Monsters, Inc.

An overview on continuous but nowhere differentiable functions

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Université de Liège – FNRS Grantee

ComPlane: the next generation

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LIÈGE université
Mathematics

A bit of history

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Ampère (1806)
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“Proof”: It is “intuitively evident” that a continuous curve must have sections on which it is either increasing, decreasing or constant.

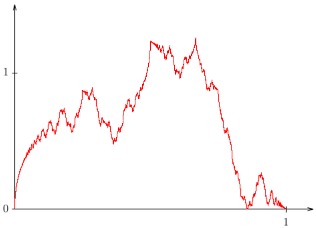
A bit of history

Riemann (1861)

The function

$$\mathcal{R} : x \mapsto \sum_{j=1}^{+\infty} \frac{\sin(j^2 \pi x)}{j^2}$$

is only differentiable at the points of the form $\frac{p}{q}$ where p and q are odd integers.



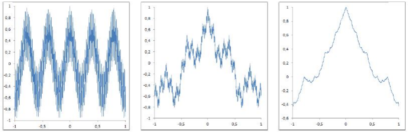
A bit of history

Weierstraß (1872) – completed by Hardy (1916)

$a \in]0,1[$, $b > 1$ and $ab > 1$, the function

$$\mathcal{W}_{a,b} : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{j=0}^{+\infty} a^j \cos(b^j \pi x)$$

is continuous but nowhere differentiable.



A bit of history

> **Poincaré**
> “Such functions are monsters”
>
> “In the past, when a new function was invented, it was with practical perspectives; nowadays, they are invented on purpose to show our ancestors’ reasoning is at fault, and we shall never get anything more out of them”

Pointwise Hölder spaces

The r-oscillation of f at x_0 is the quantity

$$\mathrm{osc}_r(x_0) = \mathrm{osc}_r(x_0; f) = \mathrm{diam} f(rB + x_0) \tag{1}$$

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Let $f : \mathbb{R}^n \to \mathbb{R}^{n'}$ be a locally bounded function, $x_0 \in \mathbb{R}^n$ and $0 < \alpha \leq 1$; $f \in \Lambda^\alpha(x_0)$ if there exist $R, C > 0$ such that

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Let $x_0 \in \mathbb{R}^n$ and $\alpha > 0$; a function $f : \mathbb{R}^n \to \mathbb{R}^{n'}$ belongs to $I^\alpha(x_0)$ if there exists $C, R > 0$ such that

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$$x \in RB + x_0 \Rightarrow |f(x_0) - f(x)| \leq C|x - x_0|^\alpha, \quad (2)$$

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Pointwise Hölder exponents

It is straightforward to show that as soon as $\alpha < \beta$, we have the embeddings $\Lambda^{\beta}(x_0) \subseteq \Lambda^{\alpha}(x_0)$ and $I^{\alpha}(x_0) \subseteq I^{\beta}(x_0)$.

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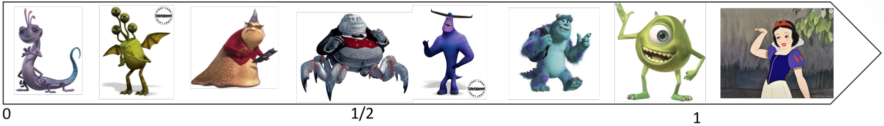
A function f will be called strongly monoHölder of exponent α ($f \in SM^{\alpha}$) if there exists $\alpha \geq 0$ such that $h_f(x_0) = \overline{h}_f(x_0) = \alpha$ for any x_0.

A tool to classify the monsters



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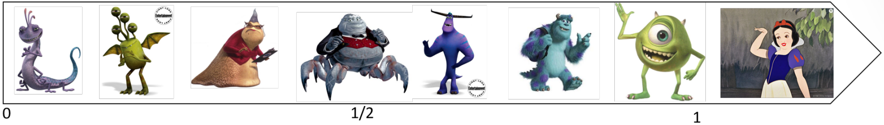
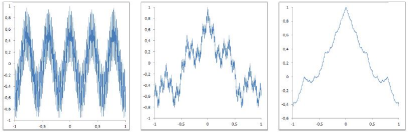


