A Harmonic Balance-Based Tracking Procedure for Amplitude Resonances

Ghislain Raze, Martin Volvert, and Gaetan Kerschen

Aerospace and Mechanical Engineering Department
University of Liege, Belgium

ABSTRACT
In a number of applications, predicting the maximum displacement, velocity or acceleration amplitude that can be undergone by a forced nonlinear system is of crucial importance. Existing resonance tracking methods rely on phase resonance or approximate the response via a single-harmonic Fourier series, which constitutes an approximation in both cases. This work addresses this problem and proposes a harmonic balance tracking procedure to follow the amplitude extrema of nonlinear frequency responses. Means to compute the amplitude of multi-harmonic Fourier series and their time derivatives are first outlined. A set of equations describing the local extrema of the amplitude of a nonlinear frequency response is then derived. The associated terms and their derivatives can be computed via an alternating frequency-time procedure without resorting to finite differences. The whole method can be embedded in an efficient predictor-corrector continuation framework to track the evolution of amplitude resonances with a changing parameter such as the external forcing amplitude. The proposed approach is illustrated on two examples: a Helmholtz-Duffing oscillator and a doubly clamped von Kàrmàn beam with a nonlinear tuned vibration absorber.

Keywords: Amplitude Resonance, Harmonic Balance, Companion Matrix, Bordered System, Alternating Frequency-Time

INTRODUCTION

Nonlinearity is a frequent encounter in structural vibrations. Numerical approaches such as the harmonic balance (HB) method [1, 2] were developed to simulate the forced periodic responses of nonlinear systems. By coupling this method with a continuation scheme, it is possible to obtain the nonlinear frequency response (NFR) of the system for a given forcing amplitude.

A feature of interest of the NFR is the maximum amplitude undergone by some degree of freedom (DoF) of the system. While efficient algorithms exist to compute the amplitude resonances of linear systems (e.g. [3]), this is still an open problem for nonlinear systems. Procedures that track the extrema of single-harmonic Fourier series were coupled with the HB in [4, 5]. Alternatively, a phase resonance approach was proposed in [6] and shown to be an efficient approach for lightly-damped structures with well-separated resonance frequencies. However, there does not exist a method able to track the amplitude extrema of general (multi-harmonic) responses of nonlinear systems in the literature, and this works aims to fill this gap.

The computation of NFRs with the HB method is first reviewed, and means to compute the amplitude of a Fourier series are outlined. A procedure to track the amplitude extrema of NFRs is then proposed. The method is finally illustrated with two examples.

NONLINEAR FREQUENCY RESPONSE AND AMPLITUDE

Before deriving the equations characterizing an extremum of amplitude, the HB method is briefly recalled, and means to compute the amplitude of a Fourier series are outlined.

The harmonic balance method

The periodic solutions of the equations of motion of a nonlinear structure can be computed with the harmonic balance method. These equations read

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)+\mathbf{f}_{\text{nl}}(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)=\lambda\mathbf{f}_{\text{ext}}(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the structural mass, damping and stiffness matrices, respectively, $\mathbf{x}$ is the vector of generalized DoFs, and $\mathbf{f}_{\text{ext}}$ and $\mathbf{f}_{\text{nl}}$ are the associated generalized loading vectors related to the external and nonlinear and/or parametric forces, respectively. λ is a parameter used to describe the amplitude of the external forcing, and an overdot denotes derivation with respect to time (t).

