Optimal quantum states for quantum information and how to prepare them

Eduardo Serrano Ensástiga

IPNAS, CESAM, University of Liège, Belgium

with J. Martin (ULiège), J. Denis (ULiège) and C. Chryssomalakos (ICN, UNAM)

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Outline of the talk

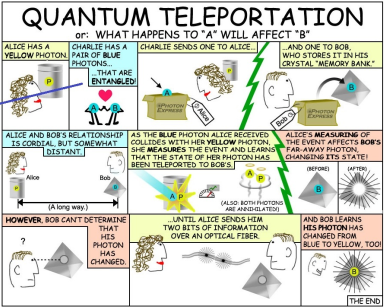
1. Basic concepts
   a. Mixed quantum states: a more realistic model
   b. Optimal quantum states by spectrum
2. Entanglement and separability
   a. Symmetric 2-qubit system
   b. Symmetric 3-qubit system (Numerical results)
   c. SAS witnesses for symmetric N-qubit systems
3. Quantum metrology
   a. Mixed Optimal Quantum Rotosensors
4. Conclusions

Quantum technology

Quantum Technology: The Second Quantum Revolution, Dowling and Milburn (2003)

The goal of quantum technology is to deliver useful devices and processes that are based on quantum principles: entanglement, quantum superposition, etc.

Maximally entangled state



In summary, we use quantum correlations of a quantum system to create/enhanced technology. In most of the cases, for a particular task, the optimal quantum states of a system are those who maximize a particular quantum correlation: entanglement, QFI, anticoherence, fidelity, etc.

Optimal quantum states

Pure states, $\mathcal{H}$

$$\rho = |\psi\rangle\langle\psi|$$

Mixed states, $\mathcal{HS}(\mathcal{H})$

$$\rho = \sum_{\alpha=1}^{d} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

$\lambda_\alpha \geqslant 0, \qquad \sum_\alpha \lambda_\alpha = 1$

$\{\psi_\alpha\}$ orthonormal basis

Most of the quantum correlations are convex with respect to mixture

$$Q\left(\sum_\alpha \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|\right) \leqslant \sum_\alpha \lambda_\alpha Q\left(|\psi_\alpha\rangle\langle\psi_\alpha|\right).$$

Consequently, the optimal quantum states for quantum technologies, i.e., the states that maximize a quantum correlation, would be only pure states. However...

Pure vs mixed states

Ideal vs Reality

Pure states of a quantum system are ideal scenarios. They face many experimental challenges

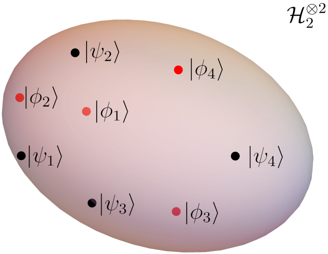
Mixedness

- Open quantum systems (Decoherence) $|\psi\rangle \Rightarrow_t \rho$
- Finite temperatures

$$\rho = \sum_\alpha \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \frac{e^{-H/k_B T}}{Z}$$

- Marginal densities $\rho_{\text{spin}} = \text{Tr}_{\text{spatial}} \left(|\Psi\rangle\langle\Psi|\right)$
- Particle loss $\rho = \text{Tr}_1(|\Psi\rangle\langle\Psi|)$

Maximizing a quantum correlation for a mixed state



Global unitary transformation

$$\rho = \sum_{k=1}^{d} \lambda_k |\phi_k\rangle\langle\phi_k| \, ,$$

$$U\rho U^\dagger = \sum_{k=1}^{d} \lambda_k |\psi_k\rangle\langle\psi_k| \, ,$$

Maximum Q. correlation by spectrum

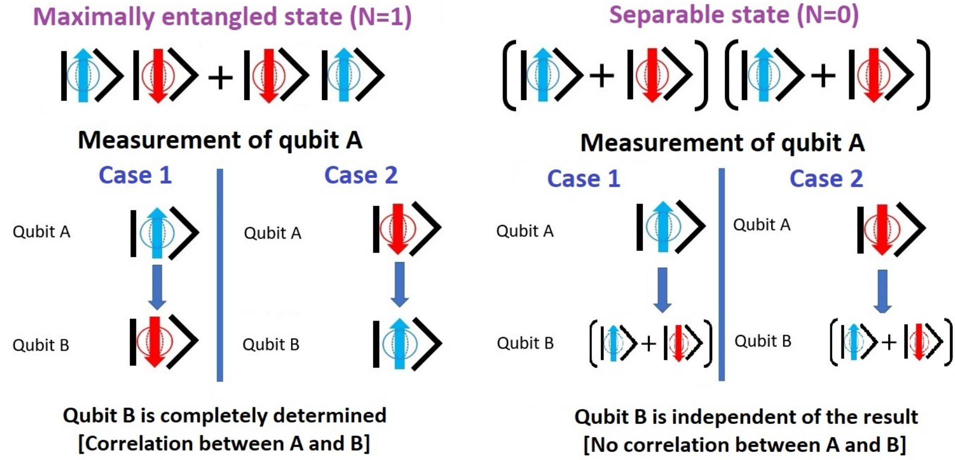
$$\max_{U \in SU(d)} Q\left(U\rho U^\dagger\right)$$

Global unitary transformations = Group of Schrödinger evolutions
= Set of thermal states
with fixed T and fixed eigenenergies
= Group of unitary gates
of a quantum protocol

Entanglement

Entanglement

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$



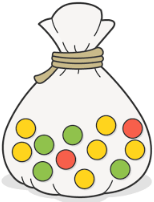
Entanglement of mixed states

Separable mixed states [Werner (1989)]

ρ is separable if

$$\rho = \int_{\mathcal{H}_2^{\otimes 2}} P(\boldsymbol{n}_1\,,\boldsymbol{n}_2)|\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2|\,\mathrm{d}\boldsymbol{n}_1\,\mathrm{d}\boldsymbol{n}_2\,.$$

with $P(\boldsymbol{n}_1, \boldsymbol{n}_2) \geq 0$. Otherwise is entangled.



Measure of entanglement

- $E(\rho) = 0$ if and only if ρ is separable.
- Invariant under local unitary transformations.
- Other properties...