Hardy

If $a \in]0,1[$, $b > 1$ and $ab > 1$, for all $x_0 \in \mathbb{R}$,

$$h_{\mathcal{W}_{a,b}} = -\frac{\log(a)}{\log(b)}$$



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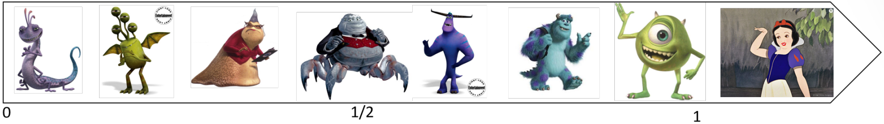
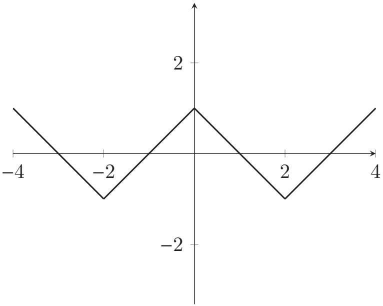


Let g be the function of period 4 defined on $[-2, 2]$ by $g(x) = 1 - |x|$



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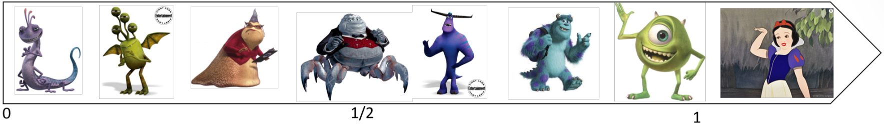

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Jaffard-Nicolay (2009)

If $b > a > 1$ the function

$$x \mapsto \sum_{j=1}^{\infty} a^{-j} g(b^j x)$$

is strongly monoHölder of exponent $-\frac{\log(a)}{\log(b)}$.

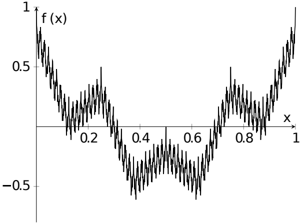
The McCarthy function

McCarthy (1953)

The function

$$x \mapsto \sum_{j=1}^{\infty} 2^{-j} g(2^{2^j} x)$$

is continuous but nowhere differentiable.



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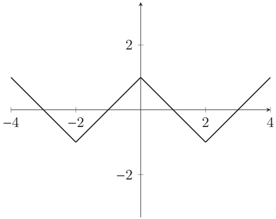
Fix $x_0 \in \mathbb{R}$. For all $k \in \mathbb{N}$, let $h_k \in \mathbb{R}$ be such that $|h_k| = 2^{-2^k}$ and the sign of h_k is chosen such that $2^{2^k} x_0$ and $2^{2^k}(x_0 + h_k)$ are in the same interval of the form $[0 + 2m, 2 + 2m]$ (for some $m \in \mathbb{Z}$).

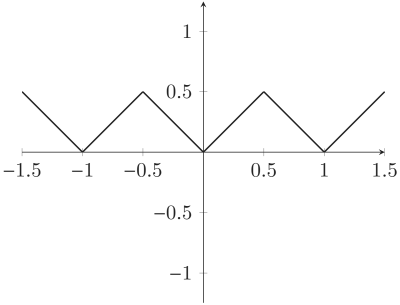


$$|\mathcal{M}(x_0 + h_k) - \mathcal{M}(x_0)| \geq 2^{-k} - \frac{(k-1)}{2} 2^{-2^{k-1}}.$$

The Tagaki function

Let

$$\phi : \mathbb{R} \to [0, \frac{1}{2}] : x \mapsto \mathsf{dist}(x, \mathbb{Z})$$

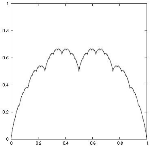


The Tagaki function

Loosveldt-Nicolay (2021)
The Takagi function

$$x \mapsto \sum_{j=0}^{+\infty} 2^{-j}\phi(2^j x)$$

is strongly monoHölder of exponent 1.



Typical continuous function

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Prevalence

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If μ is a Borel measure on an infinite dimensional real normed vector space which is translation invariant and finite on bounded Borel sets, then it is necessarily null.
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$\Rightarrow$ One cannot define a proper counterpart to the Lebesgue measure on such spaces.

In the Euclidean space, it is well known that one can associate a probability measure μ to a Borel set B such that $\mu(B + x)$ vanishes for very $x \in \mathbb{R}^n$ if and only if the Lebesgue measure $\mathcal{L}(B)$ of B also vanishes

Prevalence

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Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B+x)=0$ for every $x \in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set.