A truncated Fourier series is assumed for the generalized DoFs as

$$\mathbf{x}(t)=(\mathbf{Q}(\omega t)\otimes\mathbf{I})\,\mathbf{z}, \tag{2}$$

where $\mathbf{z}$ is the vector of Fourier coefficients, $\mathbf{I}$ is the identity matrix, $\otimes$ denotes the Kronecker product and $\mathbf{Q}$ is the vector of harmonic functions

$$\mathbf{Q}(\omega t)=\begin{bmatrix}\frac{1}{\sqrt{2}} & \sin(\omega t) & \cos(\omega t) & \cdots & \sin(N_h\omega t) & \cos(N_h\omega t)\end{bmatrix}, \tag{3}$$

N_h being the number of harmonics considered in the Fourier series. Inserting the ansatz (2) into Equation (1) and using a Galerkin procedure, one obtains the dynamic equilibrium in the frequency domain,

$$\mathbf{A}(\omega)\mathbf{z}+\mathbf{b}(\mathbf{z},\omega)=\lambda\mathbf{b}_{\text{ext}}. \tag{4}$$

The full developments under this formalism are given in [1]. Equation (4) is typically solved for a fixed value of λ with a continuation scheme [1,2].

Amplitude of a Fourier series

We are now interested in a particular DoF u whose Fourier coefficients $\mathbf{z}_u$ can be obtained from $\mathbf{z}$ and ω (if u is obtained by derivation of the generalized DoFs) given by

$$u(\tau)=\mathbf{Q}(\tau)\mathbf{z}_u(\mathbf{z},\omega), \tag{5}$$

where $\tau=\omega t$. The amplitude of a Fourier series corresponds to the maximum absolute value of this series and thus occurs at one of its extrema. A necessary condition for an extremum is

$$u'(\tau)=0, \tag{6}$$

where a prime denotes derivation with respect to τ. Since $u'(\tau)$ is a trigonometric polynomial, it can be cast into a rational polynomial in the variable $s=e^{\mathrm{i}\tau}$ [7] (with $\mathrm{i}^2=-1$). The roots of the numerator of this rational polynomial in s can be found as the eigenvalues of the companion matrix associated to this ordinary polynomial [7]. The amplitude a of the Fourier series can eventually be found with the value $\tau_{\max}=\arg(s)$ (where $\arg(\cdot)$ is the complex argument) that maximizes the absolute value of the Fourier series, yielding

$$a=|\mathbf{Q}(\tau_{\max})\mathbf{z}_u(\mathbf{z},\omega)|. \tag{7}$$

The procedure is illustrated with a Helmholtz-Duffing oscillator governed by

$$\ddot{x}+0.1\dot{x}+x+1.6x^2+x^3=\lambda\sin(\omega t). \tag{8}$$

Figure 1 features the NFR amplitude at $\lambda=0.5$ obtained with the outlined semi-analytical procedure. It is also compared with a procedure where the amplitude is obtained by sampling the Fourier series and selecting the maximum absolute value among the discrete set of computed values. An advantage of this second approach is that its output can be obtained as a by-product of an alternating frequency-time (AFT) procedure [8]. The two approaches agree well overall, although the sampling procedure results in a spurious nonsmooth character of the amplitude, which is not due to aliasing [9] but is rather a discretization error.

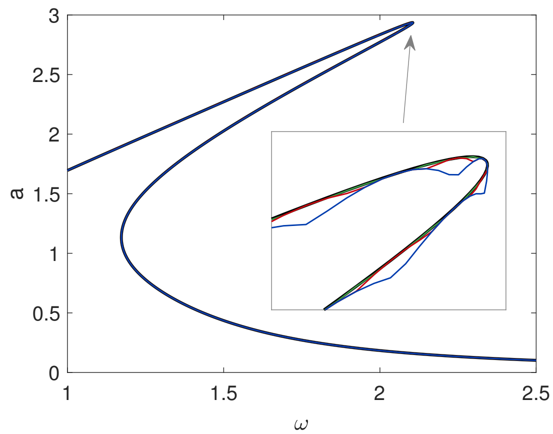


Figure 1: NFR amplitude of a Helmholtz-Duffing oscillator: semi-analytical amplitude (—), and amplitude computed using a sampling procedure with 64 (—), 128 (—) and 256 (—) points.