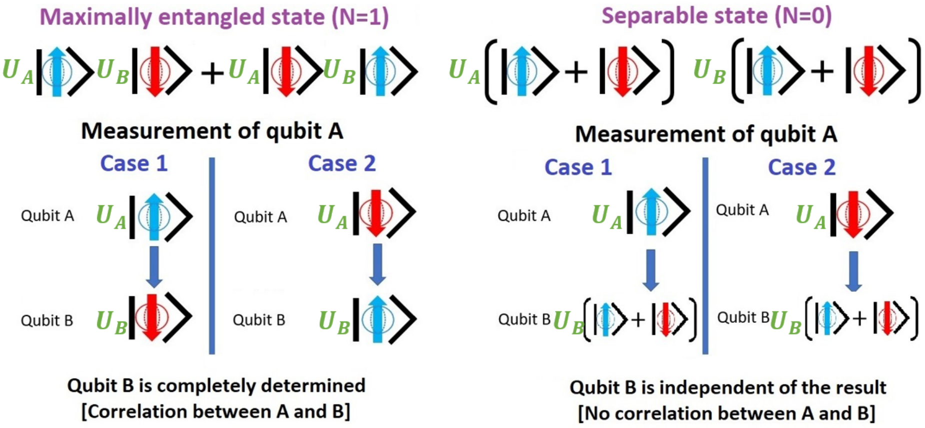
For qubit-qubit and qubit-qutrit: **Negativity** [Peres (1996)], [Horodecki et al (1996)]

The sum of the negative eigenvalues Λ_k of ρ^{T_A}

$$\mathcal{N}(\rho) = -2 \sum_{\Lambda_k < 0} \Lambda_k\,,$$

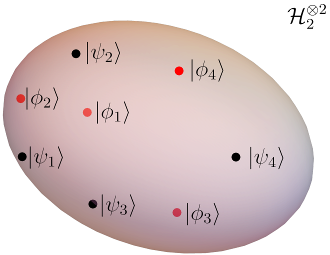
Entanglement

Invariant under local unitary transformations $U_A \otimes U_B \in SU(2) \otimes SU(2)$



Entanglement (Pure state case)

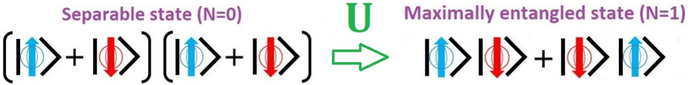
Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \, ,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \, ,$$

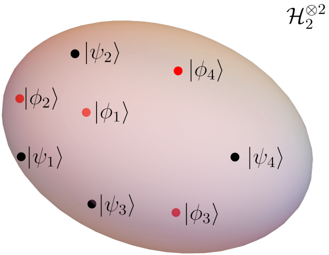


Pure state ρ_{pure}

$$\lambda_0 = 1 \, , \lambda_1 = \lambda_2 = \lambda_3 = 0 \, , \qquad \max_{U \in SU(4)} \mathcal{N}(U\rho_{pure}U^\dagger) = 1 \, ,$$

Entanglement (Maximally mixed state case)

Not-invariant under **global** unitary transformations $SU(4)$



Global unitary transformation

$$\rho = \sum_{k=0}^{3} \lambda_k |\phi_k\rangle\langle\phi_k| \,,$$

$$U\rho U^\dagger = \sum_{k=0}^{3} \lambda_k |\psi_k\rangle\langle\psi_k| \,,$$

$$\rho_* = U\rho_* U^\dagger = \frac{1}{4}\mathbb{1} = \frac{1}{4}\int_{S^2\otimes S^2} |\boldsymbol{n}_1\rangle|\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_1|\langle\boldsymbol{n}_2| \, \mathrm{d}^2\boldsymbol{n}_1 \, \mathrm{d}^2\boldsymbol{n}_2 \,.$$

Maximally mixed state ρ_*

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1/4 \,, \qquad \max_{U\in SU(4)} \mathcal{N}(U\rho_* U^\dagger) = 0 \,,$$

Maximum entanglement in the unitary orbit of ρ

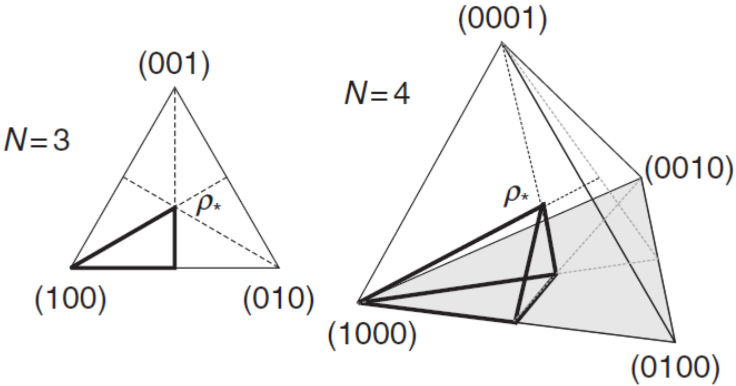


Figure taken from [Bengtsson and Życzkowski (2017)]

Questions

- What is the maximum entanglement of ρ attained in its $SU(4)$-orbit?
- Is ρ_* the unique state that is absolutely separable (AS) over all its unitary orbit?

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$

$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

$$\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right) = \max\left(0, \sqrt{(\lambda_0 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2} - \lambda_1 - \lambda_3\right),$$

ρ is AS iff $\lambda_0 \leqslant \lambda_2 + 2\sqrt{\lambda_1 \lambda_3}$.

[Verstraete, Audenart & De Moor (2001)].

Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$

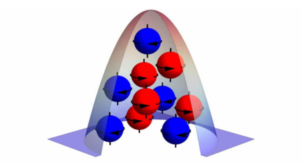
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$

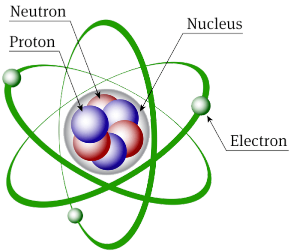
$$\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$$

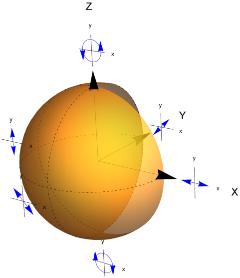
Open question. Partial results [Mendonça, Marchiolli, Herdemann (2017)]

Statement of the problem

Bosons: BEC, spin-j system, multiphotons systems, etc.







New question

For a symmetric qubit-qubit state ρ_S,

- What is the maximum entanglement achievable under a global unitary transformation U_S restricted in the symmetric subspace ?
- What is the spectrum of the symmetric states that remains separable after any global unitary transformation U_S?

