S^μ_{low, J_1}, if necessary, and assume

$$\mathbb{P}(N_{J_1}^\mu \geq (\frac{3}{2})^{J_1}) > 1 - \varepsilon.$$

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m such that $\frac{1}{m} < h$, μ great enough.

For all $J \geq J_1$ For all $\varepsilon > 0$, J_1 great enough and $J \geq J_1$

$$\mathbb{P}(N_J^\mu \geq 1) \geq (\frac{3}{2})^{J_1}) \left(\prod_{j=J_1}^{J} (1 - (\frac{2}{3})^j) \right) \quad \prod_{j=J_1}^{+\infty} (1 - (\frac{2}{3})^j) > 1 - \varepsilon.$$

and if μ is large enough

$$\mathbb{P}(N_{J_1}^\mu \geq (\frac{3}{2})^{J_1}) > 1 - \varepsilon$$

and thus

$$\mathbb{P}(\bigcap_{J \in \mathbb{N}_0} (N_J^\mu \geq 1)) \geq \mathbb{P}(N_{J_1}^\mu \geq (\frac{3}{2})^{J_1}) \left(\prod_{j=J_1}^{\infty} (1 - (\frac{2}{3})^j) \right) > (1 - \varepsilon)^2.$$

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Slow points – Kahane procedure

m such that $\frac{1}{m} < h$, μ great enough.

In total, we showed that, for all $0 < \varepsilon < \frac{1}{2}$,

$$\mathbb{P}\Big(\bigcup_{\mu \in \mathbb{N}} \bigcap_{J \in \mathbb{N}_0} (N_J^\mu \geq 1)\Big) > (1 - \varepsilon)^2.$$

Slow points

C. Esser – L.L. (2021)
Almost surely, there exists $t \in (0, 1)$ such that

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{|x|^h} < +\infty.$$

Idea of the proof :

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$$\Lambda_j^0(t) = \{0 \leq k < 2^j \ : \ |k_j(t) - k| \leq 1\}$$

and, for all $1 \leq l$

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$$\sum_{l=0}^{+\infty} \sum_{k \in \Lambda^l(t)} |\xi_{j,k}| \frac{1}{(3 + |2^j x - k|)^4}$$

Irregularity properties

C. Esser – L.L. (2021)

1. If $\psi \in \mathcal{S}(\mathbb{R})$ and ω is a modulus of continuity such that

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} < \infty$$

then

$$\limsup_{j \to +\infty} \frac{|c_{j,k_j(t)}|}{\omega(2^{-j})} < \infty$$

2. If ψ is compactly supported then, for all j,

$$|c_{j,k_j(t)}| \leq C \sup_{x \in B(t, R2^{-j})} |f_h(x) - f_h(t)|$$

where R is computed from the support of the wavelet and C is a positive deterministic constant.

Behaviours of i.i.d $\mathcal{N}(0,1)$ random variables

C. Esser – L.L. (2021)

1. Almost surely, for every $t \in \mathbb{R}$, one has
$$\limsup_{j\to+\infty} |\xi_{j,k_j(t)}| \geq 2^{-3/2}\sqrt{\pi}.$$
2. Almost surely, for almost every $t \in \mathbb{R}$, one has
$$\limsup_{j\to+\infty} \frac{|\xi_{j,k_j(t)}|}{\sqrt{\log j}} > 0\,.$$
3. Almost surely, for every non-empty open interval I of $\mathbb{R}$, there is $t \in I$ such that
$$\limsup_{j\to+\infty} \left\{\frac{|\xi_{j,k_j(t)}|}{\sqrt{j}}\right\} > 0\,.$$

Slow, ordinary and rapid points for Gaussian Wavelets Series

C. Esser – L.L. (2021)

Let I denote any non-empty interval of $\mathbb{R}$. Almost surely, the random wavelets series f_h satisfies the following property:

1. For almost every $t \in I$,

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{\omega_o^{(h)}(|s-t|)} < +\infty \tag{4}$$

and if ω is a modulus of continuity such that $\omega = o(\omega_o^{(h)})$, then

$$\limsup_{s \to t} \frac{|f_h(s) - f_h(t)|}{\omega(|s-t|)} = +\infty, \tag{5}$$

Such points are called *ordinary points*.

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Such points are called *slow points*.

Slow points are exceptional !