TRACKING PROCEDURE FOR AMPLITUDE EXTREMA

The maximum amplitude featured by an NFR is a quantity of interest but it generally depends on λ. Evaluating it for multiple values of λ requires computing several NFRs with fixed λ, which can become computationally expensive. Instead of computing multiple NFRs, the proposed tracking procedure considers λ as a variable and directly computes the locus of extrema. This section derives the characterizing equations for an extremum of amplitude, and discusses how to obtain their derivatives.

Equations for amplitude extrema

To characterize the amplitude extrema of NFRs, a scalar quantity g_p is appended to Equation (4) as

$$\mathbf{h}_p = \begin{bmatrix} \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}(\mathbf{z},\omega) - \lambda \mathbf{b}_{\text{ext}} \\ g_p(\mathbf{z},\omega) \end{bmatrix}, \tag{9}$$

and the condition $g_p = 0$ indicates an extremum of amplitude. Mathematically, g_p can either be the total derivative of the amplitude with respect to ω with fixed λ [4] or the singularity of the Jacobian of an extended system [5]. The latter approach is used herein for its robust character, and is extended to the case of a frequency-dependent amplitude. Following developments similar to [5], one eventually obtains

$$g_p(\mathbf{z},\omega) = 0 \qquad \Leftrightarrow \qquad \det(\tilde{\mathbf{J}}) = \det\left(\begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{z}} & \frac{\partial \mathbf{h}}{\partial \omega} \\ \frac{\partial a}{\partial \mathbf{z}} & \frac{\partial a}{\partial \omega} \end{bmatrix}\right) = 0. \tag{10}$$

g_p is not directly equated to the determinant of the Jacobian $\tilde{\mathbf{J}}$ in this work because its value and its derivatives would be computationally expensive to evaluate. It is instead given implicitly as the solution of the bordered system [1, 10]

$$\begin{bmatrix} \tilde{\mathbf{J}} & \mathbf{p} \\ \mathbf{q}^H & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ g_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \tag{11}$$

where $\mathbf{p}$ and $\mathbf{q}$ are arbitrary nontrivial vectors[1], and superscript H denotes a Hermitian transpose.

Equations (9)-(11) fully define a nonlinear system whose roots correspond to the extrema of the NFR amplitude. Using λ as a variable, a branch of solutions that corresponds to a one-parameter family of extrema can be computed using, e.g., a continuation approach.

[1]When $\tilde{\mathbf{J}}$ is singular, $\mathbf{p}$ and $\mathbf{q}$ must not lie in its range and the range of its Hermitian transpose, respectively [10], which is generically the case for arbitrary vectors.

Derivatives

To solve Equation (9), a Newton-Raphson procedure may be used. This requires the derivatives of Equation (9) with respect to the parameters $\mathbf{z}$, ω and λ. Computing the derivatives of $\mathbf{h}$ can efficiently be done using state-of-the-art methods such as the AFT procedure [1,2,8]. The derivatives of a featured in Equation (10) can be obtained following [11]. However, the derivatives of the last line are more complicated to compute, because this requires to know the second derivatives of $\mathbf{h}$ (since g_p is defined with its first derivatives, cf. Equation (10)). A similar issue is encountered in the bifurcation tracking literature, in which finite differences are typically used [1]. It is also possible to compute the full third-order tensor of second derivatives [12], but this procedure is both computationally expensive and memory intensive.

It is however possible to compute these derivatives semi-analytically with an AFT procedure, provided one has access to the functional form of the nonlinearities and can compute their second derivatives. Using a decomposition of the nonlinear forces into a sum of scalar contributions, the required derivatives can be computed as matrix products and circumvent the need for third-order tensors. The full derivations are not given here for brevity but will be available in the journal article associated to this publication [13].

EXAMPLES

The proposed procedure is illustrated with two examples: a single-DoF Helmholtz-Duffing oscillator, and a clamped-clamped von Kàrmàn beam.