Symmetric bipartite systems

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ $\max_{U \in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, 0)$ $\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	**Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$**
$\rho-$spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ $\max_{U \in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$	ρ_S-spectrum: $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0, 0)$ $\max_{U_S \in SU(4)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right)$

Symmetric 2-qubit system

Symmetric 2-qubit system

Theorem [ESE, Martin (2023)]

Let $\rho_S \in \mathcal{B}(\mathcal{H}_2^{\vee 2})$ with spectrum $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2$. It holds that

$$\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max\left(0, \sqrt{\lambda_0^2 + (\lambda_1 - \lambda_2)^2} - \lambda_1 - \lambda_2\right).$$

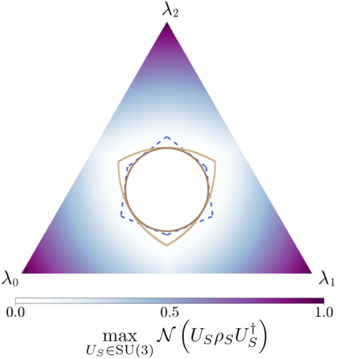


Imagen taken from [Denis, Davis, Mann, Martin (2023)]

Maximally entangled state

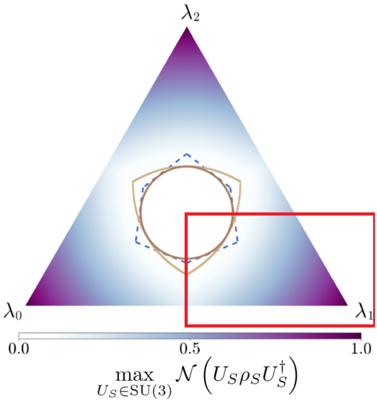
$$\rho_S = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

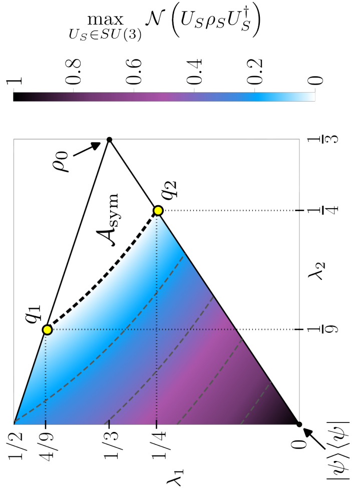
Details in the proof

- Bistochastic matrices $B \in \mathcal{B}_{N+1}$.
- (Birkhoff's theorem) Any bistochastic matrix is a linear combination of permutation matrices.

Symmetric qubit-qubit system

Main result





SAS states

$\mathcal{A}$
Absolutely separable (AS) states [Życzkowski (1999)]

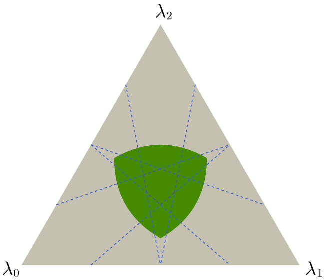
$$\max_{U \in SU(4)} \mathcal{N}(U \rho_S U^\dagger) = 0$$

$$\mathcal{A}(\mathcal{H}_2^{\vee 2}) = \{\rho_0\}$$

$\mathcal{A}_{\mathrm{sym}}$
Symmetric absolutely separable (SAS) states [Giraud et al (2008)]

$$\max_{U \in SU(3)} \mathcal{N}(U_S \rho_S U_S^\dagger) = 0$$

$$\mathrm{d}(\mathcal{A}_{\mathrm{sym}}(\mathcal{H}_2^{\vee 2})) = 2$$



Corollary [ESE, Martin (2023)]

$\rho_S \in \mathcal{A}_{\mathrm{sym}}$ iff

$$\sqrt{\lambda_1} + \sqrt{\lambda_2} \geq 1\,.$$

Applications

Symmetric qubit-qubit system at finite temperature

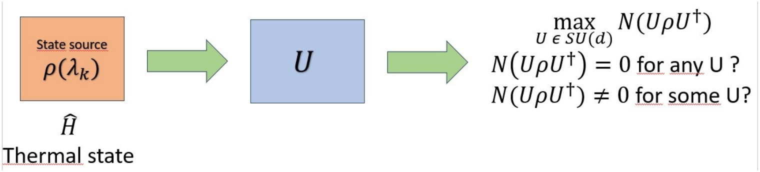
Hamiltonian: BEC [Ribeiro, Vidal Mosseri (2007)], Lipkin-Meshkov-Glick model (1965)

$$H = gJ_z + \gamma_x J_x^2 + \gamma_z J_z^2 \,,$$

with eigenenergies ϵ_j.

State at finite temperature T

$$\lambda_k = \frac{e^{-\beta \epsilon_{2s+2-k}}}{Z}\,, \quad \text{with} \quad Z = \mathrm{Tr}\left(e^{-\beta H}\right)\,.$$

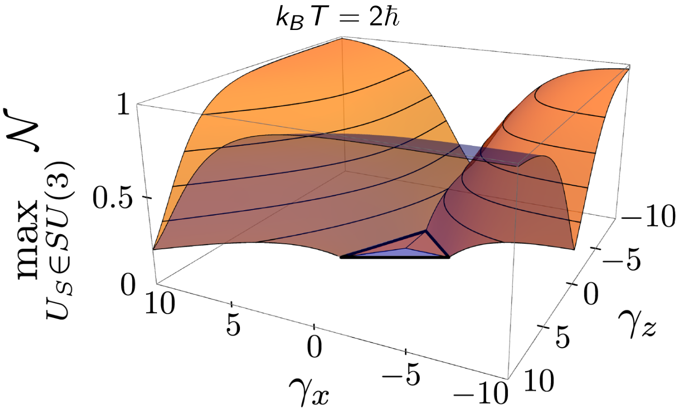


Maximum entanglement

Spectrum ϵ_j of H

For $g = 0$

$$\{\gamma_x\,,\gamma_z\,,\gamma_x+\gamma_z\}\;.$$



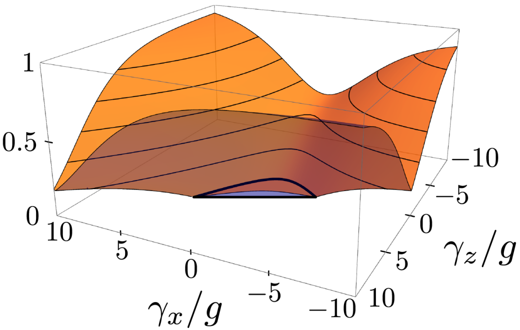
Maximum entanglement

Spectrum ϵ_j of H

For $g \neq 0$

$$\left\{\gamma_x\,,\frac{1}{2}\left(\gamma_x + 2\gamma_z - \sqrt{4g^2+\gamma_x^2}\right)\,,\frac{1}{2}\left(\gamma_x + 2\gamma_z + \sqrt{4g^2+\gamma_x^2}\right)\right\}.$$