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- the intersection of a countable collection of prevalent sets is still prevalent.

Prevalence and continuous functions

Loosveldt-Nicolay (2021)
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For all m, $\{a\mathcal{M} \;:\; a \in \mathbb{R}\}$ is a probe for $\Lambda^{1/m}(x_0)$

$$\{f \text{ continuous } : h_f(x_0) \neq 0\} \subseteq \bigcup_{m \in \mathbb{N}} \Lambda^{1/m}(x_0).$$

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Hunt (1994)
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Limitations of Hölder spaces

If T is the Tagaki function then there exists a constant $C > 0$ such that, for every $x, y \in \mathbb{R}$,

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A function $f \in \Lambda^{\alpha}(x_0)$ if there exist $R, C > 0$ and a polynomial P_{x_0} of degree less than α such that for all $r \leq R$

$$\sup_{x \in B(x_0, r)} |f(x) - P_{x_0}(x)| \leq Cr^{\alpha}.$$

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A function $f \in \Lambda^{\alpha}(x_0)$ if there exist J, $C > 0$ and a polynomial P_{x_0} of degree less than α such that for all $j \geq J$

$$\sup_{x \in B(x_0, 2^{-j})} |f(x) - P_{x_0}(x)| \leq C 2^{-\alpha j}.$$

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A function $f \in \Lambda^\alpha(x_0)$ if

$$(2^{\alpha j} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h^{[\alpha]}(x_0, 2^{-j}))})_j \in \ell^\infty.$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} = \Delta_h^1 \Delta_h^n f(x),$$

and $B_h^M(x_0, 2^{-j}) = \{x : [x_0, x_0 + (M+1)h] \subset B(x_0, 2^{-j})\}$.

Admissible sequences

A sequence $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ of real positive numbers is called admissible if there exists a positive constant C such that

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One sets

$$\underline{\sigma}_j := \inf_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

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so that for any $\varepsilon > 0$, there exists $C > 0$ s.t. for all j, k

$$C^{-1}2^{j(\underline{s}(\sigma)-\varepsilon)} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

Admissible sequences

Example

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$$\lim_{t \to 0} \frac{\psi(rt)}{\psi(t)} = 1,$$

for any $r > 0$.

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Example

If ψ is a slowly varying function and $u \in \mathbb{R}$, the sequence $\boldsymbol{\sigma} = (2^{ju}\psi(2^j))_j$ is admissible with $\underline{s}(\boldsymbol{\sigma}) = \overline{s}(\boldsymbol{\sigma}) = u$.

Spaces of generalized smoothness

First step

Let $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > 0$, $f \in L^\infty_{\text{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{\infty,\infty}(x_0)$ whenever

$$(\sigma_j \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^\infty(B_h(x_0,2^{-j}))})_j \in \ell^\infty.$$

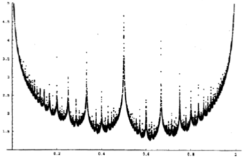
Spaces of generalized smoothness

First step

Let $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > 0$, $f \in L^{\infty}_{\text{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{\infty,\infty}(x_0)$ whenever

$$\Big(\sigma_j \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^{\infty}(B_h(x_0, 2^{-j}))}\Big)_j \in \ell^{\infty}.$$

One can also consider non-locally bounded function such as the Brjuno function.



Spaces of generalized smoothness

Generalized Hölder spaces

Let $p \in [1, \infty]$ and $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,\infty}(x_0)$ whenever

$$(\sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))})_j \in \ell^\infty .$$

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Wavelets

Under some general assumptions, there exist a real-valued function ϕ and $2^d - 1$ real-valued functions $(\psi^{(i)})_{1 \le i < 2d}$ defined on $\mathbb{R}^d$, called wavelets, such that

$$\{\phi(\cdot - k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j \cdot -k) : 1 \le i < 2^d,\ k \in \mathbb{Z}^d,\ j \in \mathbb{N}_0\}$$

form an orthogonal basis of $L^2(\mathbb{R}^d)$.