A Helmholtz-Duffing oscillator

The Helmholtz-Duffing oscillator governed by Equation (8) is considered as a first example. Figure 2 features several NFRs for forcing levels up to $\lambda = 0.15$. The proposed extremum tracking procedure is able to accurately follow the amplitude resonance associated with the primary resonance of the oscillator. As long as they are strong enough to create a local amplitude maximum, non-primary amplitude resonances can also be tracked, such as the 2:1 superharmonic resonance featured in Figure 2.

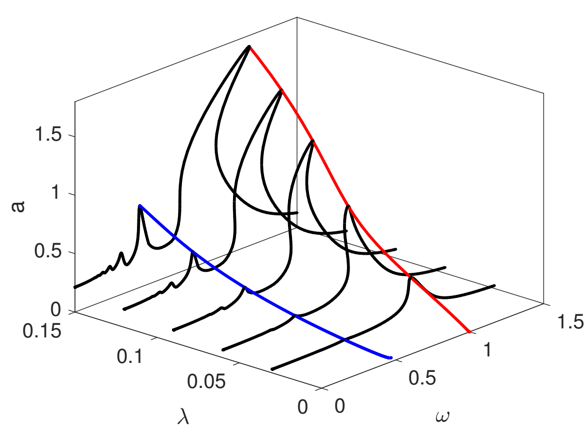


Figure 2: Set of NFRs of the Helmholtz-Duffing oscillator (—) and peak loci of the primary (—) and 2:1 superharmonic (—) resonances.

A von Kàrmàn beam with a nonlinear tuned vibration absorber

A clamped-clamped von Kàrmàn beam depicted in Figure 3 is considered as a second example. The parameters of the system are given in Table 1. The beam's response was simulated using a nonlinear finite element model [14]. 10 elements were used, thereby yielding a system with 27 DoFs (accounting for the clamped boundary conditions) for the bare beam, plus one for the absorber. Proportional modal damping of 0.5% was added to the structure for the first two bending modes.

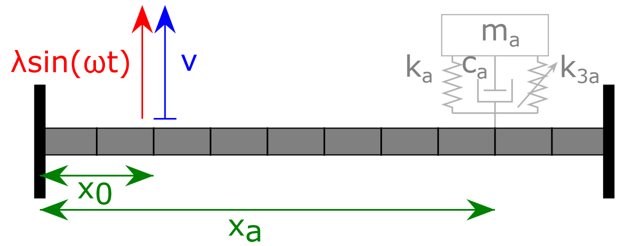
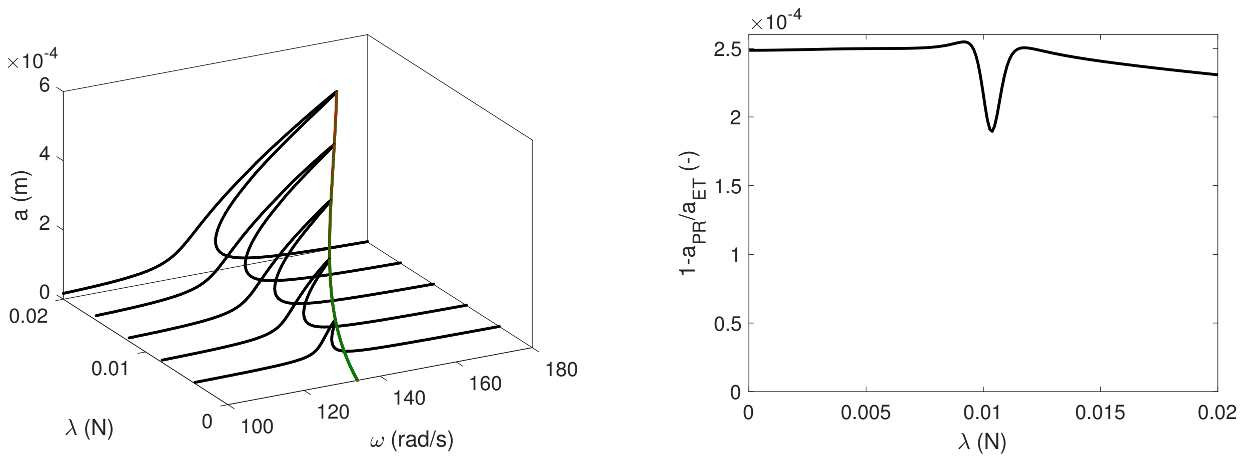


Figure 3: A clamped-clamped von Kàrmàn beam with a nonlinear tuned vibration absorber.