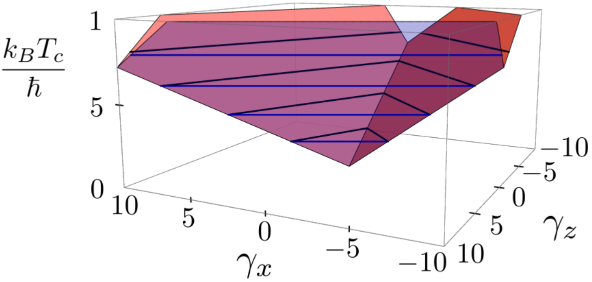
$k_B T = 3\hbar$



Condition of SAS states

$$\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow \frac{k_B T}{\hbar} \geqslant \frac{\epsilon_1 + \epsilon_2 - 2\epsilon_3}{2 \ln 2},$$

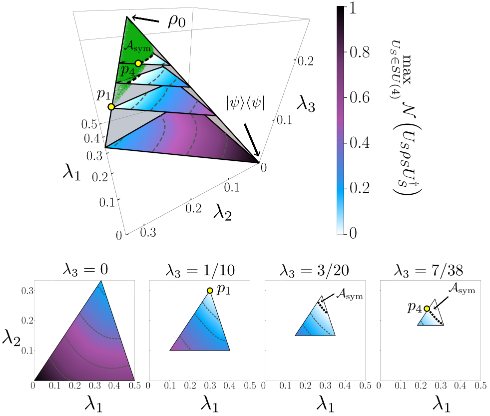
$$g = 0, \qquad k_B T = 2\hbar,$$



Symmetric 3-qubit system

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



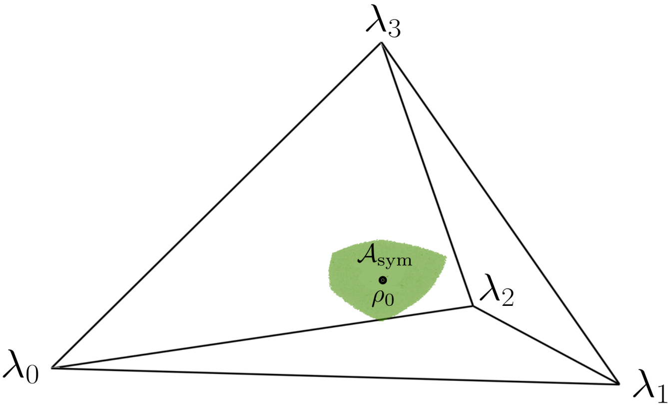
Maximally entangled states in the $SU(4)$-orbit

$$\rho_S = \begin{pmatrix} \tau_4 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_2 \end{pmatrix},$$

$$\rho_S = \begin{pmatrix} \frac{\tau_1+\tau_4}{2} & 0 & 0 & \frac{\tau_1-\tau_4}{2} \\ 0 & \frac{\tau_2+\tau_3}{2} & \frac{\tau_2-\tau_3}{2} & 0 \\ 0 & \frac{\tau_2-\tau_3}{2} & \frac{\tau_2+\tau_3}{2} & 0 \\ \frac{\tau_1-\tau_4}{2} & 0 & 0 & \frac{\tau_1+\tau_4}{2} \end{pmatrix}$$

Symmetric 3-qubit system

$\mathcal{H}_2^{\vee 3} \subset \mathcal{H}_2 \otimes \mathcal{H}_3$, numerical results



Set of SAS states in the spectra polytope of symmetric 3-qubit states.

SAS witnesses for symmetric N-qubit states

Separability condition for symmetric N-qubit states
[Giraud, Braun, Braun (2008)]

SAS states

Let $\rho \in \mathcal{B}(\mathcal{H}_2^{\vee N})$, ρ is separable $\Leftrightarrow$ there exists $P(\rho; \boldsymbol{n})$ such that

$$\rho = \int_{S^2} P(\rho; \boldsymbol{n}) |\boldsymbol{n}\rangle^{\otimes n} \langle \boldsymbol{n}|^{\otimes n} \, \mathrm{d}^2 \boldsymbol{n} \,,$$

and

$$\min_{\boldsymbol{n} \in S^2} P(\rho; \boldsymbol{n}) \geqslant 0 \,,$$

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho \,, \boldsymbol{n}) = \underbrace{\sum_{L=0}^{N} \sum_{M=-L}^{L} y_{LM} Y_{LM}(\boldsymbol{n})}_{P_0 = \mathrm{Tr}(\rho \omega^{(1)}(\boldsymbol{n})) \text{ , unique for } \rho} + \underbrace{\sum_{L=N+1}^{\infty} \sum_{M=-L}^{L} y_{LM} Y_{LM}(\boldsymbol{n})}_{P' \text{ , arbitrary } y_{LM}} \,,$$

SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2023)]

SAS states

Let $\rho \in \mathcal{B}(\mathcal{H}_2^{\vee N})$, $\rho \in \mathcal{A}_{\text{sym}} \Leftrightarrow$ there exists $P(U\rho U^\dagger; \boldsymbol{n})$ such that

$$U\rho U^\dagger = \int_{S^2} P(U\rho U^\dagger; \boldsymbol{n}) |\boldsymbol{n}\rangle^{\otimes n} \langle \boldsymbol{n}|^{\otimes n} \, \mathrm{d}^2 \boldsymbol{n} \,,$$

and

$$\min_{\substack{U \in SU(N+1) \\ \boldsymbol{n} \in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant 0 \,,$$

SAS-witness $\mathcal{W}$ [Bohnet-Waldraff, Giraud, Braun (2017)]

$$\rho \in \mathcal{A}_{\text{sym}} \quad \text{if} \quad \mathrm{Tr}(\rho^2) \leqslant \frac{1}{N+1} \left(1 + \frac{1}{2(2N+1)\binom{2N}{N} - (N+2)} \right) \,,$$