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If $\alpha \in (0,1)$, a function f belongs to Hölder space of order α on $[0,1]$, $C^\alpha([0,1])$ if, there exists a constant $C > 0$ such that, for all $s, t \in [0,1]$,

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Remark: $C^{\nearrow h}$ is a Fréchet space.

Two commonly used notions of genericity

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2. In the sense supplied by the Baire category theorem. Let us recall that a subset A of a Baire space X is of first category (or meagre) if it is included in a countable union of closed sets of X with empty interior. The complement of a set of first category is Baire-residual; it contains a countable union of dense open sets of X.

Genericity of slow points

Let $0 < h < 1$ and $t \in [0,1]$, a function f belongs to the pointwise Hölder space of order h at t, $C^h(t)$ if there exists $R > 0$ and $C > 0$ such that, for all $s \in B(t,R)$,

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C. Esser – L.L. (2021)
Let $h > 0$. The set of functions f such that $f \notin C^h(t)$ for every $t \in [0,1]$ is prevalent in $C^{\nearrow h}$.

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Extension 1 : multifractal case

If $h < h'$, $C^{h'}(t) \subseteq C^h(t)$ so one can define the pointwise Hölder of a locally bounded function f at t by

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When the function $t \mapsto h_f(t)$ is not constant we say that f is multifractal.

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$$|H(x) - H(y)| \leq \frac{C_H}{|\log|x-y||}. \tag{6}$$

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If H is the Hölder function of a continuous function then there exists $(P_j)_j$ sequence of polynomials such that

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If $\exists C > 0$ s.t., for all t, $|H(t) - P_j(t)| \leq Cj^{-1}$, we have $\log$-regularity condition.

A multifractal process

With such a function, we define the multifractal random wavelets serie

$$f_H = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \xi_{j,k} 2^{-H(k2^{-j})j} \psi(2^j \cdot -k).$$

A multifractal process

C. Esser – L.L. (2021)

Let I denote any non-empty interval of $\mathbb{R}$. If $H : \mathbb{R} \to K$ satisfies the $\log$-regularity condition, almost surely, the multifractal random wavelets series f_H satisfies the following property:

1. For almost every $t \in I$,

$$\limsup_{s \to t} \frac{|f_H(s) - f_H(t)|}{\omega_o^{(H(t))}(|s-t|)} < +\infty \tag{7}$$

and if ω is a modulus of continuity such that $\omega = o(\omega_o^{(H(t))})$, then

$$\limsup_{s \to t} \frac{|f_H(s) - f_H(t)|}{\omega(|s-t|)} = +\infty, \tag{8}$$

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A multifractal process

This $\log$ regularity condition comes from the fact that, in the proof, we need to deal with terms of the form

$$|H(t) - H(k2^{-j})|$$

where $k2^{-j}$ is a dyadic number close to t.

Extension II : the Rosenblatt process (WIP)

The (generalized) fractional Rosenblatt motion is a real-valued non-Gaussian self-similar process with stationnary increments which belongs to the second order Wiener chaos.

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If $(H_1, H_2) \in (1/2, 1)$ and $H_1 + H_2 > \frac{3}{2}$, $\{R_{H_1,H_2}(t)\}_{t\in\mathbb{R}^+}$ is defined by

$$R_{H_1,H_2}(t) = \int_{R^2}' K_{H_1,H_2}(t, x_1, x_2)\, dB(x_1)\, dB(x_2)$$

where, for all $(t, x_1, x_2) \in R^+ \times \mathbb{R}^2$

$$K_{H_1,H_2}(t, x_1, x_2) = \frac{1}{\Gamma(H_1 - 1/2)\Gamma(H_2 - 1/2)} \int_0^t (s - x_1)_+^{H_1 - 3/2} (s - x_2)_+^{H_2 - 3/2}\, ds.$$

A wavelet type expansion for the gfRm

Ayache – Esmili (2020)

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_0^t \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

where, if, $(j,k)\in\mathbb{Z}^2$

$$\varepsilon_{j,k} = 2^{j/2}\int_{\mathbb{R}}\psi(2^j x-k)\,dB(x)$$

is a $\mathcal{N}(0,1)$ random variables and $(\varepsilon_{j,k})_{j,k}$ is a sequence of independent random variables and, if $j_1\neq j_2$ or $k_1\neq k_2$