Table 1: Material and geometrical parameters of the doubly clamped beam.

Young's modulus (GPa)	Density (kg/m^3)	Length (mm)	Width (mm)	Thickness (mm)	x_0 (mm)
210	7800	500	20	1	100

Focusing first on the dynamics of the bare beam, Figure 4a shows that the proposed procedure is again able to accurately track the amplitude resonance associated with the first mode of the beam. The results were compared to phase resonance tracking, which also yields a remarkably accurate characterization of the amplitude resonance owing to the light damping present in the structure [6], as observed in Figure 4b.



(a) (b)

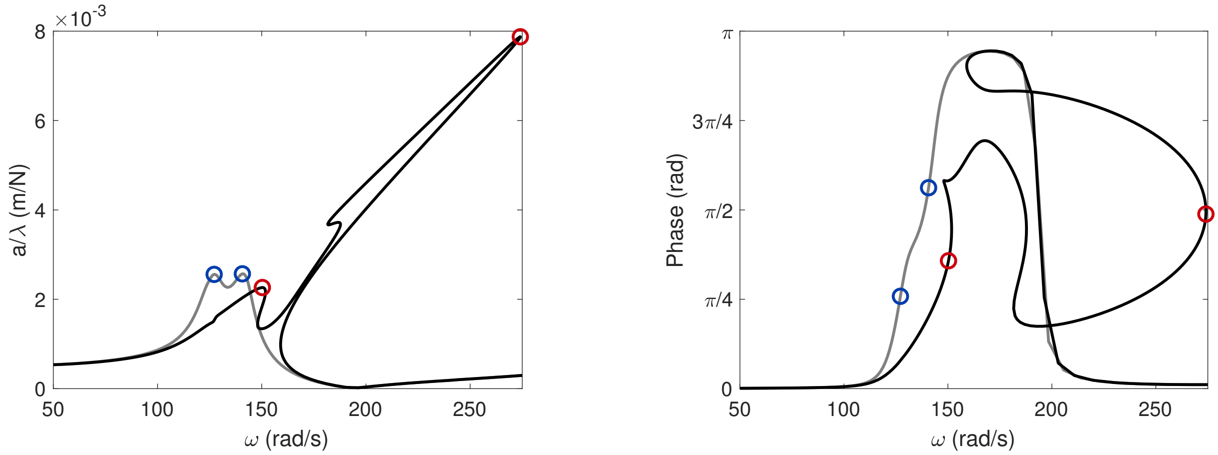
Figure 4: NFRs of the beam (—), phase resonance (—) and extremum tracking (—) (a); relative error of the phase resonance amplitude a_{PR} with respect to the extremum a_{ET} (b).

A nonlinear tuned vibration absorber (NLTVA) [15] was then added to the structure in order to mitigate the vibration amplitude around the first mode. The position of the absorber was chosen arbitrarily. The mass of the absorber was chosen to be 5% of that of the structure, and the other parameters were tuned following [16], using the implicit condensation approach [17] to obtain an effective cubic bending stiffness (including membrane stretching effects). The resulting parameters are gathered in Table 2.