SAS witnesses for symmetric N-qubit states

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{L=0}^{N}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P_0=\mathrm{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\text{ , unique for }\rho} + \underbrace{\sum_{L=N+1}^{\infty}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P'\text{ , arbitrary }y_{LM}}\,,$$

Proposal

To consider $P(\rho, \boldsymbol{n})$ such that
i) They are covariant

$$P(U\rho U^\dagger, \boldsymbol{n}) = P(D(\mathrm{R})^\dagger U\rho U^\dagger D(\mathrm{R}), \boldsymbol{z}) = P(V\rho V^\dagger, \boldsymbol{z})\,.$$

ii) We built $P(U\rho U^\dagger, \boldsymbol{n})$ that their explicit expressions depend only on (or can be approximated) the (unistochastic) bistochastic matrices $B \in \mathcal{B}_{N+1}$

$$B_{ij} = |V_{ij}|^2\,, \quad B_{ij} \geqslant 0\,, \quad \sum_i B_{ij} = \sum_j B_{ij} = 1\,.$$

SAS witness $\mathcal{W}_1$: A polytope of SAS states

$P = P_0$ [Denis, Davis, Mann, Martin (2023)]

Observation 1

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P_0(U\rho U^\dagger; \boldsymbol{n}) = \min_{V\in SU(N+1)} \operatorname{Tr}\left[\rho V \omega^{(1)}(\boldsymbol{z}) V^\dagger\right]$$

$$\left(\rho_{jk} = \tau_j \delta_{jk}\,, \omega^{(1)}(\boldsymbol{z})_{jk} = \Delta_j \delta_{jk}\right) = \min_{B\in\mathcal{B}_{N+1}} \boldsymbol{\lambda} B \boldsymbol{\Delta}^T\,,$$

B a bistochastic matrix, $B \in \mathcal{B}_{N+1}$.

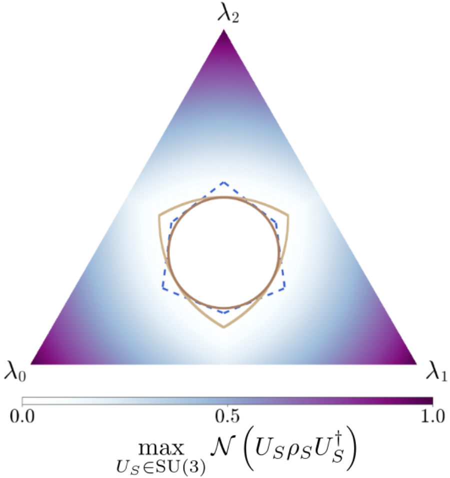
Observation 2 (Birkhoff's Theorem)

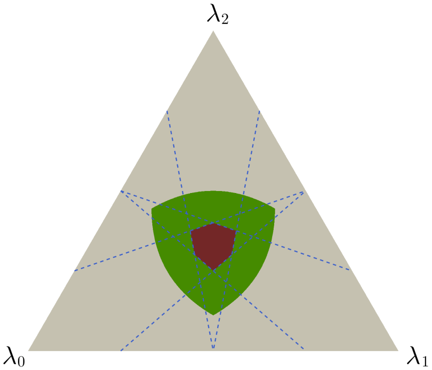
Permutations matrices achieve extremal values of a convex function $f(B)$

$$\min_{B\in\mathcal{B}_{N+1}} \boldsymbol{\lambda} B \boldsymbol{\Delta}^T = \min_{\Pi\in S_{N+1}} \boldsymbol{\lambda} \Pi \boldsymbol{\Delta}^T\,,$$

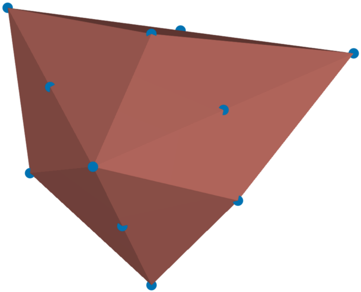
SAS witness $\mathcal{W}_1$

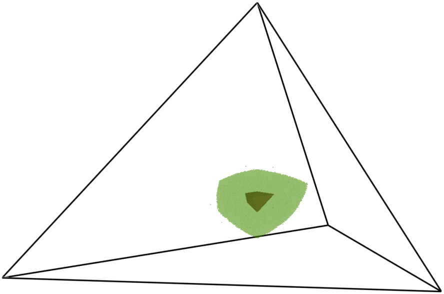
$$\rho \in \mathcal{A}_{\text{sym}} \qquad \text{if} \qquad \boldsymbol{\lambda}^{\downarrow}\boldsymbol{\Delta}^{\uparrow T} \geqslant 0\,, \qquad \Delta_k = (-1)^{N-k}\binom{N+1}{k}\,,$$





Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 2$.





Polytope of SAS states detected by $\mathcal{W}_1$ for $N = 3$.

SAS witness $\mathcal{W}_2(\{y_L\})$

Non-uniqueness of the P-function [Giraud, Braun, Braun (2008)]

$$P(\rho\,,\boldsymbol{n}) = \underbrace{\sum_{L=0}^{N}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P_0=\text{Tr}(\rho\omega^{(1)}(\boldsymbol{n}))\text{ , unique for }\rho} + \underbrace{\sum_{L=N+1}^{\infty}\sum_{M=-L}^{L} y_{LM}Y_{LM}(\boldsymbol{n})}_{P'\text{ , arbitrary }y_{LM}}\,,$$

We add some quadratic SU(2)-covariant terms of ρ (with $j = N/2$)

$$Q_L(\rho,\boldsymbol{n}) = \sum_{M=-L}^{L} \text{Tr}\left(\rho T_{LM}^{(j)\dagger}\right) Y_{LM}(\boldsymbol{n})\,,$$

$$P_L(\rho,\boldsymbol{n}) \equiv Q_L^2 - \sum_{\sigma=0}^{N}\sum_{\nu=-\sigma}^{\sigma}\left(\int Q_L^2\, Y_{\sigma\nu}^{*}(\boldsymbol{n}')\,\mathrm{d}\boldsymbol{n}'\right) Y_{\sigma\nu}(\boldsymbol{n})\,,$$

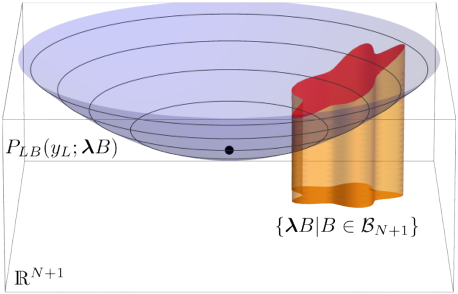
$$P'(\rho,\boldsymbol{n}) = \sum_{L>N/2} y_L P_L\,,$$

SAS witness $\mathcal{W}_2(\{y_L\})$

$P = P_0 + P'(\rho, \boldsymbol{n})$

$$\min_{\substack{U\in SU(N+1)\\ \boldsymbol{n}\in S^2}} P(U\rho U^\dagger; \boldsymbol{n}) \geqslant \min_{\substack{\boldsymbol{\lambda}B\\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L; \boldsymbol{\lambda}B)$$

P_{LB} is a quadratic function on the entries of B and linear on the $\{y_L\}$'s added by P'.