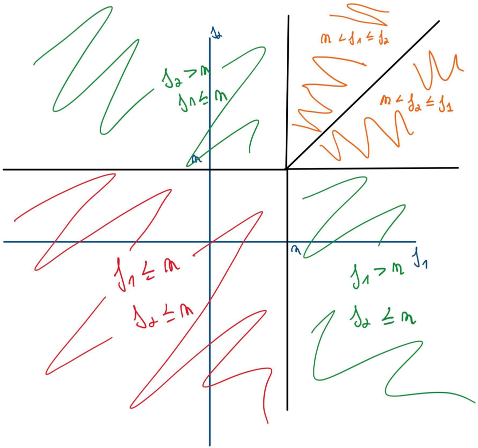
$$\varepsilon_{j_1,j_2}^{k_1,k_2} = \varepsilon_{j_1,k_1}\varepsilon_{j_2,k_2}$$

and

$$\varepsilon_{j,j}^{k,k} = (\varepsilon_{j,k})^2 - 1.$$

Methods for the regularity

If n is such that $2^{-n-1} < |t-s| \le 2^{-n}$



Methods for the regularity

If n is such that $2^{-n-1} < |t-s| \leq 2^{-n}$
If $j_1 \leq n$ and $j_2 \leq n$, we can use mean value thm and turn to bound

$$\sum_{k_1,k_2} \varepsilon_{j_1,j_2}^{k_1,k_2} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)$$

which easily reduces to what have been done for the FBM.

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Methods for the regularity

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If $j_1 \leq n$ and $j_2 > n$ or $n < j_1 \leq j_2$ we need to make appear some postive powers of 2^{-j_2} so that the sum over j_2 is finite. We have to bound the sums

$$\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \varepsilon_{j_1, j_2}^{k_1, k_2} \int_t^s \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx$$

and consider some cases

1. we first deal with the sums over $k_2 \leq 2^{j_2} t$ and $k_2 > 2^{j_2} s$ which easily reduce to the FBM situation by bounding the sum of k_1 directly in the integral and the sum aver k_2 after integration.

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$$2^{j_1(1-H_1)}2^{j_2(1-H_2)}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_t^s \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

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3. The integrals $\int_{-\infty}^t$ and $\int_s^{+\infty}$ can, again, be reduced to the FBM situation.

The last integrals

It remains us to consider

$$2^{j_1(1-H_1)}2^{j_2(1-H_2)}\sum_{k_1\in\mathbb{Z}}\sum_{2^{j_2}t\le k_2\le 2^{j_2}s}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_{\mathbb{R}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

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but,

$$F_{j_1,j_2}^{k_1,k_2}=\int_{\mathbb{R}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\neq 0$$

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if and only if $|j_1-j_2|\le 1$ and, in this case, for all $L\in\mathbb{N}_0$

- $|F_{j+1,j}^{k_1,k_2}|\le C_L 2^{-j}(3+|k_1-2k_2|)^{-L}$
- $|F_{j,j}^{k_1,k_2}|\le C_L 2^{-j}(3+|k_1-k_2|)^{-L}$
- $|F_{j,j+1}^{k_1,k_2}|\le C_L 2^{-j}(3+|2k_1-k_2|)^{-L}$

The last integrals

It remains us to consider

$$2^{j_1(1-H_1)}2^{j_2(1-H_2)}\sum_{k_1\in\mathbb{Z}}\sum_{2^{j_2}t\leq k_2\leq 2^{j_2}s}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_{\mathbb{R}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

but,

$$F_{j_1,j_2}^{k_1,k_2}=\int_{\mathbb{R}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\neq 0$$

if and only if $|j_1-j_2|\leq 1$ and, in this case, for all $L\in\mathbb{N}_0$

- $|F_{j+1,j}^{k_1,k_2}|\leq C_L 2^{-j}(3+|k_1-2k_2|)^{-L}$
- $|F_{j,j}^{k_1,k_2}|\leq C_L 2^{-j}(3+|k_1-k_2|)^{-L}$
- $|F_{j,j+1}^{k_1,k_2}|\leq C_L 2^{-j}(3+|2k_1-k_2|)^{-L}$

This very nice fast decay property helps us to reduce the sum over $k_1<2^{j_1}t$ and $k_1>2^{j_1}s$ to the FBM case.

The very last integral!!

It remains us to bound

$$2^{j_1(1-H_1)}2^{j_2(1-H_2)}\sum_{2^{j_1}t\le k_1\le 2^{j_1}s}\;\sum_{2^{j_2}t\le k_2\le 2^{j_2}s}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_{\mathbb{R}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx$$

with $|j_1-j_2|\le 1$.