Table 2: Parameters of the NLTVA.

m_a (kg)	c_a (kg/s)	k_a (N/m)	$k_{3,a}$ (N/m^3)	x_a (mm)
0.0039	0.09	68.3503	1.3764×10^7	400

The amplitude resonance of the controlled system cannot be accurately characterized by phase resonance. Indeed, Figure 5 shows the phase of the first harmonic is far from being constant and drastically changes between the quasi-linear ($\lambda = 0.03$ N) and strongly nonlinear ($\lambda = 0.15$ N) regimes of motion. This result is to be expected, given the high modal damping and closely-spaced resonance frequencies of the controlled structure.



(a)

(b)

Figure 5: Normalized NFRs of the controlled beam at $\lambda = 0.03$ (—: NFR, ○: amplitude peaks) and $\lambda = 0.15$ (—: NFR, ○: amplitude peaks): normalized multi-harmonic amplitude (a), and phase of the first harmonic (b).

The proposed tracking procedure allows once again for an accurate characterization of the peaks, as illustrated in Figure 6. Similarly to the case of a single-DoF host [15], the NLTVA is able to maintain nearly equal peaks in nonlinear regimes of motion (and therefore effective vibration mitigation), but an isola eventually merges with the rightmost peak of the controlled NFR, marking the end of the working range of the absorber. The extremum tracking procedure not only offers a way to assess the performance of the absorber, but also captures this phenomenon.

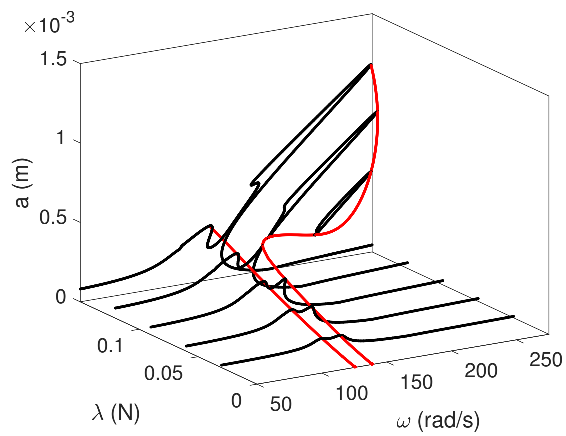


Figure 6: Set of NFRs of the controlled beam (—) and locus of resonant peaks (—).

CONCLUSION

A harmonic balance-based extremum tracking procedure was proposed in order to efficiently compute the evolution of amplitude resonances. An approach exploiting a companion matrix was used to compute the amplitude of multi-harmonic Fourier series. The equations characterizing an amplitude extremum of an NFR were derived, and the associated residuals and their derivatives can be computed efficiently with bordered systems and an AFT procedure. The proposed approach showed its accurate character with two examples, namely a single-DoF Helmholtz-Duffing oscillator and a clamped-clamped von Kàrmàn beam with an NLTVA.

ACKNOWLEDGEMENTS
Ghislain Raze is a Postdoctoral Researcher of the Fonds de la Recherche Scientifique - FNRS which is gratefully acknowledged.

REFERENCES

[1] T. Detroux, L. Renson, L. Masset, and G. Kerschen, “The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems,” *Comput. Methods Appl. Mech. Eng.*, vol. 296, pp. 18–38, nov 2015. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0045782515002297

[2] M. Krack and J. Gross, *Harmonic Balance for Nonlinear Vibration Problems*, ser. Mathematical Engineering. Cham: Springer International Publishing, 2019. [Online]. Available: http://link.springer.com/10.1007/978-3-030-14023-6

[3] N. Bruinsma and M. Steinbuch, “A fast algorithm to compute the H∞ norm of a transfer function matrix,” *Syst. Control Lett.*, vol. 14, no. 4, pp. 287–293, apr 1990. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/016769119090049Z

[4] E. P. Petrov, “Direct Parametric Analysis of Resonance Regimes for Nonlinear Vibrations of Bladed Disks,” *J. Turbomach.*, vol. 129, no. 3, pp. 495–502, jul 2007. [Online]. Available: https://asmedigitalcollection.asme.org/turbomachinery/article/129/3/495/477969/Direct-Parametric-Analysis-of-Resonance-Regimes