SAS witnesses $\mathcal{W}_2(\{y_L\})$

SAS witness $\mathcal{W}_2(\{y_L\})$: A symmetric $2j = N$-qubit state ρ is SAS if for some values of $\{y_L\}$

$$\min_{\substack{\boldsymbol{\lambda}B \\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda}B) = \min_{\substack{\boldsymbol{\lambda}B \\ B\in\mathcal{B}_{N+1}}} f + \sum_{L=1}^{2j}\left[g_L\boldsymbol{\lambda}B\,\mathbf{t}_L^T + h_L\left(\boldsymbol{\lambda}B\,\mathbf{t}_L^T\right)^2\right] \geqslant 0\,,$$

$$f = \frac{1}{N+1} + \left(\frac{y_N F(N,1)}{2}\right)\left(\mathrm{Tr}(\rho^2) - \frac{1}{N+1}\right)^2\,,$$

$$g_L = \sqrt{\frac{2L+1}{N+1}}\left(C_{jjL0}^{jj}\right)^{-1}\,, \quad h_L = y_L F(L,0)\Theta(L-j) - \frac{y_{2j}F(2j,1)}{2}\,,$$

$$\mathbf{t}_L = (C_{jj,j-j}^{L0}, -C_{jj-1,j1-j}^{L0}, \ldots, (-1)^{2j}C_{j-j,jj}^{L0})\,,$$

$$F(L,\mu) \equiv \begin{cases} 1 - \displaystyle\sum_{\substack{\sigma=0 \\ \sigma \text{ even}}}^{2j} \left(C_{L0L0}^{\sigma 0}\right)^2 & \text{if } \mu = 0 \\ 2(-1)^{\mu+1} \displaystyle\sum_{\substack{\sigma=0 \\ \sigma \text{ even}}}^{2j} C_{L0L0}^{\sigma 0} C_{L\mu L-\mu}^{\sigma 0} & \text{if } \mu \neq 0 \end{cases}$$

The variables h_L must be positive, restricting the domain of the free parameters $\{y_L\}$.

Example: $\mathcal{W}_2(\{y_2\})$ for $N = 2$

A symmetric 2-qubit state ρ with spectrum $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$ is SAS if

$$\min_{\substack{\boldsymbol{\lambda} B \\ B \in \mathcal{B}_3}} P_{LB}(y_L; \boldsymbol{\lambda} B) = \min_{\substack{\boldsymbol{\lambda} B \\ B \in \mathcal{B}_3}} f + \sum_{L=1}^{2} \left[g_L \boldsymbol{\lambda} B \, \mathbf{t}_L^T + h_L \left(\boldsymbol{\lambda} B \, \mathbf{t}_L^T \right)^2 \right] \geqslant 0$$

for some $y_2 \in \mathbb{R}^+$ and

$$f = \frac{1}{3} - \frac{12}{35} y_2 \left(\mathrm{Tr}(\rho^2) - \frac{1}{3} \right) ,$$

$$(g_1, g_2) = \left(\sqrt{2}, 5\sqrt{\frac{2}{3}} \right) , \quad (h_1, h_2) = \frac{6}{35} (2y_2, 5y_2) ,$$

$$\boldsymbol{t}_L = (C^{L0}_{11,1-1}, -C^{L0}_{10,10}, C^{L0}_{1-1,11}) ,$$

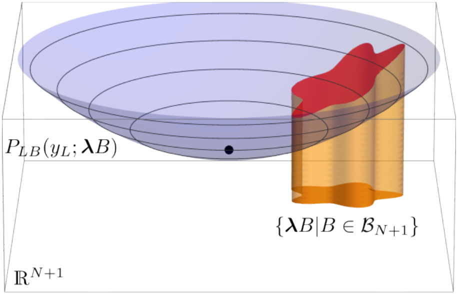
Instead of jabbering math, let us see a video.

SAS witness $\mathcal{W}_3$

A ball of SAS states detected by the $\mathcal{W}_2(\{y_L\})$ witnesses

$$\min_{\substack{\boldsymbol{\lambda} B \\ B\in\mathcal{B}_{N+1}}} P_{LB}(y_L;\boldsymbol{\lambda} B) \geqslant \min_{\boldsymbol{v}\in\mathbb{R}^{N+1}} P_{LB}(y_L;\boldsymbol{v}) \geqslant 0\,.$$

A function that depends only in the purity of the state $\mathrm{Tr}(\rho^2)$. Moreover, we can maximize the purity attained over the $\{y_L\}$ variables.



SAS witness $\mathcal{W}_3$

$\mathcal{W}_3$: A symmetric N-qubit state ρ is SAS if

$$r^2 \leqslant \frac{1}{(2j+1)^2} \left(\sum_{L=1}^{2j} \frac{g_L^2}{1 - 2\Theta(L-j)\frac{F(L,0)}{F(L,1)}} \right)^{-1} ,$$

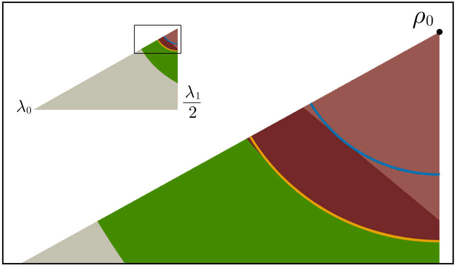
where $r^2 \equiv \left\|\rho - \rho_0\right\|_{\text{HS}}^2 = \text{Tr}(\rho^2) - (N+1)^{-1}$.