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with $|j_1-j_2|\le 1$.
Therefore, we consider the random variables

$${}^{1}_{j}\sum\nolimits^{k^{(2)},K^{(2)}}_{k^{(1)},K^{(1)}}=\sum_{k^{(1)}\le k_1\le K^{(1)}}\ \sum_{k^{(2)}\le k_1\le K^{(2)}}\varepsilon^{k_1,k_2}_{j+1,j}F^{k_1,k_2}_{j+1,j}$$

for all $j\ge n$, $(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S^1_j(\lambda)$, for $\lambda\in 3\lambda_n(t)$ where

$$S^1_j(\lambda)=\left\{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in\mathbb{Z}^4\ :\ \frac{k^{(1)}}{2^{j+1}},\frac{K^{(1)}}{2^{j+1}},\frac{k^{(2)}}{2^{j}},\frac{K^{(2)}}{2^{j}}\in\lambda\right\}$$

The very last integral!!

It remains us to bound

$$2^{j_1(1-H_1)}2^{j_2(1-H_2)} \sum_{2^{j_1}t \le k_1 \le 2^{j_1}s} \sum_{2^{j_2}t \le k_2 \le 2^{j_2}s} \varepsilon_{j_1,j_2}^{k_1,k_2} \int_{\mathbb{R}} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx$$

with $|j_1 - j_2| \le 1$.

$$ {}^0_j\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} = \sum_{k^{(1)} \le k_1 \le K^{(1)}} \sum_{k^{(2)} \le k_1 \le K^{(2)}} \varepsilon_{j,j}^{k_1,k_2} F_{j,j}^{k_1,k_2}$$

for all $j \ge n$, $(k^{(1)}, K^{(1)}, k^{(2)}, K^{(2)}) \in S_j^0(\lambda)$, for $\lambda \in 3\lambda_n(t)$ where

$$S_j^0(\lambda) = \left\{ (k^{(1)}, K^{(1)}, k^{(2)}, K^{(2)}) \in \mathbb{Z}^4 \; : \; \frac{k^{(1)}}{2^j}, \frac{K^{(1)}}{2^j}, \frac{k^{(2)}}{2^j}, \frac{K^{(2)}}{2^j} \in \lambda \right\}$$

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with $|j_1-j_2|\le 1$.

$$\,^{2}_{j}\sum\nolimits^{k^{(2)},K^{(2)}}_{k^{(1)},K^{(1)}}=\sum_{k^{(1)}\le k_1\le K^{(1)}}\ \sum_{k^{(2)}\le k_1\le K^{(2)}}\varepsilon^{k_1,k_2}_{j,j+1}F^{k_1,k_2}_{j,j+1}$$

for all $j\ge n$,$(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S^2_j(\lambda)$, for $\lambda\in 3\lambda_n(t)$ where

$$S^2_j(\lambda)=\left\{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in\mathbb{Z}^4\ :\ \frac{k^{(1)}}{2^j},\frac{K^{(1)}}{2^j},\frac{k^{(2)}}{2^{j+1}},\frac{K^{(2)}}{2^{j+1}}\in\lambda\right\}$$

Regularity property

For the rapid points we want an uniform bound so we consider the events

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$$A_n = \left\{ \forall \lambda \in \Lambda_n, \sup_{j \geq n} \max_{\ell \in \{0,1,2\}} \sup_{(k^{(1)},K^{(1)},k^{(2)},K^{(2)}) \in S_j^{\ell}(\lambda)} \frac{\left| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right|}{\left\| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2}} \leq \kappa(j-n+1)n \right\}$$

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$$\begin{aligned} \mathbb{P}(A_n) &\leq C 2^n \exp(-\kappa \widehat{C} n) \sum_{j \geq n} 2^{4(j-n)} \exp(-\kappa \widehat{C}(j-n)) \\ &\leq C' 2^n \exp(-\kappa \widehat{C} n) \end{aligned}$$

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By Borel-Cantelli, on an event of probability 1, one can bound the sum over $j \geq n$ of our last integrals by

$$C_1 \sum_{j \geq n} 2^{j(\frac{3}{2}-H_1-H_2)}(j-n+1)2^{-\frac{n}{2}} n \leq C_2 2^{n(1-H_1-H_2)} n \leq c_3 |t-s|^{H_1+H_2-1} |\log|t-s||.$$