[5] A. Renault, O. Thomas, and H. Mahé, “Numerical antiresonance continuation of structural systems,” *Mech. Syst. Signal Process.*, vol. 116, pp. 963–984, feb 2019. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0888327018304060

[6] A. Förster and M. Krack, “An efficient method for approximating resonance curves of weakly-damped nonlinear mechanical systems,” *Comput. Struct.*, vol. 169, pp. 81–90, jun 2016. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0045794916300499

[7] J. P. Boyd, “Computing the zeros, maxima and inflection points of Chebyshev, Legendre and Fourier series: solving transcendental equations by spectral interpolation and polynomial rootfinding,” *J. Eng. Math.*, vol. 56, no. 3, pp. 203–219, jan 2007. [Online]. Available: http://link.springer.com/10.1007/s10665-006-9087-5

[8] T. M. Cameron and J. H. Griffin, “An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems,” *Journal of Applied Mechanics*, vol. 56, no. 1, p. 149, 1989. [Online]. Available: http://appliedmechanics.asmedigitalcollection.asme.org/article.aspx?articleid=1409535

[9] L. Woiwode, N. N. Balaji, J. Kappauf, F. Tubita, L. Guillot, C. Vergez, B. Cochelin, A. Grolet, and M. Krack, “Comparison of two algorithms for Harmonic Balance and path continuation,” *Mech. Syst. Signal Process.*, vol. 136, p. 106503, feb 2020. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0888327019307241

[10] W. J. F. Govaerts, *Numerical Methods for Bifurcations of Dynamical Equilibria*. Society for Industrial and Applied Mathematics, jan 2000. [Online]. Available: http://epubs.siam.org/doi/book/10.1137/1.9780898719543

[11] E. P. Petrov, “Sensitivity Analysis of Nonlinear Forced Response for Bladed Discs With Friction Contact Interfaces,” in *Vol. 4 Turbo Expo 2005*, vol. 5. ASMEDC, jan 2005, pp. 483–494. [Online]. Available: https://asmedigitalcollection.asme.org/GT/proceedings/GT2008/43154/355/324553https://asmedigitalcollection.asme.org/GT/proceedings/GT2005/47276/483/312435

[12] M. Krack, L. Panning-von Scheidt, and J. Wallaschek, “A high-order harmonic balance method for systems with distinct states,” *J. Sound Vib.*, vol. 332, no. 21, pp. 5476–5488, oct 2013. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0022460X13004057

[13] G. Raze, M. Volvert, and G. Kerschen, “Tracking Amplitude Extrema of Nonlinear Frequency Responses using the Harmonic Balance Method,” *Int. J. Numer. Methods Eng.*, in press.

[14] A. Lazarus, O. Thomas, and J.-F. Deü, “Finite element reduced order models for nonlinear vibrations of piezoelectric layered beams with applications to NEMS,” *Finite Elem. Anal. Des.*, vol. 49, no. 1, pp. 35–51, feb 2012. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0168874X11001673

[15] G. Habib, T. Detroux, R. Viguié, and G. Kerschen, "Nonlinear generalization of Den Hartog's equal-peak method," *Mech. Syst. Signal Process.*, vol. 52-53, no. 1, pp. 17–28, 2015. [Online]. Available: http://dx.doi.org/10.1016/j.ymssp.2014.08.009

[16] G. Raze and G. Kerschen, "Multimodal vibration damping of nonlinear structures using multiple nonlinear absorbers," *Int. J. Non. Linear. Mech.*, vol. 119, p. 103308, mar 2020. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0020746219303427

[17] M. McEwan, J. Wright, J. Cooper, and A. Leung, "A combined modal/finite element analysis technique for the dynamic response of a non-linear beam to harmonic excitation," *J. Sound Vib.*, vol. 243, no. 4, pp. 601–624, jun 2001. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0022460X00934345