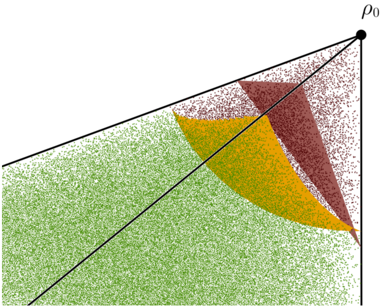
SAS witnesses for symmetric N-qubit states

[ESE, Denis, Martin (2024)]

Number of qubits $N = 2j$	$\begin{cases} \text{Witness } \mathcal{W}_1 \\ \text{Witness } \mathcal{W}_3 \end{cases}$
2	$\begin{cases} \boldsymbol{\lambda}\,(-3,\ 1,\ 3)^T \geqslant 0 \\ r^2 \leqslant \frac{1}{78} \approx 0.01282 \end{cases}$
3	$\begin{cases} \boldsymbol{\lambda}\,(-6,\ -1,\ 4,\ 4)^T \geqslant 0 \\ r^2 \leqslant \frac{1}{354} \approx 0.002825 \end{cases}$
4	$\begin{cases} \boldsymbol{\lambda}\,(-10,\ -5,\ 1,\ 5,\ 10)^T \geqslant 0 \\ r^2 \leqslant \frac{11}{25390} \approx 0.0004332 \end{cases}$
5	$\begin{cases} \boldsymbol{\lambda}\,(-15,\ -15,\ -1,\ 6,\ 6,\ 20)^T \geqslant 0 \\ r^2 \leqslant \frac{1595}{16058598} \approx 0.00009932 \end{cases}$

Table: SAS witnesses $\mathcal{W}_1$ and $\mathcal{W}_3$ for a state with eigenspectrum $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_N)$ sorted in descending order $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_N$.





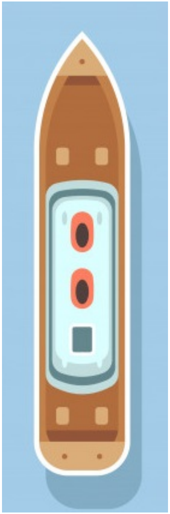
Dark Brown $= \mathcal{S}_2(\{y_L\})$
Light Brown $= \mathcal{S}_1$

Orange surface $=$ Bound of $\mathcal{S}_3$
Blue surface $=$ Bound of $\mathcal{S}$
[Bohnet-Waldraff, Giraud, Braun (2017)]

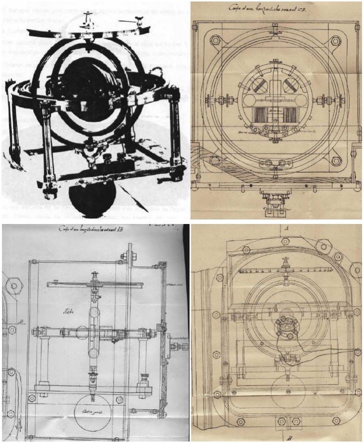
Green $=$ Unwitnessed SAS states by $\mathcal{W}_k$

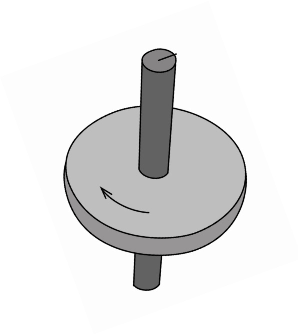
Optimal quantum rotosensors with mixed states

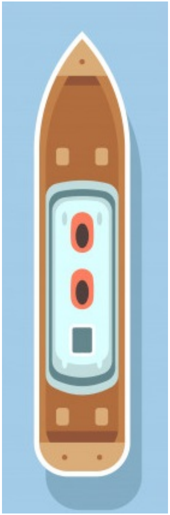
Metrology: The science of measurements



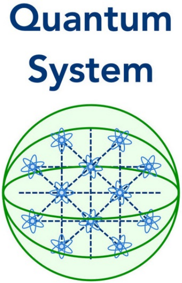
How can we measure a rotation on a boat in the see?







Which states of a quantum system could measure a rotation? *Quantum rotosensors*



Which is the optimal state of a quantum system to measure an infinitesimal rotation over an arbitrary axis? *Optimal quantum rotosensors* [Hernández-Coronado, Chryssomalakos (2017)]

Optimal quantum rotosensors

Quantum Fisher information (QFI) [Braunstein and Caves (1994)]

Initial state: $|\psi\rangle$ Final state: $e^{-i\theta S_n}|\psi\rangle$

Uhlmann-Jozsa Fidelity (Bures distance):

$F(\rho_i, \rho_f) = \mathrm{Tr}(\sqrt{\sqrt{\rho_i}\rho_f\sqrt{\rho_i}})$, Pure case $F = |\langle\psi|e^{-i\theta S_n}|\psi\rangle|^2$

Fidelity of an infinitesimal rotation (QFI)

$$QFI_{J_n}(|\psi\rangle) = g_{FS}(V_n, V_n) = \langle \Delta J_n \rangle^2 = \langle\psi|J_n^2|\psi\rangle - \langle\psi|J_n|\psi\rangle^2 .$$

$$QFI_{J_n}(\rho) = g_B(V_n, V_n) = \mathrm{Tr}(\rho J_n^2) - 2\sum_{l,m=1}^{k} \frac{\lambda_l \lambda_m}{\lambda_l + \lambda_m} |\langle\psi_l|J_n|\psi_m\rangle|^2 .$$

Quantum Cramer-Rao bound

$$\mathrm{Var}[\theta] \geq QFI_{J_z}(\rho)^{-1} .$$

Therefore, an *optimal quantum rotosensor over a fixed axis* maximizes $QFI_{J_z}(\rho)$.

Quantum Fisher information and Cramer-Rao bound

Averaged Quantum Cramer-Rao bound

$$\overline{\mathrm{Var}[\theta]} \geq \overline{QFI_{J_n}(\rho)^{-1}} = \int_{S^2} \frac{1}{QFI_{J_n}(\rho)} \, \mathrm{d}\boldsymbol{n} \, ,$$

By the Jensen inequality,

$$\overline{\mathrm{Var}[\theta]} \geqslant \overline{QFI_{J_n}(\rho)^{-1}} \geqslant \overline{QFI_{J_n}(\rho)}^{-1} \, ,$$

Hence, the conditions to minimize the r.h.s. of the Q Cramer Rao bound are

- Maximize $\overline{QFI_{J_n}(\rho)}$
- Saturate the Jensen Inequality $\Rightarrow QFI_{J_n}(\rho)$ independent of $\boldsymbol{n}$

Conditions of OQR

Anticoherence (AC)