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form an orthogonal basis of $L^2(\mathbb{R}^d)$.
Any function $f \in L^2(\mathbb{R}^d)$ can be decomposed as

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{i=1}^{2^d - 1} c_{j,k}^{(i)} \psi(2^j x - k).$$

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A dyadic cube at scale j is a cube of the form

$$\lambda = \frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^{j+1}})^d \qquad (j \in \mathbb{N}, k \in \mathbb{Z}^d, 0 \le i < 2^d).$$

We will write c_λ instead of $c_{j,k}^{(i)}$ and ψ_λ instead of $\psi^{(i)}(2^j \cdot - k)$.

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Wavelet leaders

Given a dyadic cube $\lambda \in \Lambda_j$ at scale j, the p-wavelet leader of λ ($p \in [1, \infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} \big(2^{(j-j')d/p} |c_{\lambda'}|\big)^p \Big)^{1/p}.$$

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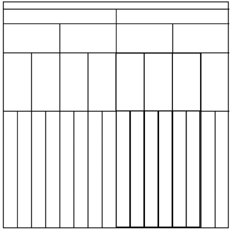
$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$



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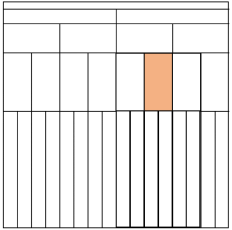

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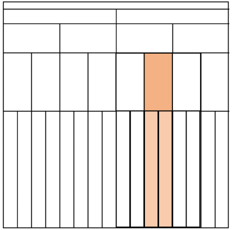

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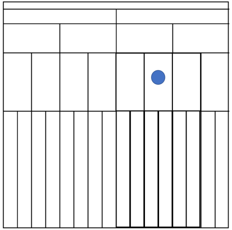

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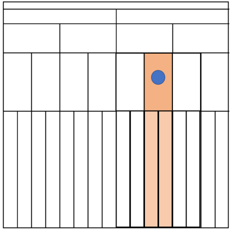

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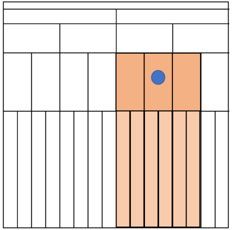

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L.L. & S. Nicolay (2020)

If f belongs to the space $T_{p,q}^{\boldsymbol{\sigma}}(x_0)$, then

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

Conversely, if $2^{-jd/p} \sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d/p}$, if f belongs to $B_{p,q}^s(\mathbb{R}^d)$ for some $s > 0$, then $(\sigma_j d_j^p(x_0))_j \in \ell^q$ implies $f \in T_{p,q,\log}^{\boldsymbol{\sigma}}(x_0)$.

Application to the detection of Fractional Brownian Motion

> A stochastic process $X = \{X(t) : t \in I\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family, indexed by I of real random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

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A stochastic process $X = \{X(t) : t \in I\}$ is gaussian if for all $a_1, \ldots, a_J \in \mathbb{R}$ and $t_1, \ldots, t_J \in I$,

$$\sum_{j=1}^{J} a_j X(t_j)$$

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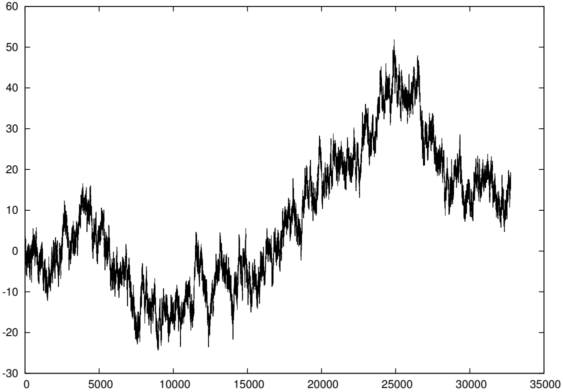
If $\alpha \in (0,1)$, $\{B_\alpha(t) : t \in \mathbb{R}\}$ is a Fractional Brownian Motion of order α if it is a gaussian process with expectation 0, which vanishes at 0 and such that

$$\mathbb{E}[B_\alpha(s)B_\alpha(t)] = \frac{\mathbb{E}[B_\alpha(1)^2]}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |s-t|^{2\alpha}).$$