Regularity property

L. Daw – L.L. (2021)
Almost surely, there exists a constant $C_1 > 0$ such that, for all $t, s \in (0,1)$ we have

$$\limsup_{s\to t} |R_{H_1,H_2}(s) - R_{H_1,H_2}(t)| \le C_1 |t-s|^{H_1+H_2-1} \log|t-s|^{-1}.$$

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For the slow points, we want to kill intervals so, in the Kahane procedure, we also kill a dyadic interval λ at scale n if it does not satisfy the condition

$$\sup_{j\geq n} \max_{\ell\in\{0,1,2\}} \sup_{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S_j^\ell(\lambda)} \frac{\left| {}^\ell_j \sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right|}{\left\| {}^\ell_j \sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2}} \leq (j-n+1)\mu$$

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On an event of probability 1, one can found a slow point and bound the sum over $j\geq n$ of our last integrals by

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Irregularity : creating independence

Now we work with a compactly supported wavelet ψ, with support in $(-N, N)$.

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$$c_{j,k} = 2^j \int_{\mathbb{R}} \psi(2^j x - k) R_{H_1,H_2}(x)\, dx$$

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$$c_{j,k} = \int_{-N}^{N} \psi(x) \left(R_{H_1,H_2}\left(\frac{x+k}{2^j}\right) - R_{H_1,H_2}\left(\frac{k}{2^j}\right) \right) \, dx$$

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$$c_{j,k} = C_{H_1,H_2} \int_{-N}^{N} \psi(x) \left(\int_{\mathbb{R}^2}' \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} (s - x_1)_+^{H_1 - 3/2} (s - x_2)_+^{H_2 - 3/2} \, ds dB(x_1) dB(x_2) \right) dx$$

with $C_{H_1,H_2} = \frac{1}{\Gamma(H_1 - 1/2)\Gamma(H_2 - 1/2)}$

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$$c_{j,k} = C_{H_1,H_2} \int'_{]-\infty, \frac{k+N}{2^j}]^2} \int_{-N}^{N} \psi(x) \left(\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} (s - x_1)_+^{H_1 - 3/2} (s - x_2)_+^{H_2 - 3/2} ds \right) dx \, dB(x_1) dB(x_2)$$

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We fix $M \in \mathbb{N}$ and set $c_{j,k} = \widetilde{c_{j,k}^M} + \widehat{c_{j,k}^M}$ where, if $A_1 =]\frac{k-NM}{2^j}, \frac{k+N}{2^j}]^2$ and $A_2 =]-\infty, \frac{k+N}{2^j}]^2 \setminus]\frac{k-NM}{2^j}, \frac{k+N}{2^j}]^2$,

$$\widetilde{c_{j,k}^M} = C_{H_1,H_2} \int'_{A_1} \int_{-N}^{N} \psi(x) \left(\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} (s-x_1)_+^{H_1-3/2} (s-x_2)_+^{H_2-3/2} ds \right) dx \, dB(x_1) dB(x_2)$$

and

$$\widehat{c_{j,k}^M} = C_{H_1,H_2} \int'_{A_2} \int_{-N}^{N} \psi(x) \left(\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} (s-x_1)_+^{H_1-3/2} (s-x_2)_+^{H_2-3/2} ds \right) dx \, dB(x_1) dB(x_2)$$

Irregularity : creating independence

Now we work with a compactly supported wavelet ψ, with support in $(-N, N)$. The sequence

$$\frac{\widetilde{c_{j,k}^M}}{\|\widetilde{c_{j,k}^M}\|_{L^2}}$$

is identically distributed and, as soon as $(\frac{k-NM}{2^j}, \frac{k+N}{2^j}) \cap (\frac{k'-NM}{2^{j'}}, \frac{k'+N}{2^{j'}})$ the associated coefficients are independent.

$$\|\widetilde{c_{j,k}^M}\|_{L^2} \geq C_{\psi,H_1,H_2} 2^{-j(H_1+H_2-1)}$$

and

$$\|\widehat{c_{j,k}^M}\|_{L^2} \leq C'_{\psi,H_1,H_2} 2^{-j(H_1+H_2-1)} M^{\max\{H_1,H_2\}-1}$$

Points lents, ordinaires et rapides pour les séries gaussiennes d'ondelettes et applications à l'étude de processus stochastiques

Laurent Loosveldt

Séminaire Cristollien d'Analyse Multifractale

2 décembre 2021

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