- ρ is 1-AC if $\langle\psi|\boldsymbol{n}\cdot\boldsymbol{J}|\psi\rangle = 0$
- ρ is 2-AC if $\langle\psi|\boldsymbol{n}\cdot\boldsymbol{J}|\psi\rangle = 0$ and $\langle\psi|(\boldsymbol{n}\cdot\boldsymbol{J})^2|\psi\rangle = j(j+1)/3$

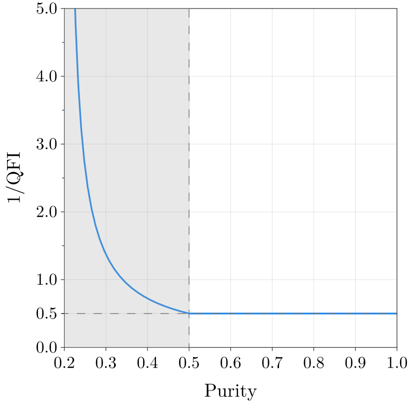
	Pure states $\|\psi\rangle$	Mixed states ρ
ρ maximizes $\overline{QFI_{J_{\mathbf{n}}}(\rho)}$	$\|\psi\rangle$ is 1-AC	$\mathrm{im}(\rho)$ is a 1-AC subspace
$QFI_{J_{\mathbf{n}}}(\rho)$ is independent of $\mathbf{n}$	$\|\psi\rangle$ is 2-AC	ρ is 2-AC

In both cases, the maximum estimation to measure a rotation has variance equal to

$$\overline{\mathrm{Var}[\theta]} \geq 3/j(j+1)\,.$$

Pure case [Goldberg, James (2017)]
Mixed case [ESE, C. Chryssomalakos, J. Martin (2024)]

Mixed OQRs



$$\rho(\xi) = \begin{cases} \xi\rho_{\psi_1} + (1-\xi)\rho_{\psi_2}, & \xi \in \left[\frac{1}{2}, 1\right] \\ \left(\frac{5\xi-1}{3}\right)(\rho_{\psi_1} + \rho_{\psi_2}) + \left(\frac{1-2\xi}{3}\right)\mathbb{1}, & \xi \in \left[\frac{1}{5}, \frac{1}{2}\right] \end{cases},$$

Conclusions

Maximum entanglement (negativity) over the unitary orbit for $N = 2, 3$

ESE and John Martin, SciPost Phys. **15**, 120 (2023)

SAS witnesses in terms of the spectrum

of the symmetric N-qubit states

ESE, Jérôme Denis and John Martin, PRA **109**, 022430 (2024)

Optimal quantum rotosensors with mixed states

ESE, John Martin and Chryssomalis Chryssomalakos, arXiv.2404.15548 (2024)

Thank you very much for your attention!

Observation 1

$$\mathcal{N}(\rho) = -2 \sum_{\Lambda_k < 0} \Lambda_k = -2\,(0\,, \Lambda_{\min})\ ,$$

Observation 2

$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{U \in SU(4)} \min_{|\psi\rangle \in \mathcal{H}_2^{\otimes 2}} \mathrm{Tr}\left[\rho U^\dagger (|\psi\rangle\langle\psi|)^{T_A} U\right]$$

Observation 3

$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{V \in SU(4)} \min_{\alpha \in [0,\pi]} \mathrm{Tr}\left[\rho V D V^\dagger\right]$$

$$\left(\rho_{jk} = \lambda_j \delta_{jk}\,, D_{jk} = \sigma_j(\alpha)\delta_{jk}\right) = \min_{B \in \mathcal{U}} \min_{\alpha \in [0,\pi]} \boldsymbol{\lambda}^T B \boldsymbol{\sigma}\,,$$

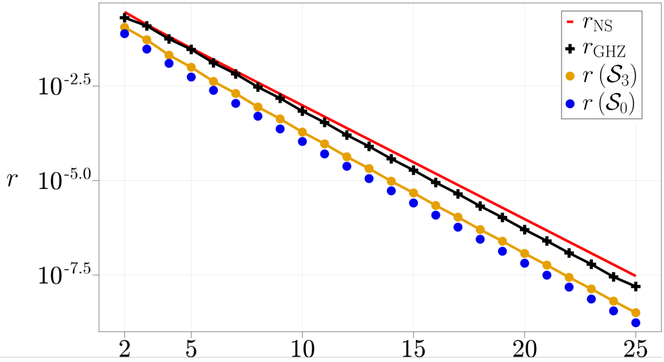
B an unistochastic matrix, $B \in \mathcal{U} \subset \mathcal{B}$.

Observation 4 (Birkhoff's Theorem)

Permutations matrices achieve extremal values of a convex function $f(B)$

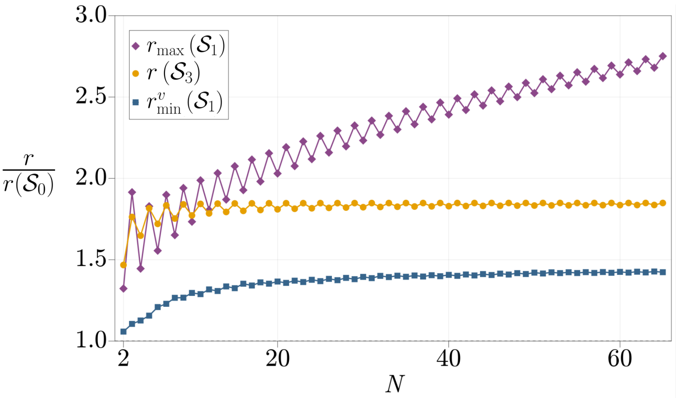
$$\min_{U \in SU(4)} \Lambda_{\min} = \min_{\Pi \in S_3} \min_{\alpha} \sum_{j=1}^{4} \boldsymbol{\lambda}^T \Pi \boldsymbol{\sigma}\,.$$

Comparison between the SAS witnesses



Comparison between the maximal distances of several sets of SAS states. The black crosses are defined by the furthest away SAS state in the ray ρ_0 and the GHZ pure state, with distance r_{GHZ}. The red line shows the radius $r_{\mathrm{NS}} = (2^N(2^N-1))^{-1/2}$ of the largest ball containing only AS states in the full Hilbert space [Gurvits and Barnum (2002)].

Comparison between the SAS witnesses



Distances $r_{\max}(\mathcal{S}_1)$ (purple), $r^v_{\min}(\mathcal{S}_1)$ (blue) and $r(\mathcal{S}_3)$ (orange), rescaled by the distance of the witness $r(\mathcal{S}_0)$.