Fractional Brownian Motion

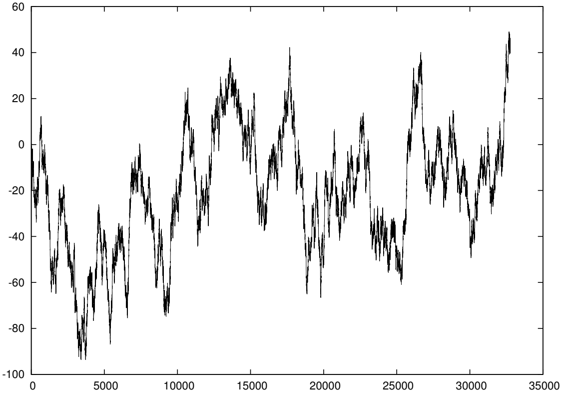
If $0 < \alpha < \frac{1}{2}$,



the increment process $\{B_\alpha(t+1) - B_\alpha(t) \;:\; t \in \mathbb{R}\}$ are negatively correlated.

Fractional Brownian Motion

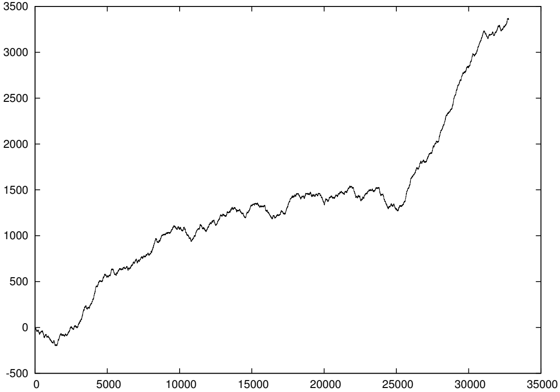
If $\alpha = \frac{1}{2}$ we have the standard Brownian motion,



the increment process $\{B_\alpha(t+1) - B_\alpha(t) \ : \ t \in \mathbb{R}\}$ are independent.

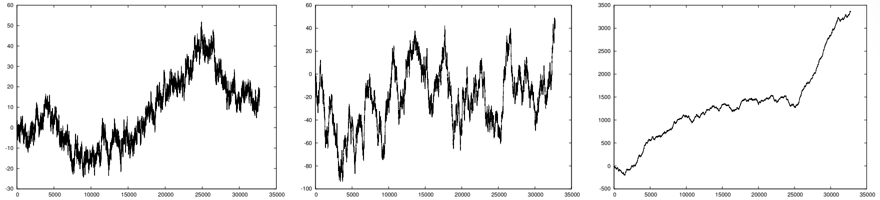
Fractional Brownian Motion

If $\frac{1}{2} < \alpha < 1$,



the increment process $\{B_\alpha(t+1) - B_\alpha(t) \ : \ t \in \mathbb{R}\}$ are positively correlated.

Fractional Brownian Motion



Fractional Brownian Motion appears in many applications

- Modeling turbulence in liquids
- Motion of particles in the Pollen
- Motion of Molecules
- Modeling the DNA
- Modeling the dynamics of a financial market.

For all these reasons, determining is a process is a Fractional Brownian and finding the associated exponents are **crucial** questions.

Pointwise behaviour of FBM

Esser-Loosveldt (2021)

If $\{B_\alpha(t) : t \in \mathbb{R}\}$ is a α-FBM then for every open interval $I \neq \emptyset$, almost surely

1. almost every $t \in I$ is an **ordinary point**:

$$0 < \limsup_{j \to +\infty} \frac{d_j^\infty(t)}{2^{-\alpha j}\sqrt{\log(j)}} \text{ and } \limsup_{s \to t} \frac{|B_\alpha(t) - B_\alpha(s)|}{|t-s|^\alpha \sqrt{|\log|\log|t-s||}} < \infty.$$

2. there exists a **rapid point** $t \in I$:

$$0 < \limsup_{j \to +\infty} \frac{d_j^\infty(t)}{2^{-\alpha j}\sqrt{j}} \text{ and } \limsup_{s \to t} \frac{|B_\alpha(t) - B_\alpha(s)|}{|t-s|^\alpha \sqrt{|\log|t-s||}} < \infty.$$

3. there exists a **slow point** $t \in I$:

$$0 < \limsup_{j \to +\infty} \frac{d_j^\infty(t)}{2^{-\alpha j}} \text{ and } \limsup_{s \to t} \frac{|B_\alpha(t) - B_\alpha(s)|}{|t-s|^\alpha} < \infty.$$

Monsters, Inc.

An overview on continuous but nowhere differentiable functions

Laurent Loosveldt
Université de Liège – FNRS Grantee

ComPlane: the next generation

17th June 2021